Spreadsheet-Enhanced Problem Solving in Context as Modeling

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Abstract

This paper is prompted by a recent call by the International Commission on Mathematical Instruction (ICMI) for the study of mathematical modeling as technology-enhanced didactic inquiry into relations between mathematics and the real world. It reflects on activities designed for a teacher education course that focuses on the computer spreadsheet as a tool for concept development through situated mathematical problem solving. Modeling activities described in this paper support the epistemological position regarding the interplay that exists between the development of mathematical concepts and available methods of calculation. The spreadsheet used is Microsoft®Excel 2001.

Keywords: modeling, problem solving, recursion, measurement model for division, teacher education.

1 Introduction

The theme of mathematical modeling and applications, whose educational importance most recently came into light in an ICMI discussion document [11], has been in the focus of research in mathematics education for the last four decades [21] [20] [36] [29] [10] [15] [34] [28] [16]. Fascinating advances in technology-enhanced applications of mathematics to the study of the real world called for appropriate changes in curriculum and pedagogy of school mathematics [6] [8]. Connection of mathematics to other sciences, its relevance to the outside world, learning concepts in context and connecting them through applications, teaching conceptually through helping students construct their own meanings grounded in real-life experiences – these are some of the basic ideas that underpinned this new vision of mathematics education at the pre-college level. In particular, it has been argued that a pedagogical approach utilizing modeling activities has great potential to create learning environments conducive to mathematics discovery experience [36], and serve as useful vehicle in understanding mathematical concepts [33] [35] [15].

As curriculum and didactic changes have been realized in the form of standards (e.g., [30] [32]) teachers have come to be increasingly recognized as major agents and key players in the implementation of the standards ([31] [14] [13]). This made programs for preparing prospective teachers of mathematics uniquely accountable for providing appropriate milieu for learning new pedagogy, including technology-enhanced training in modeling-oriented discovery. Teachers, who have had experience of mathematical discovery as part of their studies, are more likely to impart such experience to their students than those with education confined essentially to the production of correct answers.

This paper reflects on modeling activities designed for a computer-enhanced mathematics teacher education course taught by the author at SUNY Potsdam over the last five years. This course was intended to serve as an introduction to computational methods for concept development in school mathematics using a spreadsheet. It demystifies the stereotype of using the software as a pure computation oriented and/or record keeping tool. It provides an alternative to simply transmitting disconnected concepts and, instead, exposes students to the same concepts through context-bounded problematic situations. The students enrolled in this (elective) course usually range from experienced teachers of secondary mathematics pursuing their masters degree to preservice elementary education majors and have different beliefs and expectations about using a spreadsheet as a mathematical/pedagogical tool. Hereafter both groups are referred to as the teachers.

It is interesting to note that, designed originally for non-educational purposes, a spreadsheet, according to [38], was conceptualized by its inventor as "an electronic blackboard and electronic chalk in a classroom". Today, such a vision of the utilization of the software sounds strikingly accurate if one attempts to browse through the numerous literature on the use of spreadsheets in mathematics education [7]. Indeed, during the last twenty-five years, the software has proved to be an amazingly useful and cost effective educational tool supporting teaching and learning of mathematics across all educational levels. As far as teacher education is concerned, many authors reported success with preservice and in-service teachers' learning mathematical concepts through modeling in spreadsheet-enhanced environments [18] [19] [25] [2] [26] [3] [17] [23] [5]. In particular, through modeling activities a spreadsheet naturally becomes an agent of meaningful engagement into mathematical problem solving by teachers [4].

2 Modeling as problem solving with technology

It has been repeatedly argued that modeling and problem solving are closely related mathematical activities [29] [10] [22] [35] [28] [16]. From a didactic perspective, training in modeling pedagogy is ultimately structured by one's engagement in formulating and resolving problematic situations through the use of a variety of models that represent those situations [40]. This suggests a fundamental relationship that exists between modeling and problem posing. Furthermore, viewing problem solving and posing as two sides of the same coin [27] [9], suggests the importance of providing teachers of mathematics with experiences in modeling through formulating, exploring, and resolving problematic situations that lead to new mathematical ideas and concepts.

The presence of technology in the teacher education classroom has great potential to enrich this kind of modeling pedagogy by having teachers explore computer-enhanced models and formulate questions about those models [12]. While learning to use technology as an amplifier of mathematical modeling activities, one can come across many computationally driven problematic situations which may have no apparent relevance to the original contextual inquiries for which technology-enhanced model was designed. To address new inquiries, one might have to develop new computational environments that, in turn, prompt new inquiries and stimulate search for new problem solving strategies. As far as a spreadsheet is concerned, its computational nature enables immediate feedback so that one can test emerging strategies and see results in ways that were never possible with more traditional, pencil and paper materials. When such use of a spreadsheet is a part of technology-motivated mathematics teacher education coursework, the course instructor's role is to encourage teachers to take intellectual risk through the formulation of mathematically meaningful questions about numerical patterns observed. In such an intellectual milieu the instructor's ability to possess 'the answer' may not be an imperative [41], thus both parties could work as equal partners towards generating new knowledge in this technological paradigm.

3 Two approaches to extending problematic situations

In the problematic situations discussed below, two ways of formulating their extensions can be distinguished. The first way is to extend a problematic situation by altering its corresponding context. In doing so, after a mathematical model of contextual inquiry has been developed (and, perhaps, computerized), one goes back and changes the inquiry, develops a relevant model involving a number of parameters, and refines corresponding problem-solving tools. Apparently, this approach does not require a full grasp of the generalized meanings of the parameters involved in the construction of the original model. As a result, mathematical concepts that emerge through the multiple implementation of this approach may not be formally connected to each other.

Another way is to change the parameters of the model within the model itself and to formulate new contextual inquiry through interpreting this change in terms of the context. The second approach, however, does require one's conceptualization of the parameters that structure the original model. Mathematical concepts that emerge through the second approach are likely to be connected to each other through layers of consecutive generalizations. In other words, the first approach describes a situation where the model is dependent on contextual inquiry but not vice versa; the second approach describes situation where contextual inquiry results from the meaningful change of parameters of a model, a process requiring a higher level of mathematical thinking. The pedagogy of computerization of modeling activities plays an important role in formulating generalized contextual inquiries and developing context-bounded interconnected mathematical concepts. Figures 1 and 2 illustrate the difference in the two approaches to formulating problematic situations and developing mathematical concepts as means of computerization.

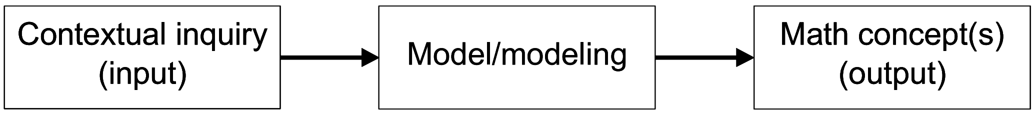

Figure 1: Model does not change unless contextual inquiry changes.

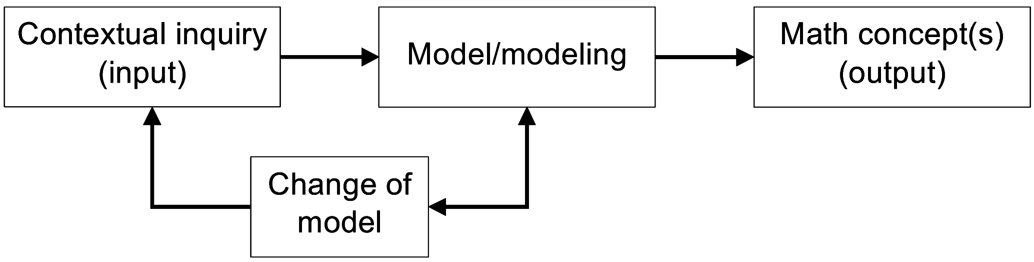

Figure 2: Change of model within a model affects contextual inquiry.

4 Setting a context

In this paper, mathematical modeling will be considered, using Pollak's [36] terminology, in a whimsical context. While it may be argued that problems of whimsy have only superficial connection to the real world [15] the most recent use of the term modeling embraces all possible relations between mathematics and the world outside it [11]. It appears that one's perception of what is whimsical and what is not largely depends on one's experience; in fact, many of today's real-life situations seemed like fictions yesterday. Furthermore, the strong relationship that exists between modeling and problem solving suggests the importance for the former to be explored in a whimsical context that very often provides a powerful cognitive milieu for the latter.

For example, Engel [20] explored such 'whimsical' contexts as fishing, book reading and coin tossing. It has been shown that each of these contexts is conducive for using modeling strategies as means of instruction and developing the habit of seeing possible applications of mathematics. Note that context itself does not account for the mathematical content – the latter usually begins

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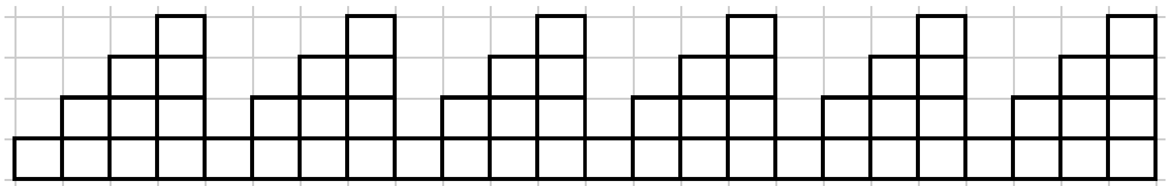

Figure 3: A blueprint of the four-storied hotel.

with a quantitative inquiry into the former, something that may be referred to as mathematization. Although the mathematical content of this paper is rather elementary and limited mostly to arithmetic, this area of mathematics "at all levels of sophistication provides a tremendous opportunity for experimentation" [39, p142]. Following is an example of context that will be used as a milieu for such mathematization and experimentation throughout the paper.

> *An architect was very creative while designing a new hotel. (A blueprint of its beginning is depicted in Figure 3.) The hotel was made up of a number of buildings adjacent to each other. Each building had one, two, three, or four floors with one room on each floor...*

Below different problematic situations (contextual inquiries) that stem from this context will be created and then resolved through spreadsheet-enhanced modeling activities. In doing so, mediated environments will be constructed to visually support contextual inquiries. These environments will be referred to as meta-context. (A simple example of meta-context is the blueprint depicted in Figure 3.) This would make it possible to extend the use of technology in modeling to include spreadsheet-generated meta-contexts that will be used as mediators between context and model. Thus, any alteration of context will be supported by the interactive alteration of meta-context (blueprint).

In referring to context (as well as to meta-context), the following terminology will be used throughout the paper. Any one-cell unit will be referred to as a *room*. A combination of one or several vertically arranged rooms will be referred to as a *building*. A combination of different buildings adjacent to each other will be referred to as a *block*. Finally, a *hotel* is a combination of several blocks. In such a way there are sixty rooms, twenty-four buildings, and six blocks in the blueprint of the four-storied hotel depicted in Figure 3.

5 Measurement model for division as emerging mathematical model

Observing the blueprint of the 4-storied hotel depicted in Figure 3, one can see that the number of rooms in each building varies from one to four. Because such a variation occurs in a regular pattern, one may wonder if there is any relationship between a building number and the number of rooms in this building. With this in mind, the following simple problematic situation (PS) can be formulated:

PS 1.0 *How many rooms are in the 22nd building of the 4-storied hotel?*

An answer to this question can be obtained from Figure 3 through simple counting, something that does not require the use of any mathematical model. However the limitation of one-by-one counting as a problem solving strategy becomes obvious if a much bigger (say, a three-digit) number replaces 22. Thus, on a more general level, one may wonder (**PS 1.1**): *How many rooms are in the k-th building of the 4-storied hotel?* This new level of generality suggests the need for the replacement of intuitive reasoning by formal reasoning, something that does require the development of a mathematical model. The measurement model for division or, in other words, the process of grouping a certain number of objects into equal sets, can provide such a model [4]. This process brings about the associated notion of remainder which becomes a crucial tool in describing a model that allows for the variation of building number and formulating the following strategy of resolving PS 1.1.

In order to find the number of rooms in the k-th building of the 4-storied hotel, one has to divide 4 into k and, if remainder is 0, replace it by 4, otherwise a non-zero remainder (that is, using formal notation, $MOD(k, 4)$) represents the number of rooms in this building.

The importance of the MOD function as tool in developing mathematical model will be demonstrated by exploring other problematic situations throughout the paper.

Note that in the context of the 4-storied hotel the inquiry into the number of rooms as a function of building number (that is, PS 1.1) has been resolved completely. As was mentioned above, the diagrams presented in Figures 1 and 2 suggest two ways of extending this inquiry. One way is to change context (and, perhaps, inquiry into it); another way is to change one of the parameters in the model and interpret this change in contextual terms. In its most simple form, such a change in context may result from making the numerical component in the model `MOD(k,4)` a variable, thus extending the context to hotels of different number of stories. In such a way, the following generalized problematic situation can be explored.

PS 1.2 *How many rooms are in the k-th building of the n-storied hotel?*

In terms of modeling, one can see that a dynamic model structured by variable parameters, replaces a static model in which all parameters are fixed.

It should be noted that the process of generalization requires a set of blueprints, something that may be construed as the set of meta-contexts. By exploring various meta-contexts one can see that the number of stories in a hotel coincides with the number of buildings in each block of the hotel. Furthermore, dividing the number of stories into a building number yields a remainder which, in all cases but one, coincides with the number of rooms in this building. This leads to the following general statement:

The number of rooms in the k-th building of the n-storied hotel equals $MOD(k, n)$ if $MOD(k, n) \neq 0$, otherwise it equals n.

Figure 4 shows a simple spreadsheet environment that can easily be designed to explore PS 1.2 using a spreadsheet function MOD from the tool kit of available computing devices. This environment has two slider-controlled variables—hotel type (i.e., the number of stories) and building number. The variables are connected via the spreadsheet formula `=IF(MOD(A3, G3)=0, G3, MOD(A3, G3))` which, being defined in cell `I3`, generates the number of rooms in the building.

Several mathematical concepts can be discussed in the framework of the modeling activities associated with PS 1.2 and its computerization. Among them: divisor, quotient, remainder, measurement model for division, and modular arithmetic. In particular, the spreadsheet-based calculation of remainders illustrates the usefulness of this mathematical concept in developing a model and the use of computerization as a vehicle for generalization. As mentioned elsewhere [3], in a spreadsheet-enhanced mathematics education course the MOD function becomes a tool for the teachers rather than a notation in a number theory course. In general, through the practice of computerization one can better appreciate mathematical concepts by using them as computational *tools* rather than abstract entities alone. Further inquiries in the context of PS 1.2 may deal the exploration of the function $r_k(n)$ which describes the change of the number of rooms in building number k of the n-storied hotel as n varies. This, however, requires more sophisticated use of spreadsheet than the computation of remainders and will be discussed later in this paper in connection with a pedagogical idea of revealing hidden contextual messages through interpreting results of spreadsheet graphics.

6 Numeration of rooms prompts mathematical explorations

A next step in the development of modeling activities within the framework of the paper may be prompted by the extension of the original meta-context to include a blueprint of the 4-storied hotel with all rooms numerated throughout (Figure 5). This new meta-context brings about the following inquiry:

PS 2.0 *On what floor does room number 78 in the 4-storied hotel belong?*

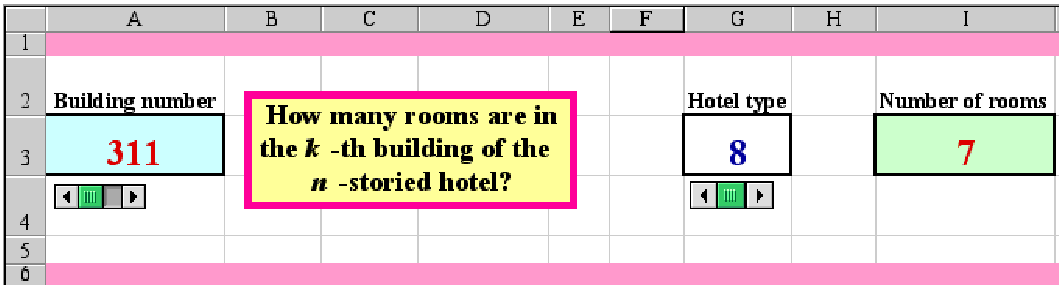

Figure 4: In the 8-storied hotel there are seven rooms in the 311th building.

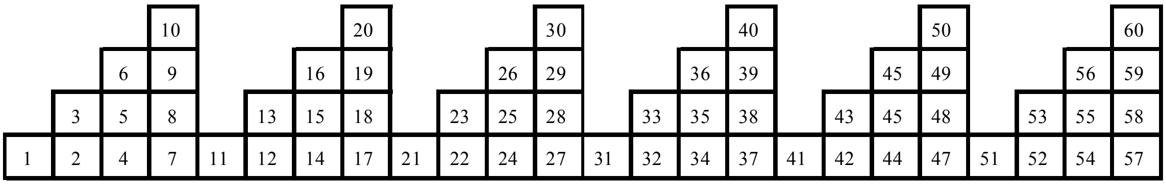

Figure 5: Meta-context produced through intuitive numeration technique.

In order to address this inquiry, the measurement model for division once again can be utilized as the basic model. This time, however, the variation of context suggests dividing room number by the number of rooms in a block rather than by the number of buildings. Such a grouping makes sense because, similar to the model of PS 1.2 that was structured by two related parameters, the room number in PS 2.0 is measured by the number of rooms in a block. This shows how context can mediate one's understanding of a model and can control the choice of parameters for it.

Apparently, the number of rooms in the first block coincides with the room number on its top floor. This shows the importance of meta-context as a mediator in developing a mathematical model of the inquiry into the corresponding meta-context. Also, this suggests that the worthwhile change of model may not occur until one understands mathematical meaning of each of the parameters involved.

By exploring this new model one can come to the following conclusion: if $MOD(N, 10) = 6$ or 9—it is the third floor; if $MOD(N, 10) = 3$, or 5, or 8—it is the second floor; if $MOD(N, 10) = 0$—it is the fourth floor, otherwise it is the first floor. Therefore, because $MOD(78, 10) = 8$, room number 78 belongs on the second floor. This model, being similar to the one used before, has yet a different parameter that describes it—the number of rooms in each block. Apparently, referencing to corresponding meta-context indicates that changing 10 to 11, that is using $MOD(N, 11)$ as a new model, would not be supported by any meaningful context.

It should be noted that the activities described in this section required the use of multiple blueprints, the development of which was based on the use of skills that, to some extent, can be referred to as intuitive or informal. At that point one may wonder: Could blueprints be generated by a spreadsheet? This question prompts the idea of the development of mathematical model that would allow for the formalization of the intuitive skills. In turn, through such formalization, a computerized mathematical model of meta-context would be developed and then used as a cultural amplifier of modeling new problematic situations allowing for the introduction of new notions and concepts. In such a way, the problem of developing computer-generated meta-contexts (in other words, blueprints of different hotels numerated throughout) as mediators in the transition from a static model to a dynamic one appears to be meaningful. While this problem will be addressed later in this paper, a (spreadsheet-generated) blueprint of the 5-storied hotel depicted in Figure 8 indicates that the value of $MOD(78, 15)$ can be used as a model in the locating floor number in the 5-storied hotel on which room 78 belongs.

How are the number of rooms in each block in a hotel and the number of stories in it related? More specifically, how are the numbers 10 and 4 (as well as 15 and 5) related? Further investigation of the meta-context produces pairs 6 and 3; 21 and 6, which, in turn, bring about the modeling tools $MOD(78,6)$ and $MOD(78,21)$. Depending on the type of hotel, these and like tools map any room number into the first block. At this point the concept of triangular numbers as essential components of the mathematical model emerges. Consequently, a relationship between triangular number t_n and its rank n, namely, $t_n = n(n+1)/2$, may come into play. Known as the closed-form representation of triangular numbers, this relationship becomes part of the tool kit associated with the model. Indeed, this new tool can generate various $MOD(N,t_n)$, and therefore meaningful change of model may now occur within the model itself without recourse to the meta-context. Generalizing from special cases stemming from the change of model, one can come up with the following inquiry:

PS 2.1 *Given room number N in the n-storied hotel, on what floor does it belong?*

In order to resolve PS 2.1, one has to explore the emerging dynamic model provided by the family of models $MOD(N,t_n)$. In other words, one has to investigate the behavior of $MOD(N,t_n)$ for different values of N and n. Once again, this investigation can be computerized using a spreadsheet. Through the process of computerization the closed-form definition of triangular numbers would change its status from abstract mathematical artifact to concrete computational tool. However, this tool alone is not enough to provide for a slider-controlled environment in which, given hotel type, room number and floor number serve as input and output respectively. What is needed is another tool that would map room number into floor number on which it belongs. In such a way, a spreadsheet becomes an agent through which a new mathematical model can be developed.

7 Technology as an agent of mathematical modeling

To begin, note that as n (and consequently t_n) grows larger, the range of the function $MOD(N,t_n)$ for sufficiently large N increases. This justifies the use of a spreadsheet in relating N to the corresponding floor number based on the value of $MOD(N,t_n)$. To this end, one has to 'teach' the software to identify sequences like $1,2,4,7,11,\ldots$ (the sequence of room numbers on the first floor of the first block of the hotel with at least five stories). It is the need for such an identification that turns a spreadsheet into an agent of mathematical modeling activities. Consider the following auxiliary problematic situation (APS).

APS 1 *Given floor number m of the first block of the n-storied hotel, find a formula for the sequence that represents room numbers on this floor.*

Using appropriate meta-context (e.g., Figure 8), one can observe that each room number on the first floor ($m=1$) is one more than a corresponding triangular number. In formal algebraic notation this observation can be expressed as follows: The case of $m=1$ (the first floor) yields the sequence of room numbers $1,2,4,7,11,\ldots$ which can be represented in the following closed form $x_k = (k-1)k/2+1, k=1,2,\ldots,n$.

Similarly, one may note that the case of $m=2$ (the second floor) yields the sequence of room numbers $3,5,8,12,17,\ldots$ (each of which is two more than a corresponding triangular number) which, in turn, can be represented as $x_k = k(k+1)/2+2, k=1,2,\ldots,n-1$.

Next, the case of $m=3$ (the third floor) yields the sequence of room numbers $6,9,13,18,\ldots$ (each of which is three more than a corresponding triangular number) which can be represented as $x_k = (k+1)(k+2)/2+3, k=1,2,\ldots,n-2$.

In general, the sequence of room numbers on the m-th floor can be represented as

$$x_k = (k+m-2)(k+m-1)/2+m, \qquad k=1,2,\ldots,n-m+1 \tag{1}$$

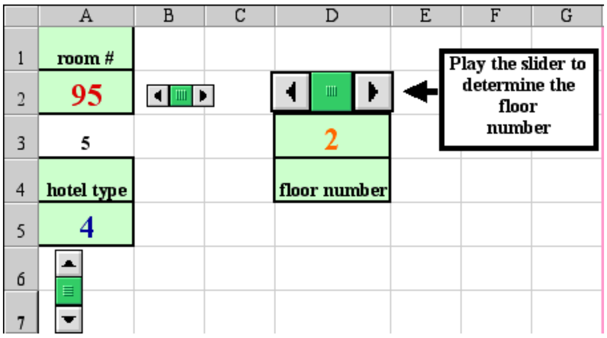

Figure 6: Floor number as the function of room number and hotel type.

Using meta-context as a situational referent one may note that as m grows larger approaching n, the number of terms in this sequence decreases by one approaching a single term; namely, $x_1 = t_n$.

Because relation (1) might be considered a solution to APS 1, it is important to emphasize at this point that the very reason for this auxiliary problem to come into play was the need to develop a mathematical model enabling for the spreadsheet-based association of room number in the first block with floor number on which it belongs. With this in mind, relation (1) can be transformed into the following equivalent form

$$x_k = 0.5k^2 + (m - 1.5)k + 0.5(m^2 - m + 2), \qquad k = 1, 2, \ldots, n - m + 1 \tag{2}$$

which may be considered as a mathematical model of room numbers on the m-th floor of the n-storied hotel. This model, a quadratic trinomial in k with coefficients depending on m, is suitable for spreadsheet-based computerization because it enables the software, given integer P, to identify its location within the first block of the hotel. Indeed, in order to find positive integers k and m for which $x_k = P$, one has to equate the right-hand side of relation (2) to P, solve the resulting equation in k, and connect m and k sought through the formula

$$k = -m + 1.5 + (-2m + 2P + 0.25)^{0.5} \tag{3}$$

This leads to the following criterion:

In order for room N in the n-storied hotel to belong on the m-th floor, the right hand side of (3) with $P = MOD(N, t_n)$, that is, the value of

$$-m + 1.5 + (-2m + 2MOD(N, t_n) + 0.25)^{0.5} \tag{4}$$

has to be a positive integer.

Computerization of this criterion is presented by a spreadsheet depicted in Figure 6. More specifically, the spreadsheet formula `=IF(MOD(A2,A5*(A5+1)/2)=0,A5,MOD(A2,A5*(A5+1)/2))` which maps any room number (cell `A2`) into the first block is defined in cell `A3` and has reference to cell `A5` (hotel type); cell `D3` contains the formula `=IF(AND(INT(1.5-D2+SQRT(-2*D2+0.25+2*A3))`
`=1.5-D2+SQRT(-2*D2+0.25+2*A3),1.5-D2+SQRT(-2*D2+0.25+2*A3)>0),D2," ")`
which has a reference to hidden cell `D2` designated for a floor number (the variable m in expression (4)). In particular, as Figure 6 shows, in the 4-storied hotel room number 95 belongs on the second floor. To conclude this section note that different teachers may come up with different designs for spreadsheet-enhanced models of PS 2.1. In fact, having teachers take intellectual risk in exploiting the semiotic heterogeneity of a spreadsheet has been a part of the course pedagogy, allowing them to advance technological creativity in the apprenticeship mode of learning [3].

8 The spreadsheet as a tool for modeling meta-context

As was mentioned above, many modeling activities described in this paper can be amplified by the use of meta-context in the form of dynamic, spreadsheet-generated blueprints. This section shows how a spreadsheet can generate blueprints and interactively transform one blueprint into another upon the change of a single slider-controlled variable—the number of stories in the hotel (that is, hotel type). Through the process of constructing such a tool, new mathematical activities emerge. These activities, being similar to those previously discussed, are structured by yet another concept known as recursive definition—a strategy of defining a current state of a discrete system in terms of the preceding state (or states) of this system. More specifically, this strategy will define the sequence of room numbers on the m-th floor of the n-storied hotel that span across several blocks through that of on the $(m-1)$-th floor. Once such a definition is found, it can be transformed into a computational tool which draws on the ease of recurrent counting within a spreadsheet.

To this end note that according to Figure 5, the sequence

$$1, 2, 4, 7, 11, 12, 14, 17, 21, 22, 24, 27, 31, 32, 34, 37, \ldots \tag{5}$$

represents room numbers on the first floor of the 4-storied hotel spanned across four blocks. How can this sequence be represented through recursive definition? How can such a definition be generalized to include different hotel types?

For the sake of brevity this section presents final formulas only (both specialized and generalized) that resulted from trial and error explorations of pencil-and-paper blueprints. In doing so, recursive definition for sequence 5 can be found under the guidance of the instructor:

$$x_{k+1} = \begin{cases} x_k + MOD(k,4) & \text{if} \quad MOD(k,4) \neq 0 \quad \text{and} \quad MOD(x_k,10) \neq 1 \\ x_k + 1, & \text{if} \quad MOD(k,4) \neq 0 \quad \text{and} \quad MOD(x_k,10) = 1 \\ (k/4) \times 10 + 1 & \text{if} \quad MOD(k,4) = 0 \end{cases} \tag{6}$$

In general, the sequence of room numbers on the first floor that starts with room number one and spans over several blocks of the n-storied hotel can be defined through the following recursive definition:

$$x_{k+1} = \begin{cases} x_k + MOD(k,n) & \text{if} \quad MOD(k,n) \neq 0 \quad \text{and} \quad MOD(x_k,t_n) \neq 1 \\ x_k + 1, & \text{if} \quad MOD(k,n) \neq 0 \quad \text{and} \quad MOD(x_k,t_n) = 1 \\ (k/n) \times t_n + 1 & \text{if} \quad MOD(k,n) = 0 \end{cases} \tag{7}$$

Visualization provided by the blueprint of Figure 5 suggests that room numbers on the first floor may serve as seed values for the corresponding room numbers on higher floors. This observation brings about the idea of representing room numbers beginning from the second floor recursively through corresponding room numbers on a previous floor, leaving without numeration non-existent (according to the blueprint) rooms.

To this end, one can represent the sequence of room numbers on the second floor of the 4-storied hotel through the following code

$$0, 3, 5, 8, 0, 13, 15, 18, 0, 23, 25, 28, \ldots \tag{8}$$

where zeros substitute for room numbers that are actually absent in the blueprint. These zeros occur in a regular pattern which can be uniquely described in terms of the corresponding room numbers of the first floor. Indeed, each of the numbers 1, 11, 21, 31, 41, ... has the same remainder when divided by 10. Once again, a model involving the concept of congruence modulo triangular number emerges. Furthermore, each non-zero term in sequence (8) is one greater than the corresponding term of sequence (5).

In much the same way, the sequence of room numbers on the third floor can be represented as

$$0, 0, 6, 9, 0, 0, 16, 19, 0, 0, 26, 29, \ldots \tag{9}$$

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
1	5		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2																						
3	17																					
14	6																					
15	5						15					30					45					60
16	4					10	14				25	29				40	44				55	59
17	3				6	9	13			21	24	28			36	39	43			51	54	58
18	2			3	5	8	12		18	20	23	27		33	35	38	42		48	50	53	57
19	1		1	2	4	7	11	16	17	19	22	26	31	32	34	37	41	46	47	49	52	56

Figure 7: A spreadsheet-generated blueprint of the 5-storied hotel.

where zeros correspond to those room numbers on the first floor that are congruent to either one or two modulo 10. It appears that, in general, as the floor number increases, zeros that supersede room numbers can be identified through the appropriate congruence of room numbers on the first floor modulo triangular number while actual room numbers can be defined through simple recursion.

In a spreadsheet environment this emergent mathematical model of the blueprints can be translated into a computerized mathematical artifact allowing one, by playing a slider, to generate different blueprints that can be used for further explorations. To this end, a spreadsheet (like those depicted in Figures 7 & 8) can be programmed as follows.

Cell `A1` is slider-controlled and its content determines hotel type (i.e., the number of stories). In row 1, beginning from cell `C1` natural numbers that numerate buildings are defined. In the range `A3:A19` natural numbers that numerate stories of the hotel are defined. Cell `C19` contains number one – the smallest room number. Cell `D19` contains the formula

```
=IF(MOD(D1-1,$A$1)=0,((D1-1)/$A$1)*$A$1*($A$1+1)/2+1,
IF(MOD(C19,$A$1*($A$1+1)/2)=1,C19+1,C19+MOD(C1,$A$1)))
```

which, when replicated across row 19, generates room numbers on the first floor that span across several blocks. Cell `C18` contains the formula

```
=IF(OR($A18>$A$1,AND(MOD(C$1,$A$1)<$A18, MOD(C$1,$A$1)>0)), " ",C19+1)
```

which is replicated across the columns and up the rows. As a result, this formula, given a hotel's type, generates room numbers and through this process creates a blueprint of the hotel. Figures 7 and 8 show blueprints of 5-and 6-storied hotels respectively.

There are many interesting, computationally driven mathematical activities (besides those already mentioned) made possible by the ease of production of spreadsheet-generated blueprints. Such activities can be organized around the following inquiries:

- What is the sum of room numbers on the m-th floor of the first block of the n-storied hotel?
- What is the sum of room numbers on the first floor of the n-storied hotel spanning across several blocks?
- Given the n-storied hotel, for what values of n can one find at least two buildings with the same sum of room numbers?
- How can one find an answer to the last question a) mathematically; b) computationally?

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	6		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
2																										
3	17																									
14	6							21						42						63						84
15	5						15	20					36	41					57	62					78	83
16	4					10	14	19				31	35	40				52	56	61				73	77	82
17	3				6	9	13	18			27	30	34	39			48	51	55	60			69	72	76	81
18	2			3	5	8	12	17		24	26	29	33	38		45	47	50	54	59		66	68	71	75	80
19	1		1	2	4	7	11	16	22	23	25	28	32	37	43	44	46	49	53	58	64	65	67	70	74	79

Figure 8: A spreadsheet-generated blueprint of the 6-storied hotel.

9 Finding building number given room number

From a didactic perspective, the availability of spreadsheet-generated blueprints have a potential to support the transition from context to meta-context when the possession of just intuitive skills for the production of the latter is not adequate in dealing with the emerging complexity of mathematical model for the former. The following problematic situation is such an example.

PS 3.0 *To what building in the 4-storied hotel does room number 1239 belong?*

How can one mathematize this situation so that the building number sought could be found without counting buildings one by one on a blueprint? Polya's [37] famous heuristic guidance says that if one does not know how to solve a problem one can start with a simpler but related problem and solve it first. With this in mind, one can use the blueprint of Figure 5 and replace 1239 with, say, 59. Visualization suggests that there are five blocks plus three buildings prior to the building that houses room number 59. Multiplying five by four yields the number of buildings within five blocks. At a formal level, grouping fifty-nine rooms into the sets of ten (that is, dividing 10 into 59) allows one to utilize the resulting quotient (or, using formal notation, INT(59/10)) in determining the total number of blocks sought. Once again, the measurement model for division becomes a mathematical model for PS 3.0; yet a different concept, namely, the quotient, is utilized as modeling tool. In such a way, the building number to which room 59 belongs can be found as the sum of $4 \times INT(59/10) + 4$.

Because the model $MOD(59, 10)$ maps room number 59 into room number 9, this auxiliary (simpler) problem is essentially reduced to determining building number within the first block where room number $MOD(59, 10)$ is located. As far as PS 3.0 is concerned, the rule of determining building number to which room 1239 belongs can be described as $4 \times INT(1239/10)$ *plus the building number where room number MOD(1239, 10) is located.*

According to Figure 5, room number 9 belongs to building number 4; therefore building number 495 houses room number 1239. Although PS 3.0 has been resolved completely, it should be noted the above semi-intuitive rule may not be considered as a mathematical model unless meanings of its verbally defined component is formalized. To this end, the variation of meta-context made possible by the use of a spreadsheet can be suggested as a means of such formalization.

To begin, consider the following general inquiry:

PS 3.1 *In what building of the n-storied hotel does room number N belong?*

By exploring various spreadsheet-generated blueprints (e.g., Figures 7 and 8), one can observe that the top room number in each building of the first block is a triangular number whose rank is the building number. In general, building number k has top room number t_k. Three distinct cases are possible.

1. Room number N is on the top floor. Such a room can be described as $MOD(N, t_n) = 0$ and it belongs to building number described as $n \times (N/t_n)$.

2. Room number N is a far-left room not on the top floor. Such a room can be described as follows: $MOD(N,t_n)$ is a triangular number different from t_n.

3. Room number N is not of the above two types. Such a room can be described as follows: $MOD(N,t_n)$ is not a triangular number.

This, together with the triangular test, provides the foundation for the following general criterion[1]. Given room number N in the n-storied hotel, the building number in which it belongs is equal to

$$\begin{cases} n\times(N/t_n), & \text{if } MOD(N,tn)=0 \\ n\times INT(N/t_n)+R+1 & \text{if } MOD(N,t_n)\neq 0 \quad \text{and} \quad (8\times MOD(N,t_n)+1)^{0.5} \quad \text{is a whole number} \\ n\times INT(N/t_n)+R & \text{otherwise} \end{cases}$$

where R is the rank of the largest triangular number not greater than $MOD(N,t_n)$.

In comparison with a semi-intuitive rule used above in resolving PS 3.0, this criterion may be considered as a mathematical model of PS 3.1; however, besides being a combination of three smaller models, it includes a component for which formal mathematical description has yet to be provided. Indeed, in order for this model to become a computerized mathematical artifact, one has to describe R through a computational formula rather than verbally. Thus the mathematical activity of finding such a formula for R may be viewed as a process of defining a model within a model; that is, designing new tool that may have no immediate relevance to the problematic situation in question. With this in mind, the following auxiliary problematic situation has to be resolved.

APS 2 *Given number N, find the rank of the largest triangular number not greater than $MOD(N,t_n)$.*

To this end, let $R(Q)$ be the rank of the largest triangular number not greater than Q. Then $INT((8\times MOD(N,t_n)+1)^{0.5})$ is an odd number

$$R(Q)=(-1+INT((8\times MOD(N,t_n)+1)^{0.5}))/2 \tag{10}$$

otherwise

$$R(Q)=(-2+INT((8\times MOD(N,t_n)+1)^{0.5}))/2 \tag{11}$$

Indeed, in order to find the rank of the largest triangular number not greater than Q, one has to find the largest whole number n such that $n(n+1)/2\leq Q$ or, alternatively, $n\leq(-1+(1+8Q)^{0.5})/2$. Note that

$(1+8Q)^{0.5}-INT((1+8Q)^{0.5})<1$ hence $(1/2)(1+8Q)^{0.5}-(1/2)INT((1+8Q)^{0.5})<1/2$ and $(1/2)(-1+(1+8Q)^{0.5})-(1/2)(-1+INT((1+8Q)^{0.5}))<1/2$.

Furthermore, one of the numbers $(1/2)(-1+INT((1+8Q)^{0.5}))$ and $(1/2)(-2+INT((1+8Q)^{0.5}))$ is an integer, and

$$\begin{aligned}&(1/2)(-1+(1+8Q)^{0.5})-(1/2)(-2+INT((1+8Q)^{0.5}))\\ &=(1/2)(-1+(1+8Q)^{0.5})-(1/2)(-1+INT((1+8Q)^{0.5}))+1/2<1.\end{aligned}$$

This proves that the largest whole number not greater than $(1/2)(-1+(1+8Q)^{0.5})$ is either

$$(1/2)(-1+INT((1+8Q)^{0.5})) \tag{12}$$

or

$$(1/2)(-2+INT((1+8Q)^{0.5})) \tag{13}$$

[1] Recall that in order for number P be a triangular number, the expression $(-1+(8P+1)^{0.5})/2$ should be an integer which also represents its rank (see e.g., [1]).

depending on whether $INT((1+8Q)^{0.5})$ is an odd or even number. Substituting $MOD(N, t_n)$ for Q in (12) and (13) yields formulas (10) and (11) respectively; that is, a mathematical model for a APS 2 has been constructed. In turn, this construction completes that development of mathematical model for PS 3.1.

Finally, following whole class discussion of formulas (10) and (11), this model can be computerized using a spreadsheet (Figure 9). To this end, cells `A1`, `D1` and slider-controlled cell `G1` can be set for hotel type (n), top room number in the first block (t_n), and room number (N) respectively. In cells `J1`, `D3` and `B6` the following spreadsheet formulas

`=IF(MOD(INT(D3),2)>0, (-1+INT(D3))/2, (-2+INT(D3))/2)`,

`=SQRT(8*MOD(G1,D1)+1)`, and

`=IF(MOD(G1,D1)=0,A1*INT(G1/D1),IF(INT(D3)=D3,A1*INT(G1/D1)+J1, A1*INT(G1/D1)+J1+1))`

are defined respectively. This environment enables one, given hotel type and room number as inputs, to generate an output – building number in which the room belongs. As Figure 9 shows, in the 49-storied hotel room number 100 belongs in building number 14.

10 Discovery of hidden meanings through graphing

As was mentioned above, the presence of a computer in a discovery-oriented classroom can enrich modeling discourse by providing opportunities for teachers to explore various patterns that the computer generates in response to modeling goals. This section provides examples of modeling activities enhanced through the use of spreadsheet graphics. To this end, a spreadsheet-based method of dynamic table representation of verbally defined functions [3] will be utilized. One such function, $r_k(n)$, which describes the change of the number of rooms in building number k of the n-storied hotel as the variable n varies was previously mentioned in connection with PS 1.2. Two other functions will be constructed in the context of PS 3.1.

To begin with, consider the latter problematic situation and construct two graphs: the graph of the function $k_n(N)$ which, in the n-storied hotel, relates room number N to building number k in which it belongs, and the graph of the function $k_N(n)$ which, given room number N, relates hotel type n to building number k that houses this room. One can discover that whereas the function $k_n(N)$ exhibits step-wise monotonous growth, the function $k_N(n)$ for each N decreases monotonously and converges to a certain number. For example, $k_{50}(n)$ converges to 10, $k_{100}(n)$ converges to 14, and $k_{200}(n)$ converges to 20. The use of dynamic spreadsheet-based blueprints enables one to mediate the grasp of this phenomenon by referring to meta-context and ultimately to context itself. Indeed, the phenomenon of convergence discovered through graphing justifies the validity of the model and indicates that as the number of stories grows larger, room number, once it gets in a certain building, stays there forever. In other words, by translating a mathematical result made possible by modeling into its original context one can use model to reveal meaningful yet hidden aspects of the context [10] [35].

The graph of the function $k_{100}(n)$ is depicted in the inset of Figure 9, where the range `L1:M8` represents a fragment of its table representation. More specifically, column `L` contains values of n (set as a slider-controlled variable in cell `A1`) while column `M` contains numerical values of $k_{100}(n)$. These values are generated through the spreadsheet formula `=IF(A$1=1, " ", IF(A$1=L1,B$6,M1))` defined in cell `M1` and replicated down column `M`. The use of a circular reference in this formula (i.e., a reference to a cell in which it is defined) enables the spreadsheet to keep the value of an already computed building number unchanged as the content of cell `A1` changes.

In much the same way, the environment depicted in Figure 4 can be extended to include both table and graphic representations of the function $r_k(n)$ mentioned in the beginning of this section. By exploring such an extended environment one can come to the following conclusion: the function $r_k(n)$, given the value of k (i.e., building number), always converges to k as the variable n (i.e., hotel

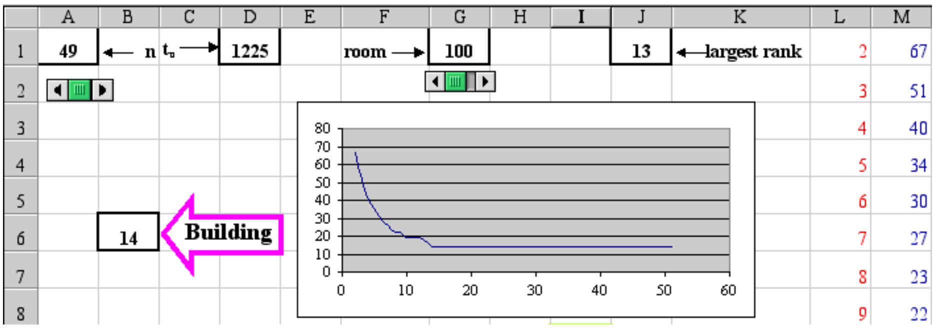

Figure 9: In the 49-storied hotel room number 100 belongs in building number 14.

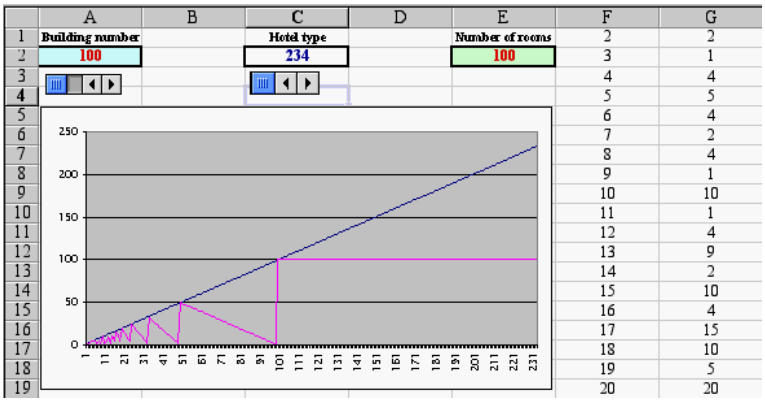

Figure 10: Convergence of the number of rooms to the building number.

type) grows larger. Furthermore, $r_k(n) \leq n$ and all factors of k serve as the fixed points for this function. Such behavior of the function can be confirmed through reference to the corresponding model; namely, $r_k(n) = n$, when $MOD(m, n) = 0$, otherwise $r_k(n) = MOD(m, n)$ where m stands for building number.

The graph of the function $r_{100}(n)$ and its table representation are depicted in Figure 10 thus providing an illustration of the above mentioned analytical behavior. A useful activity for teachers is to interpret the behavior of the function $r_k(n)$ using context as a situational referent and meta-context as mediational means. In other words, by using spreadsheet-generated blueprints one can visualize in an alternative environment that all factors of building number are fixed points of $r_k(n)$ and to conceptualize the mathematical phenomenon discovered in contextual terms. Indeed, as blueprints depicted in Figures 7 and 8 show, each building number that has the number of rooms equal to hotel type is a multiple of the latter.

To conclude this section note that, as was demonstrated above, mathematical results obtained with the help of a technology-enhanced model may go beyond those for which the model was originally created. For example, the meanings of the functions $k_N(n)$ and $r_k(n)$ were hidden in context and revealed through the extension of spreadsheet-based modeling to include graphing. Generally speaking, through the process of developing technology in support of a mathematical model, new ideas and concepts that were difficult if not impossible to predict at an earlier stage of modeling could emerge. In the context of technology-enabled mathematics, this supports the epistemological position that views the progress in the development of mathematical concepts, in part, as a function of the methods of calculation available [24].

11 Suggestions for further use of a spreadsheet in an architectural context

Further spreadsheet-enhanced modeling activities could center on quantitative explorations of qualitatively different types of hotels in which each building has $2, 3, 4, \ldots, k$ more stories than the previous one, so that a block with n buildings has, respectively, $2n - 1, 3n - 2, 4n - 3, \ldots, kn - k + 1$ stories. Apparently, in such a mathematically rich architectural context, different polygonal numbers will serve as room numbers on the top floor of each building. More specifically, if there are r buildings in a block, the largest room number in each block is a multiple of p_r—a corresponding polygonal number of rank r. For example, in the context of the first of the above mentioned extensions, the hotel with six buildings in each block (that is, the 11-storied hotel) has $2MOD(10, 6) - 1$ rooms in the 10th building. In general, in the m-th building of the $(2n - 1)$-storied hotel there are $2MOD(m, \sqrt{s_n}) - 1$ rooms where s_n is the largest room number in the first block. Extending problematic situations discussed in this paper to the new hotel types may result in many interesting and, perhaps, challenging modeling activities dealing with the exploration of what could be referred to as polygonal hotels. This, however, is well beyond the scope of this paper, the goal of which was to demonstrate a possible application of electronic spreadsheet as an amplifier of mathematical modeling activity in context in a teacher education course.

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