SPREADSHEETS in EDUCATION

Bond University

Volume 12 | Issue 1 | 2019

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Solving the Black-Scholes Partial Differential Equation via the Solution Method for a One-Dimensional Heat Equation: A Pedagogic Approach with a Spreadsheet-Based Illustration[1]

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September 2019

[1] The author wishes to thank the two anonymous reviewers for valuable comments and suggestions.

Abstract

The derivation of the Black-Scholes option pricing model, if covered in detail, is by far the most complicated among all major models in the finance curriculum. This paper presents a pedagogic approach to solve the Black-Scholes partial differential equation via the solution method for a one-dimensional heat equation. It is intended to help finance students with backgrounds in traditional business disciplines strengthen their understanding of the model derivation, without being distracted by the advanced mathematical requirements. Excel plays an important pedagogic role in this paper. The Excel illustration here not only confirms numerically some key features of the model derivation, but also connects the derived Black-Scholes formula and results from two key intermediate steps in the model derivation, thus making the analytical materials involved much easier to follow.

Keywords: Black-Scholes Partial Differential Equation; Black-Scholes Option Pricing Model; Black-Scholes Formula; Heat Equation

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1 Introduction

The impact of the option pricing model of Black and Scholes (1973) on the financial world has been profound. Having inspired many financial innovations since its publication, it has ushered in a new era of modern finance. On the educational front, it has broadened students' knowledge of modern investments and has also led to higher requirements in mathematics and statistics for them. The derived Black-Scholes formula — which allows the price of a European call option on a stock that pays no dividends to be determined for a given set of input parameters — is now covered widely, even in introductory finance textbooks, as part of the standard curriculum in business education.[1] To understand the computations involved, students are required to have working knowledge of the cumulative normal distribution in statistics. Some textbook coverage of the Black-Scholes model has included, for computational purposes, Excel's statistical function NORMDIST or NORMSDIST.[2]

The same model is also known as the Black-Scholes-Merton option pricing model, in recognition of the contributions of Merton (1973). Merton has devised a method that is very different from the original Black-Scholes approach to derive the same model. As detailed and illustrated in Hull (2018), which is the most authoritative textbook on the topic of derivative securities, the Merton method has very wide applicability. Merton has also generalized the Black-Scholes formula. For finance textbooks, especially those at the introductory level, however, the usage of the term Black-Scholes formula, as compared to the term Black-Scholes-Merton formula, is much more common. As this paper uses a pedagogic approach to cover the original derivation

[1]A call option on a stock is a financial instrument that gives its holder the right, not the obligation, to purchase from its seller (writer) one share of the underlying stock, at a predetermined price, at or before an expiry date. American and European call options differ in that the latter can be exercised only at the expiry date. In a market with zero transaction costs, as an American call option on a stock that pays no dividends will not be exercised before the expiry date, its value is the same as that of the European call option with otherwise identical features.

[2]See, for example, Ross, Westerfield, Jaffe, Roberts, and Driss (2019, Chapter 23) and Berk, DeMarzo, and Stangeland (2019, Chapter 15) for textbook coverage of the Black-Scholes formula for call options. The latter textbook also indicates the Excel functions involved in numerical illustrations.

of the Black-Scholes model, it will stay with a shorter name for expositional convenience below.

The derivation of the Black-Scholes model — which, if covered in detail, is by far the most complicated among all major models in the finance curriculum — requires two distinct tasks to be performed. The first task is to reach the Black-Scholes partial differential equation, by using stochastic calculus tools to implement analytically a crucial economic insight that involves risk-free hedging under some simplifying assumptions. The second task, which has two consecutive parts labeled as Part 1 and Part 2 in this paper, consists of the analytical steps as needed to solve the Black-Scholes partial differential equation. Specifically, Part 1 is to transform the Black-Scholes partial differential equation into a one-dimensional heat equation. Heat equations, which are well-known in physical science and engineering fields, describe how temperature is distributed over space and time as heat spreads. Part 2 is to solve a specific heat equation to reach the Black-Scholes formula.

For the first task, relevant materials from stochastic calculus, including geometric Brownian motion and Itô's lemma, are available in textbooks such as Hull (2018, Chapters 14 and 15) and Wilmott, Howison, and Dewynne (1995, Chapters 2 and 3). An Excel-based illustration of geometric Brownian motion and an informal derivation of Itô's lemma, as applied to the Black-Scholes model, is also available in Brewer, Feng, and Kwan (2012). In contrast, how the second task can be performed has remained a mystery for many finance students, as the corresponding textbook coverage is primarily intended for students who have advanced mathematical knowledge. See, for example, Wilmott, Howison, and Dewynne (1995, Chapters 4 and 5) for the analytical details. Further, Hull (2018, Chapter 13) has opted instead for an approach — known as risk-neutral valuation — that bypasses the Black-Scholes partial differential equation to derive the Black-Scholes formula.

Finance students with backgrounds in traditional business disciplines tend to have only rudimentary knowledge of the topic of differential equations. As the Black-Scholes partial differential equation is, by far, more complicated than any differential equations that these students have ever encountered in their calculus courses, there will inevitably be a huge knowledge gap to bridge if the Black-Scholes model derivation in its entirety is covered in a finance course. Thus, it is challenging for the instructor to deliver the materials involved, without having to drag many students along a burdensome path during a lengthy model derivation.

This paper, which has its focus on the aforementioned second task, is motivated by such a challenge. The materials in Part 1 — which are all about changes of variables — can still be explained to students who have some knowledge of multivariate differential calculus, though unfamiliar with the topic of differential equations. This basic requirement is expected to be exceeded by many finance students who have learned, for example, comparative statics, portfolio investments, or market equilibrium in previous finance and economics courses, given that multivariate differential calculus is deemed an essential mathematical tool for learning such topics. However, students who do not meet this basic requirement may find the materials involved too complicated to follow. To accommodate such students, the instructor may have to provide them with guidance, in advance, for reviewing some relevant materials in multivariate differential calculus involving changes of variables.

The materials in Part 2 of the aforementioned second task, which have further mathematical requirements for students, are even more challenging for the instructor to deliver. From a pedagogic standpoint, the delivery of the materials there can be basic or thorough, depending on the mathematical knowledge of the students involved. Here, a basic version is where one accepts the solution method for a one-dimensional heat equation as given. In contrast, a thorough version is where the heat equation must be solved as well; that is, without treating the solution method available from science and engineering fields as given. The level of thoroughness also depends on how well the students involved are familiar with each required mathematical tool.

This paper covers Part 1 and only a basic version of Part 2, in order to bypass various advanced mathematical materials. For a thorough coverage of Part 2 based on a well-established solution method, the materials involved include not only how to transform the heat equation into an ordinary differential equation, but also how to work with complex variables in the process. The presence of complex variables is a direct result of the Fourier transform of a function of real variables in the heat equation. To complete the derivation, the original function will have to be brought back in the end via the inverse Fourier transform. An obvious benefit of a thorough version is that, once the task is completed, there will be no more gaps in students' understanding of the Black-Scholes model derivation. To benefit fully from a thorough derivation, finance students without prior knowledge of the Fourier transform will have to learn it first. A practical concern, however, is that the learning process can be challenging for many finance students.

For a basic version of Part 2 as covered in this paper, it is unimportant whether students have any prior experience in solving differential equations. Nevertheless, they are still required to have some knowledge of the normal distribution, as well as integral calculus including integration techniques in univariate settings, so that they can follow the materials involved to reach the Black-Scholes formula. Although many finance students with backgrounds in traditional business disciplines have good knowledge of the normal distribution, even including the analytical expression of its probability density function, they have seldom encountered any finance topics that require the use of integration techniques in calculus. Notably, based on anecdotal observations by the author of this paper, finance students are seldom required to verify analytically that integrating any specific probability density function over all possible outcomes will result in unity as expected. If such observations reflect reality in general, then a basic version of Part 2 may still be very difficult for some finance students.

To address the above concerns, this paper uses a pedagogic approach to cover the materials involved. Specifically, Part 1 — which, as indicated earlier, is all about changes of variables — is viewed as having two successive steps. The first step is for reducing the Black-Scholes differential equation to a more convenient form, in order to facilitate its eventual transformation into a one-dimensional heat equation in the second step. Part 2 is viewed as having three successive steps instead. The first step of Part 2 sets up two definite integrals for the next step to commence by first reducing the boundary condition for a call option to an analytically convenient form. This step is straightforward for students who are familiar with essential properties of exponential functions of real variables. In the second step of Part 2, integration techniques in calculus are used to reach an interim formula involving the cumulative standard normal distribution. Such a formula will lead to the Black-Scholes formula in the third step there, upon further changes of variables to bring back the original set of variables.

In order to reduce the reliance on integration techniques in calculus for students to understand Part 2, this paper uses an Excel spreadsheet to illustrate numerically the equivalence of the Black-Scholes formula and the result of each of the first two steps there during its derivation. Specifically, it is shown in the first step of Part 2 that the analytical solution of the heat equation satisfying the boundary condition of a call option can be expressed as the difference of two definite integrals. Such an analytical feature is particularly useful for a numerical illustration, because a definite integral in a univariate setting can be interpreted in terms of an enclosed area.

The use of Excel to perform numerical integration for finding the enclosed area and then to establish numerically the equivalence of the integration results and the corresponding analytical results in the remaining steps is helpful for students. Each of the two definite integrals in the first step of Part 2, once integrated analytically in the second step, can be expressed in terms of a cumulative distribution of the standard normal distribution, thus revealing the same key feature of the Black-Scholes formula even before the completion of its derivation. The numerical equivalence of the corresponding results in the three steps of Part 2 will enable students to boost their confidence in following the analytical materials there, even if they are initially unfamiliar with integration techniques in calculus.

As it will be clear later in the paper, some additional computations in the same Excel spreadsheet will be beneficial for students whose mathematical barriers include also their inadequate knowledge of multivariate differential calculus involving changes of variables. For example, some materials in the first step of Part 2 are also illustrated numerically in the Excel spreadsheet that accompanies this paper. Students can benefit more fully from the illustration by verifying numerically, on their own, the robustness of the results as described in detail later in this paper, with different sets of the input parameters.

This paper is organized in the following manner: With the Black-Scholes partial differential equation being the starting point, Section 2 summarizes and explains pedagogically the key features of the Black-Scholes model derivation. The corresponding analytical details are provided in the two appendices at the end of this paper, which are based on Wilmott, Howison, and Dewynne (1995, Chapter 5). Specifically, Appendix A covers Part 1 of the aforementioned task: it shows all analytical steps as required to transform the Black-Scholes partial differential equation into a one-dimensional heat equation. Appendix B covers a basic version of Part 2 of the aforementioned task; by treating the solution method for the heat equation as given, it shows the analytical steps leading to the Black-Scholes formula. Section 3 illustrates how Excel can help in making the Black-Scholes model derivation easier to follow by finance students with backgrounds in traditional business disciplines. Finally, Section 4 provides some concluding remarks.

2 The Black-Scholes Model Derivation: Summary and Pedagogic Explanations

The Black-Scholes partial differential equation for a European call option on a stock that pays no dividends is

$$\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} + rS\frac{\partial C(S,t)}{\partial S} - rC(S,t) = 0, \tag{1}$$

satisfying the boundary condition of

$$C(S,T) = \max(S - X, 0). \tag{2}$$

Here, r is the continuously compounded annual risk-free interest rate, σ — which is a standard deviation of returns — captures the volatility of the underlying stock returns in annual terms, X is the exercise price of the option, T is the time of expiry of the option, S is the price of the underlying stock at time t, and $C(S,t)$ is the price of the option at time t. Implicitly, r, T, σ, and X are constants, and t and T are measured in proportions of a year, for $0 \leq t \leq T$. Notice that the available correspondence between the equation numbers in the main text and those in the two appendices is indicated at the end of Appendix B.

Appendix A shows that, after some changes of variables, equation (1) reduces to the following form:

$$\frac{\partial \theta(x,\tau)}{\partial \tau} = \frac{\partial^2 \theta(x,\tau)}{\partial x^2}, \tag{3}$$

where $\theta(x,\tau)$ is a function of x and τ to be solved. Equation (3) that satisfies the boundary condition of

$$\theta(x,0) = \theta_0(x), \tag{4}$$

where $\theta_0(x)$ is a known function of x, is a one-dimensional heat equation for $-\infty < x < \infty$ and $\tau > 0$. In physical science and engineering fields, the solution is known to be

$$\theta(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{w=-\infty}^{\infty} \theta_0(w) \exp\left[-\frac{(x-w)^2}{4\tau}\right] dw \tag{5}$$

or its variant.

Subsequently, Appendix B shows the analytical details of how equation (5) leads to the Black-Scholes formula. With $C(S,t)$ written simply as C for notational convenience, the Black-Scholes formula is

$$C = S\ N(d_1) - X\ \exp[-r(T-t)]\, N(d_2), \tag{6}$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S}{X}\right) + r(T-t)\right] + \frac{1}{2}\sigma\sqrt{T-t} \tag{7}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}. \tag{8}$$

With the cumulative standard normal distribution

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{y=-\infty}^{z} \exp\left(-\frac{y^2}{2}\right) dy \tag{9}$$

being a function of z, it implicitly includes $z = d_1$ and $z = d_2$ above. In finance textbooks, the term $T-t$ is often captured simply by a symbol representing, at the time when C is measured, the proportion of a year before the option expires.

2.1 Transforming the Black-Scholes Partial Differential Equation into a One-Dimensional Heat Equation

Part 1, the first step: As shown in Appendix A, the transformation of equation (1) into equation (3) requires an intermediate step. Equation (1) is first transformed into

$$\frac{\partial v(x,\tau)}{\partial \tau} = \frac{\partial^2 v(x,\tau)}{\partial x^2} + (k-1)\frac{\partial v(x,\tau)}{\partial x} - kv(x,\tau), \tag{10}$$

where

$$k = \frac{2r}{\sigma^2}, \tag{11}$$

$$\tau = \frac{\sigma^2}{2}(T-t), \tag{12}$$

$$x = \ln\left(\frac{S}{X}\right), \tag{13}$$

and

$$v(x,\tau) = \frac{C(S,t)}{X}. \tag{14}$$

With such changes in variables, the boundary condition in equation (2) is equivalent to

$$v(x,0) = \max[\exp(x) - 1, 0]. \tag{15}$$

The idea of this intermediate step is that, by first writing equation (1) equivalently as

$$\frac{\partial C(S,t)}{\partial t} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} - rS\frac{\partial C(S,t)}{\partial S} + rC(S,t), \tag{16}$$

we look for a way to change the variables involved, in order to reduce the multiplicative factor $-\sigma^2 S^2/2$ for the first term on the right hand side to unity, thus making the resulting equation closer to equation (3) in appearance. Specifically, with $C(S,t)$ expressed in terms of $\upsilon(x,\tau)$, we are interested in reaching a partial differential equation where the left hand side is $\partial\upsilon(x,\tau)/\partial\tau$ and the first term on the right hand side is exactly $\partial^2\upsilon(x,\tau)/\partial x^2$. The sign reversal, from $-\sigma^2 S^2/2$ to $\sigma^2 S^2/2$, is achieved by letting

$$t = T - \frac{2\tau}{\sigma^2}, \tag{17}$$

which is equivalent to equation (12). Here, τ is the proportion of a year before the option expires, multiplied by the factor $\sigma^2/2$. As t increases, the corresponding τ decreases instead, thus ensuring the above sign reversal. This multiplicative factor is intended to match the $\sigma^2/2$ part of the multiplicative factor $-\sigma^2 S^2/2$ for the first term on the right hand side of equation (16).

To take care of the S^2 part of the multiplicative factor there requires the following variable change:

$$S = X\exp(x), \tag{18}$$

which is equivalent to equation (13). With the variable x being the natural logarithm of the ratio S/X, which is a dimensionless quantity, the corresponding option price $\upsilon(x,\tau)$ is expressed as the ratio C/X, just as equation (14) indicates. The first term on the right hand side of equation (16), $\partial^2 C(S,t)/\partial S^2$, when expressed in terms of $\partial^2\upsilon(x,\tau)/\partial x^2$, will generate an $\exp(-2x)$ term, which is the same as $1/S^2$, to cancel the original S^2 part of the multiplicative factor $-\sigma^2 S^2/2$. As both x and $\upsilon(x,\tau)$ are dimensionless quantities, the option will be exercised if $\exp(x) > 1$, and thus the boundary condition is as indicated in equation (15).

Part 1, the second step: To transform equation (10) into equation (3) requires the following substitution:

$$\upsilon(x,\tau) = \exp(\alpha x + \beta\tau)\theta(x,\tau). \tag{19}$$

The task here is to find the specific values of α and β in terms of k, for which equation (3) is the end result. As it turns out that

$$\alpha = -\frac{1}{2}(k-1) \tag{20}$$

and

$$\beta = -\frac{1}{4}(k+1)^2, \tag{21}$$

the boundary condition is

$$\theta(x,0) = \max\left\{\exp\left[\frac{1}{2}(k+1)x\right] - \exp\left[\frac{1}{2}(k-1)x\right], 0\right\}, \tag{22}$$

which is $\theta_0(x)$ for use in equation (5). The attainment of equations (3) and (22) marks the completion of the task to transform the Black-Scholes partial differential equation into a one-dimensional heat equation, for which equation (5) is applicable.

2.2 Solving the Heat Equation to Reach the Black-Scholes Formula

To use equation (5) directly to solve equation (3) under the boundary condition in equation (22), no prior experience in solving partial differential equations is deemed necessary. This is because, with $\theta_0(w)$ being a known function of w, all that is required to reach the solution is to perform the integration in equation (5). For such a task, however, students are required to have some knowledge of integration techniques and to recognize some key expressions pertaining to the normal distribution, both in univariate settings.

Part 2, the first step: As shown in Appendix B, the expression $\exp[(k+1)w/2] - \exp[(k-1)w/2]$ has the same sign as w. With $\theta_0(w) = 0$ for $w < 0$, equation (5) reduces to

$$\theta(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{w=0}^{\infty} \theta_0(w)\exp\left[-\frac{(x-w)^2}{4\tau}\right]dw = I_1 - I_2, \tag{23}$$

where

$$I_1 = \frac{1}{2\sqrt{\pi\tau}}\int_{w=0}^{\infty} \exp\left[\frac{1}{2}(k+1)w - \frac{(x-w)^2}{4\tau}\right]dw \tag{24}$$

and

$$I_2 = \frac{1}{2\sqrt{\pi\tau}}\int_{w=0}^{\infty} \exp\left[\frac{1}{2}(k-1)w - \frac{(x-w)^2}{4\tau}\right]dw. \tag{25}$$

Notice that the lower limit of the integral in equation (23) is now $w = 0$, instead of $w = -\infty$ in equation (5), because $\theta_0(w) = 0$ for $w < 0$.

Part 2, the second step: The integrands in the integral expressions of I_1 and I_2 above are exponential functions with quadratic exponents, where the highest-degree term is negative. Thus, completing the square in each exponent will allow the corresponding integral to be expressed eventually in terms of a cumulative standard normal distribution, like the function $N(z)$

in equation (9). The end result, still in terms of x, k, and τ, is as follows:

$$I_1 = \exp\left[\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau\right] N(d_1) \tag{26}$$

and

$$I_2 = \exp\left[\frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau\right] N(d_2), \tag{27}$$

where

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau} \tag{28}$$

and

$$d_2 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}. \tag{29}$$

Part 2, the third step: The remaining task in the derivation of the Black-Scholes formula is to bring back the original variables and parameters for the expressions of I_1 and I_2. Once $\theta(x,\tau)$ is solved, so are $v(x,\tau)$ and then $C(S,t)$, given how the three bivariate functions are connected via equations (14) and (19)-(21). With $C(S,t)$ solved, the task of the Black-Scholes model derivation is completed. As expected, the derived expression of $C(S,t)$ is the same as the one in equation (6), where d_1 and d_2 are also given by equations (7) and (8), respectively.

3 An Excel-Based Illustration

The Excel-based illustration in this section is intended to help finance students with backgrounds in traditional business disciplines, who tend to be unfamiliar with the topic of differential equations, strengthen their understanding of the analytical details of the steps involved in solving the Black-Scholes partial differential equation to derive the Black-Scholes formula. The emphasis of the illustration is to confirm numerically the equivalence of three different approaches to reach each of $C(S,t)$, $v(x,\tau)$, and $\theta(S,t)$. These approaches include the use of the Black-Scholes formula directly and the use of the result from each of the first two steps in Part 2.

To compute the option price by using the Black-Scholes formula requires values of the following input parameters: S, X, $T-t$, r, and σ. Here, for computational purposes, we have treated the underlying stock price S as an input parameter, although it is actually a random variable. As the maintenance of a risk-free hedge between the call option and its underlying stock — whose price movements follow a stochastic process — is a crucial analytical feature that leads to the Black-Scholes partial differential equation, it is important to remind students that, for computational purposes, we simply use the realized price S of the underlying stock at the time of option valuation.

3.1 Selection of Input Parameters with Scroll Bars

In the Excel spreadsheet as displayed in Figure 1, five scroll bars are provided to allow the input parameters over some predetermined ranges to be selected. The selected values are displayed in B22:F22. To illustrate how each scroll bar is used for varying the corresponding input parameter, let us use the scroll bar for S, which is the first scroll bar displayed in D1:E1, as an example. This scroll bar, which allows any value in the range of $0 - 1,000$ in increments of 1 to be selected, is linked to B1. The selected value of S in B22 is based on the cell formula =B2+(B3-B2)*B1/1000, where B2 and B3 provide the range of its permissible values. Descriptions for using the remaining scroll bars to select values of the input parameters of X, $T-t$, r, and σ in C22:F22 for the illustration are analogous. In the case of $T-t$, its minimum value, as displayed in B10, is set at 0.001 years, in order to bypass a technical issue involving equations (24) and (25) for $\tau = 0$. This technical issue will be addressed later in Subsection 3.5.

3.2 Simplification of the Boundary Condition

Part 2, the first step: We first illustrate that the expression of $\theta(w, 0)$ in equation (22), where x is substituted by w, can be simplified to an analytically more convenient form for use in equation (23). The block G3:G19 shows an arbitrary range of values of w, from -4.0 to 4.0 in increments of 0.5. The corresponding values of $\max\{\exp\left[(k+1)w/2\right] - \exp\left[(k-1)w/2\right], 0\}$, denoted as $\theta(w, 0)$ for each case, are displayed in H3:H19, where the computation of k in G22 is as described in the next subsection. The cell formula for H3, which is =MAX(EXP((G$22+1)*G3/2)-EXP((G$22-1)*G3/2),0), is copied to H3:H19.

As $\exp\left[(k+1)w/2\right] - \exp\left[(k-1)w/2\right]$ has the same sign as w, the presence of some consecutive zeros in the computed values of $\theta(w, 0)$ as displayed in H3:H19 is as expected. Specifically, we always have $\theta(w, 0) = 0$ for $w < 0$. It is this feature that justifies the use of $w = 0$, instead of $w = -\infty$, for the lower limit of each definite integral in the first step of Part 2. This part of the illustration is obvious for students who are familiar with essential properties of exponential functions of real variables. It is included here in order to ensure that students who are

	A	B	C	D	E	F	G	H	I
1	scroll bar (0-1000)	537							
2	S (min)	$10.00					w	theta (w,0)	
3	S (max)	$60.00					-4.0	0	
4							-3.5	0	
5	scroll bar (0-1000)	400					-3.0	0	
6	X (min)	$10.00					-2.5	0	
7	X (max)	$60.00					-2.0	0	
8							-1.5	0	
9	scroll bar (0-1000)	499					-1.0	0	
10	T - t (min)	0.001	years				-0.5	0	
11	T - t (max)	0.500	years				0.0	0	
12							0.5	0.94517	
13	scroll bar (0-1000)	435					1.0	3.64755	
14	r (min)	1%					1.5	10.7684	
15	r (max)	6%					2.0	28.7906	
16							2.5	73.4185	
17	scroll bar (0-1000)	745					3.0	182.568	
18	sigma (min)	4%					3.5	447.597	
19	sigma (max)	20%					4.0	1088.37	
20									
21		S	X	T - t	r	sigma	k	x	tau
22	variables, parameters	$36.85	$30.00	0.25	3.18%	15.92%	2.50546	0.20566	0.00317
23									
24			d1	d2	N(d1)	N(d2)	C	v	theta
25	from Black-Scholes formula		2.72316	2.64356	0.99677	0.99590	$7.09013	0.23634	0.27861
26	from theta (x, tau) formula		2.72316	2.64356	0.99677	0.99590			
27									
28			I1	I2	theta	v	C		
29	from theta (x, tau) formula		1.44334	1.16473	0.27861	0.23634	$7.09013		
30	from area of rectangular strips		1.44334	1.16473	0.27861	0.23634	$7.09013		
31									
32		w	for I1	for I2					
33		0.0005	0.03613	0.03612					
34		0.0015	0.03739	0.03733					
35		0.0025	0.03867	0.03858					
36		0.0035	0.04000	0.03986					
37		0.0045	0.04137	0.04118					
38		0.0055	0.04277	0.04254					
39		0.0065	0.04422	0.04393					
1030		0.9975	1.9E-21	6.9E-22					
1031		0.9985	1.6E-21	6.1E-22					
1032		0.9995	1.5E-21	5.4E-22					

Figure 1 An Excel-Based Illustration

unfamiliar with such properties of exponential functions will still understand why the solution of the heat equation satisfying the boundary condition of a call option can be expressed as the difference of the two specific definite integrals in equations (7) and (8).

3.3 Computations Based on the Black-Scholes Formula

Part 2, the third step: Given the values of S, X, $T-t$, r, and σ in B22:F22, the corresponding values of k, x, τ, d_1, d_2, $N(d_1)$, and $N(d_2)$, as well as $C(S,t)$, $v(x,\tau)$, and $\theta(x,\tau)$ based on the Black-Scholes formula, are computed and displayed in G22:I22 and C25:I25, as indicated. The computed values of k, x, and τ in G22:I22 are based on equations (11)-(13), with cell formulas =2*E22/F22^2, =LN(B22/C22), and =D22*F22^2/2, respectively. The Black-Scholes formula is in equations (6)-(8). As indicated earlier, the connections of $C(S,t)$ to $v(x,\tau)$ and $\theta(x,\tau)$ are via equations (14) and (19)-(21).

The computed values of d_1 and d_2 based on the Black-Scholes formula are displayed in C25:D25, with cell formulas =(LN(B22/C22)+E22*D22)/(F22*SQRT(D22))+F22*SQRT(D22)/2 and =C25-F22*SQRT(D22), respectively. The corresponding values of $N(d_1)$ and $N(d_2)$, as displayed in E25:F25 are via cell formulas =NORMSDIST(C25) and =NORMSDIST(D25), respectively. Given S, X, $T-t$, r, σ, $N(d_1)$, and $N(d_2)$, the computation of $C(S,t)$ based on the Black-Scholes formula is straightforward; the computed value of $C(S,t)$, according to the cell formula =B22*E25-C22*EXP(-E22*D22)*F25, is displayed in G25. The corresponding values of $v(x,\tau)$ and $\theta(x,\tau)$, as deduced directly from the computed value of $C(S,t)$ in G25, are displayed in H25:I25, with cell formulas =G25/C22 and =H25/EXP(-(G22-1)*H22/2-(G22+1)^2*I22/4).

3.4 Computations Based on an Intermediate Step

Part 2, the second step: After establishing how combining equations (5) and (22) leads to equations (23)-(25), the remaining analytical materials in Appendix B are all about completing the squares and changing some variables afterwards. Students who are familiar with integration techniques in calculus and have some knowledge of the cumulative distribution function of the standard normal distribution ought to be able to recognize the equivalence of equations (24) and (26) and the equivalence of equations (25) and (27). As the attainment of equations (26)-(29) marks the completion of a major intermediate step in the Black-Scholes model derivation, these four equations can also be used to compute $\theta(x,\tau)$, from which the corresponding values of $v(x,\tau)$ and $C(S,t)$ can be deduced.

The computation of $\theta(x,\tau)$ in E29 via equations (26)-(29) requires d_1, d_2, $N(d_1)$, $N(d_2)$, I_1, and I_2, which are displayed in C26:F26 and C29:D29. The cell formulas for d_1 and d_2 in C26:D26 are =H22/SQRT(2*I22)+(G22+1)*SQRT(2*I22)/2 and =H22/SQRT(2*I22)+(G22-1)*SQRT(2*I22)/2, respectively. The cell formulas for $N(d_1)$ and $N(d_2)$ in E26:F26 are =NORMSDIST(C26) and =NORMSDIST(D26), respectively. Next, the cell formulas for I_1 and I_2 in C29:D29 are =EXP((G22+1)*H22/2+(G22+1)^2*I22/4)*E26 and =EXP((G22-1)*H22/2+(G22-1)^2*I22/4)*F26, respectively. As $\theta(x,\tau)$ in E29 is the difference between I_1 and I_2, its computation is straightforward. With the value of $\theta(x,\tau)$ known, the computations of $v(x,\tau)$ and $C(S,t)$ in F29:G29 are via the cell formulas =E29*EXP(-(G22-1)*H22/2-(G22+1)^2*I22/4) and =C22*F29, respectively.

3.5 Computations Based on Numerical Integration

Part 2, the first step: In order to use equations (24) and (25) directly, we rely on a well-known feature in integral calculus that, in a univariate case, a definite integral can be viewed as the area enclosed by the integrand and the horizontal axis, between the limits of integration. In the case of I_1 in equation (24), the integrand, which is a function of w, is $\exp\left[(k+1)w/2-(x-w)^2/(4\tau)\right]$, the horizontal axis is the w-axis, and the lower and upper limits of integration are $w=0$ and $w=\infty$, respectively. In the case of I_2 in equation (25), the only difference is the substitution of $k+1$ by $k-1$ in the expression of the integrand. For a given set of input parameters, each enclosed area must be the same as the end result from integrating the function involved, which is I_1 in equation (26) or I_2 in equation (27), as the case may be.

Notice that, although equations (26) and (27) can easily accommodate input parameters where $\tau=0$, it is not so for equations (24) and (25). This is because, if equations (24) and (25) are used directly to compute I_1 and I_2, respectively, as τ approaches zero, so does each integrand, thus resulting in the corresponding integral approaching zero as well. Meanwhile, as τ approaches zero, the multiplicative factor $1/2\sqrt{\pi\tau}$ for each integral in equations (24) and (25) approaches infinity, thus preventing the computation of $\theta(x,0)$ to be performed. Given such a limitation, the Excel-based illustration here cannot accommodate input parameters where τ is exactly zero. However, this is not a consequential limitation, as the value of τ can still be made very close to zero without being exactly zero, for the purpose of approximating the expiry date of the option in the Excel-based illustration.

To approximate numerically the above enclosed area, we simply aggregate the areas of a large number of narrow rectangular strips that are located side by side, where each height is the value of the function being integrated. For this task, let b be a predetermined common width of the individual strips. Let also n be the number of strips as needed for this approximation. The height of strip i is at its mid-point

$$w_i = \frac{b}{2} + (i-1)b, \text{ for } i = 1, 2, 3, \ldots, n. \tag{30}$$

Thus, we can approximate I_1 and I_2 as follows:

$$I_1 = \frac{b}{2\sqrt{\pi\tau}} \sum\nolimits_{i=1}^{n} \exp\left[\frac{1}{2}(k+1)w_i - \frac{(x-w_i)^2}{4\tau}\right] \tag{31}$$

and

$$I_2 = \frac{b}{2\sqrt{\pi\tau}} \sum\nolimits_{i=1}^{n} \exp\left[\frac{1}{2}(k-1)w_i - \frac{(x-w_i)^2}{4\tau}\right]. \tag{32}$$

For the purpose of the illustration in Figure 1, n and b have been preset at $1,000$ and 0.001, respectively. That is, the areas of $1,000$ narrow rectangular strips, with each width being 0.001, are to be summed. The block B33:B1032 shows the mid-point locations of the $1,000$ strips. With $w_1 = 0.0005$, $w_2 = 0.0015$, $w_3 = 0.0025$, $w_4 = 0.0035$, $\ldots$, $w_{1,000} = 0.9995$, the computations for I_1 and I_2 are based on equations (31) and (32). The block C33:C1032 contains the corresponding $1,000$ values of $\exp\left[(k+1)w_i/2 - (x-w)^2/(4\tau)\right]$, with one value for each i. The cell formula for C33, which is =EXP((G22+1)*$B33/2-($H$22-$B33)^2/(4*I22)), is copied to C33:C1032. The heights of rows 40 to 1029 have been set to nearly zero, in order to keep the display of Figure 1 within one page. The sum of the areas of the $1,000$ narrow rectangular strips, multiplied by the factor $1/(2\sqrt{\pi\tau})$ — which is I_1 in equation (31) — is displayed in C30; the corresponding cell formula is =SUM(C33:C1032)*(B34-B33)/(2*SQRT(PI()*I22)).

Likewise, for I_2, the $1,000$ values of $\exp\left[(k-1)w_i/2 - (x-w)^2/(4\tau)\right]$, with one value for each i, corresponding to the $1,000$ values of w_i in B33:B1032, are displayed in D33:D1032. The cell formula for D33, which is =EXP((G22-1)*$B33/2-($H$22-$B33)^2/(4*I22)), is copied to D33:D1032. The sum of the areas of the $1,000$ narrow rectangular strips, also multiplied by the same factor $1/(2\sqrt{\pi\tau})$ — which is I_2 equation (32) — is displayed in D30; the corresponding cell formula is =SUM(D33:D1032)*(B34-B33)/(2*SQRT(PI()*I22)). With I_1 and I_2 known, $\theta(x,\tau)$ in equation (23) is simply their difference; its value is displayed in E30. The corresponding value of $\upsilon(x,\tau)$ according to equations (19)-(21) is displayed in F30 via the cell formula =E30*EXP(-(G22-1)*H22/2-(G22+1)^2*I22/4). Finally, the corresponding value of $C(S,t)$, as displayed in G30, is simply $X\upsilon(x,\tau)$.

As indicated earlier, the use of equations (24) and (25) where $\tau = 0$ for numerical integration requires that $T - t$ be approximated by a small positive value, in order to bypass a technical issue. For the same set of input parameters in B22:F22, except that $T - t = 0$ years, as the underlying stock price on the expiry date of the option, \$36.85, exceeds the exercise price, \$30.00, the value of the call option ought to be \$6.85 (= \$36.85 − \$30.00), according to the boundary condition in equation (2). The computed value of C based on $T - t = 0.001$ years is \$6.85095, which slightly overstates its correct value on the expiry date as expected. If the expiry date is approximated by $T - t = 0.0001$ years instead, the corresponding C will become \$6.85010, which is closer to its correct value. Such overstatements are small enough to justify the use of a small positive value for $T - t = 0$ years in the Excel illustration.

3.6 Further Explanations

A comparison of the three computed values of $C(S,t)$ in G25 and G29:30, of $\upsilon(x,\tau)$ in H25 and F29:F30, and of $\theta(x,\tau)$ in I25 and E29:E30 confirms that there are total agreements of the end results in each case, regardless of whether the Black-Scholes formula or either intermediate step is involved. Such an outcome is robust, regardless of what values of the input parameters are used for the computations. To see why this is as expected, let us first review the key features of the Black-Scholes model derivation as summarized in Section 2.

In order to solve the Black-Scholes partial differential equation in equation (1), some changes in variables are needed to replace the function $C(S,t)$ there first with $\upsilon(x,\tau)$, and subsequently with $\theta(x,\tau)$, in order to reach a one-dimensional heat equation in equation (3), for which a solution method is available. There are specific connections of $\theta(x,\tau), \upsilon(x,\tau)$, and $C(S,t)$ via equations (14) and (19)-(21), so that, if one of them is known, so are the remaining two. Such connections are crucial for the transformation of the Black-Scholes partial differential equation into the heat equation.

What is also crucial in the Black-Scholes model derivation is that $\theta(x, \tau)$ satisfying the boundary condition of a call option can be expressed as the difference of two specific definite integrals, which are I_1 and I_2 in equations (24) and (25), respectively. Given this analytical feature, the remaining tasks of the derivation include the attainment of an expression for each definite integral in terms of the transformed variables, as in equations (26)-(29), and then the attainment of the Black-Scholes formula in terms of the original variables, as in equations (6)-(8). To perform such tasks does not change the fact that $\theta(x, \tau)$ is the difference between I_1 and I_2; nor does it affect the computed values of I_1 and I_2 for any given set of input parameters. Given the existing connections of $\theta(x, \tau), \upsilon(x, \tau)$, and $C(S, t)$ via equations (14) and (19)-(21), the equivalence of the computed values of any of these functions, via the Black-Scholes formula or either intermediate step, is assured.

4 Concluding Remarks

Although the impact of the Black-Scholes option pricing model on the financial world has been profound, its full derivation via the Black-Scholes partial differential equation has eluded many finance students with backgrounds in traditional business disciplines. Apparently, some advanced mathematical requirements, which are well beyond those covered in the standard finance curriculum, have prevented these students from following its full derivation. Based on textbook coverage such as that in Hull (2018), students in courses of financial derivatives still learn how to reach the Black-Scholes partial differential equation. Thus, for many finance students, the mystery in the Black-Scholes model derivation is in how to solve the Black-Scholes partial differential equation.

Cox, Ross, and Rubinstein (1979) have offered a binomial approach for option pricing as a simpler alternative, thus making the economic insights of the original Black-Scholes model more readily accessible to the finance profession. Using an Excel-based illustration, Feng and Kwan (2012) have explained pedagogically how the two models are connected, thus making the Black-Scholes model derivation less mysterious for finance students. However, so far, textbook coverage of directly solving the Black-Scholes partial differential equation has been intended for students with advanced mathematical knowledge.

To bypass as many advanced mathematical requirements as possible for solving the Black-

Scholes partial differential equation, this paper has treated the solution method for a one-dimensional heat equation in physical science and engineering fields as given. In so doing, this paper has inevitably left a knowledge gap — as to how the solution method has been derived in the first place — for finance students with traditional business backgrounds in their understanding of the Black-Scholes model derivation. Although it is feasible to cover pedagogically a more thorough derivation via the Black-Scholes partial differential equation, where any unfamiliar mathematical materials, when encountered, will have to be explained, it is highly unlikely that Excel can still play a pedagogic role, as it does in this paper. Thus, any limitations notwithstanding, the basic version of the Black-Scholes model derivation as covered in this paper, especially in conjunction with the Excel-based illustration for it, can still contribute meaningfully in dispelling its perceived mystery among many finance students.

Appendix A: Details of Transforming the Black-Scholes Partial Differential Equation into a One-Dimensional Heat Equation

Part 1, the first step: The starting point for the derivation is

$$\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} + rS\frac{\partial C(S,t)}{\partial S} - rC(S,t) = 0, \tag{A1}$$

for which

$$C(S,T) = \max(S - X, 0). \tag{A2}$$

Following the approach in Wilmott, Howison, and Dewynne (1995, Chapter 5), let

$$S = X\exp(x), \tag{A3}$$

$$t = T - \frac{2\tau}{\sigma^2}, \tag{A4}$$

and

$$C(S,t) = Xv(x,\tau). \tag{A5}$$

It is implicit that $-\infty < x < \infty$ and $\tau > 0$. The intention is to use x, τ, and the function $v(x,\tau)$, instead of S, t, and $C(S,t)$, respectively, in equation (A1). The partial derivatives $\partial C/\partial t$, $\partial C/\partial S$, and $\partial^2 C/\partial S^2$ there — implicitly for the corresponding partial derivatives of $C(S,t)$ — will have to be written in terms of the newly defined variables.

Specifically, with arguments of the functions $C(S,t)$ and $\upsilon(x,\tau)$ omitted for notational convenience, we have

$$\frac{\partial C}{\partial t} = -\frac{X\sigma^2}{2}\frac{\partial \upsilon}{\partial \tau}, \tag{A6}$$

$$\frac{\partial C}{\partial S} = \exp(-x)\frac{\partial \upsilon}{\partial x}, \tag{A7}$$

and

$$\frac{\partial^2 C}{\partial S^2} = \frac{\exp(-2x)}{X}\frac{\partial^2 \upsilon}{\partial x^2} - \frac{\exp(-2x)}{X}\frac{\partial \upsilon}{\partial x} = \frac{\exp(-2x)}{X}\left(\frac{\partial^2 \upsilon}{\partial x^2} - \frac{\partial \upsilon}{\partial x}\right). \tag{A8}$$

Substituting these expressions into equation (A1) leads to

$$\begin{aligned} & -\frac{X\sigma^2}{2}\frac{\partial \upsilon}{\partial \tau} + \frac{\sigma^2 X^2}{2}\exp(2x)\frac{\exp(-2x)}{X}\left(\frac{\partial^2 \upsilon}{\partial x^2} - \frac{\partial \upsilon}{\partial x}\right) + rX\exp(x)\exp(-x)\frac{\partial \upsilon}{\partial x} - rX\upsilon \\ = \quad & -\frac{X\sigma^2}{2}\frac{\partial \upsilon}{\partial \tau} + \frac{\sigma^2 X}{2}\left(\frac{\partial^2 \upsilon}{\partial x^2} - \frac{\partial \upsilon}{\partial x}\right) + rX\frac{\partial \upsilon}{\partial x} - rX\upsilon = 0. \end{aligned} \tag{A9}$$

As $X > 0$, this equation is equivalent to

$$\frac{\sigma^2}{2}\frac{\partial^2 \upsilon}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial \upsilon}{\partial x} - \frac{\sigma^2}{2}\frac{\partial \upsilon}{\partial \tau} - r\upsilon = 0. \tag{A10}$$

Now, let

$$k = \frac{2r}{\sigma^2}. \tag{A11}$$

Equation (A10) becomes

$$\frac{\partial \upsilon}{\partial \tau} = \frac{\partial^2 \upsilon}{\partial x^2} + (k-1)\frac{\partial \upsilon}{\partial x} - k\upsilon. \tag{A12}$$

As $t = T$ corresponds to $\tau = 0$ according to equation (A4), equation (A2) can be re-stated as

$$X\upsilon(x,0) = \max[X\exp(x) - X, 0] \tag{A13}$$

or, simply,

$$\upsilon(x,0) = \max[\exp(x) - 1, 0]. \tag{A14}$$

Part 1, the second step: Equation (A12) is now very close to the analytical form of a one-dimensional heat equation. All that is left is to reduce the right hand side of equation (A12) to a single term involving the second partial derivative. For this task, let us write

$$\upsilon(x,\tau) = \exp(\alpha x + \beta\tau)\theta(x,\tau). \tag{A15}$$

Here, the parameters α and β have yet to be determined.

With this substitution, equation (A12) becomes

$$\begin{aligned}
&\exp(\alpha x+\beta\tau)\left(\beta\theta+\frac{\partial\theta}{\partial\tau}\right)\\
=\;&\exp(\alpha x+\beta\tau)\left[\left(\alpha^2\theta+2\alpha\frac{\partial\theta}{\partial x}+\frac{\partial^2\theta}{\partial x^2}\right)+(k-1)\left(\alpha\theta+\frac{\partial\theta}{\partial x}\right)-k\theta\right].
\end{aligned}\tag{A16}$$

From

$$\beta\theta+\frac{\partial\theta}{\partial\tau}=\left(\alpha^2\theta+2\alpha\frac{\partial\theta}{\partial x}+\frac{\partial^2\theta}{\partial x^2}\right)+(k-1)\left(\alpha\theta+\frac{\partial\theta}{\partial x}\right)-k\theta,\tag{A17}$$

we have

$$\frac{\partial\theta}{\partial\tau}=\frac{\partial^2\theta}{\partial x^2}+(2\alpha+k-1)\frac{\partial\theta}{\partial x}+\left[-\beta+\alpha^2+\alpha(k-1)-k\right]\theta.\tag{A18}$$

Thus, by setting

$$\alpha=-\frac{1}{2}(k-1)\tag{A19}$$

and

$$\beta=\alpha^2+\alpha(k-1)-k=\frac{1}{4}(k-1)^2-\frac{1}{2}(k-1)^2-k=-\frac{1}{4}(k+1)^2,\tag{A20}$$

we finally have

$$\frac{\partial\theta}{\partial\tau}=\frac{\partial^2\theta}{\partial x^2}\tag{A21}$$

or, more formally with the two arguments of θ explicitly indicated,

$$\frac{\partial\theta(x,\tau)}{\partial\tau}=\frac{\partial^2\theta(x,\tau)}{\partial x^2},\tag{A22}$$

which is a one-dimensional heat equation.

As

$$\begin{aligned}
\upsilon(x,\tau)&=\exp(\alpha x+\beta\tau)\theta(x,\tau)\\
&=\exp\left[-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau\right]\theta(x,\tau),
\end{aligned}\tag{A23}$$

equation (A14) becomes

$$\exp\left[-\frac{1}{2}(k-1)x\right]\theta(x,0)=\max[\exp(x)-1,0].\tag{A24}$$

This boundary condition is equivalent to

$$\theta(x,0)=\max\left\{\exp\left[\frac{1}{2}(k+1)x\right]-\exp\left[\frac{1}{2}(k-1)x\right],0\right\}.\tag{A25}$$

With

$$\theta_0(x) = \theta(x, 0) \tag{A26}$$

being a known function of x, the solution of equation (A22) that satisfies equation (A25) is

$$\theta(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{w=-\infty}^{\infty} \theta_0(w) \exp\left[-\frac{(x-w)^2}{4\tau}\right] dw. \tag{A27}$$

Appendix B: Details of Solving the Heat Equation to Reach the Black-Scholes Formula

Part 2, the first step: Here, we continue to follow the approach in Wilmott, Howison, and Dewynne (1995, Chapter 5). Let us first rewrite equation (A25) as

$$\theta_0(w) = \theta(w, 0) = \max\left\{\exp\left[\frac{1}{2}(k+1)w\right] - \exp\left[\frac{1}{2}(k-1)w\right], 0\right\}, \tag{B1}$$

by substituting the variable x there with w. The sign of

$$\exp\left[\frac{1}{2}(k+1)w\right] - \exp\left[\frac{1}{2}(k-1)w\right] = \exp\left(\frac{kw}{2}\right)\left[\exp\left(\frac{w}{2}\right) - \exp\left(-\frac{w}{2}\right)\right] \tag{B2}$$

is the same as the sign of w. This is because $\exp(kw/2)$ is always positive, for all real values of w, and because $\exp(w/2)$, which is also positive, is an increasing function of w. If $w \geq 0$, we have $\exp(w/2) - \exp(-w/2) \geq 0$; if $w < 0$ instead, we have $\exp(w/2) - \exp(-w/2) < 0$. Thus, with $\theta_0(w) = 0$ for $w < 0$, we can write equation (A27) as

$$\theta(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{w=0}^{\infty} \left\{\exp\left[\frac{1}{2}(k+1)w\right] - \exp\left[\frac{1}{2}(k-1)w\right]\right\} \exp\left[-\frac{(x-w)^2}{4\tau}\right] dw, \tag{B3}$$

which is equivalent to

$$\theta(x, \tau) = I_1 - I_2, \tag{B4}$$

where

$$I_1 = \frac{1}{2\sqrt{\pi\tau}} \int_{w=0}^{\infty} \exp\left[\frac{1}{2}(k+1)w - \frac{(x-w)^2}{4\tau}\right] dw \tag{B5}$$

and

$$I_2 = \frac{1}{2\sqrt{\pi\tau}} \int_{w=0}^{\infty} \exp\left[\frac{1}{2}(k-1)w - \frac{(x-w)^2}{4\tau}\right] dw. \tag{B6}$$

Part 2, the second step: Now, let

$$\omega = \frac{w - x}{\sqrt{2\tau}}. \tag{B7}$$

It follows that

$$
\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi}} \int_{\omega=-x/\sqrt{2\tau}}^{\infty} \exp\left[\frac{1}{2}(k+1)\left(\omega\sqrt{2\tau}+x\right)-\frac{1}{2}\omega^2\right] d\omega \\
&= \frac{\exp\left[\frac{1}{2}(k+1)x\right]}{\sqrt{2\pi}} \int_{\omega=-x/\sqrt{2\tau}}^{\infty} \exp\left[\frac{1}{2}(k+1)\omega\sqrt{2\tau}-\frac{1}{2}\omega^2\right] d\omega \\
&= \frac{\exp\left[\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau\right]}{\sqrt{2\pi}} \int_{\omega=-x/\sqrt{2\tau}}^{\infty} \exp\left\{-\frac{1}{2}\left[\omega-\frac{1}{2}(k+1)\sqrt{2\tau}\right]^2\right\} d\omega. \quad \text{(B8)}
\end{aligned}
$$

Letting

$$u = \omega - \frac{1}{2}(k+1)\sqrt{2\tau}, \quad \text{(B9)}$$

we can write, given the symmetry of the standard normal distribution,

$$
\begin{aligned}
I_1 &= \frac{\exp\left[\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau\right]}{\sqrt{2\pi}} \int_{u=-d_1}^{\infty} \exp\left(-\frac{1}{2}u^2\right) du \\
&= \frac{\exp\left[\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau\right]}{\sqrt{2\pi}} \int_{u=-\infty}^{d_1} \exp\left(-\frac{1}{2}u^2\right) du \\
&= \exp\left[\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau\right] N(d_1), \quad \text{(B10)}
\end{aligned}
$$

where

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau} \quad \text{(B11)}$$

and

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{u=-\infty}^{d_1} \exp\left(-\frac{1}{2}u^2\right) du. \quad \text{(B12)}$$

The latter is a cumulative standard normal distribution function.

The only difference between the expressions of I_1 and I_2 is that the $k+1$ term in the former case is $k-1$ instead in the latter case. Therefore, we can directly write

$$I_2 = \exp\left[\frac{1}{2}(k-1)x+\frac{1}{4}(k-1)^2\tau\right] N(d_2), \quad \text{(B13)}$$

where

$$d_2 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}. \quad \text{(B14)}$$

The cumulative standard normal distribution function $N(d_2)$ is also given by equation (B12), but with the d_1 there substituted by d_2.

Part 2, the third step: To complete the derivation, we must express x, τ, k, and $\theta(x,\tau)$ in terms of S, X, σ, T, t, and $C(S,t)$. Based on how x, τ, k, and $\theta(x,\tau)$ are defined, we have, along with equation (A11),

$$x = \ln\left(\frac{S}{X}\right), \tag{B15}$$

$$\tau = \frac{1}{2}\sigma^2(T-t), \tag{B16}$$

and

$$C(S,t) = Xv(x,\tau) = X\exp\left[-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau\right]\theta(x,\tau). \tag{B17}$$

The last expression above can be written as

$$\begin{aligned}
C(S,t) &= X\exp\left[-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau\right]N(d_1) \\
&\quad -X\exp\left[-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau\right]N(d_2) \\
&= X\exp\left[-\frac{1}{2}(k-1)x + \frac{1}{2}(k+1)x\right]N(d_1) \\
&\quad -X\exp\left[-\frac{1}{4}(k+1)^2\tau + \frac{1}{4}(k-1)^2\tau\right]N(d_2) \\
&= X\exp\left[\ln\left(\frac{S}{X}\right)\right]N(d_1) - X\exp(-k\tau)N(d_2) \\
&= SN(d_1) - X\exp[-r(T-t)]N(d_2),
\end{aligned} \tag{B18}$$

where

$$\begin{aligned}
d_1 &= \frac{\ln(S/X)}{\sigma\sqrt{T-t}} + \frac{1}{2}\left(\frac{2r}{\sigma^2}+1\right)\sigma\sqrt{T-t} \\
&= \frac{\ln(S/X) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned} \tag{B19}$$

and

$$\begin{aligned}
d_2 &= \frac{\ln(S/X)}{\sigma\sqrt{T-t}} + \frac{1}{2}\left(\frac{2r}{\sigma^2}-1\right)\sigma\sqrt{T-t} \\
&= \frac{\ln(S/X) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t},
\end{aligned} \tag{B20}$$

thus completing the derivation of the Black-Scholes formula. The available correspondence between the equation numbers in the main text and those in the two appendices is as follows:

(1) ⇔ (A1)	(8) ⇔ (B20)	(15) ⇔ (A14)	(24) ⇔ (B5)
(2) ⇔ (A2)	(9) ⇔ (B12)	(17) ⇔ (A4)	(25) ⇔ (B6)
(3) ⇔ (A22)	(10) ⇔ (A12)	(18) ⇔ (A3)	(26) ⇔ (B10)
(4) ⇔ (A26)	(11) ⇔ (A11)	(19) ⇔ (A15)	(27) ⇔ (B13)
(5) ⇔ (A27)	(12) ⇔ (B16)	(20) ⇔ (A19)	(28) ⇔ (B11)
(6) ⇔ (B18)	(13) ⇔ (B15)	(21) ⇔ (A20)	(29) ⇔ (B14)
(7) ⇔ (B19)	(14) ⇔ (A5)	(22) ⇔ (A25)	

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