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Inequalities and Spreadsheet Modeling

Sergei Abramovich
State University of New York at Potsdam, abramovs@potsdam.edu

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Abstract

This paper presents computer spreadsheet as a milieu for high school mathematics teacher training in the use of inequalities and associated proof techniques. It reflects on activities designed in the framework of a technology-enhanced secondary mathematics education course. The main feature of the milieu is the unity of context, mathematics, and technology that enables inequalities to emerge as meaningful tools in computing applications to authentic problematic situations. In addition to using a spreadsheet as a generator of problems on computational efficiency that lead to the use of inequalities, the paper shows how one can extend a context within which such problems arise allowing for inequalities to be considered a useful component of the mathematical education of high school teachers.

1 Introduction

Inequalities are among the most useful tools of pure and applied mathematics. Yet, until recently, didactical aspects of inequalities have received little attention in mathematics education research [7]. At the 28th annual meeting of the International Group for the Psychology of Mathematics Education the importance of inequalities as pedagogical tools was recognized at the Research Forum titled "Algebraic equations and inequalities: issues for research and teaching." Presentations to the forum focused mainly on high school students' difficulties with solving inequalities. Whereas such difficulties may well be a result of inadequate training in this topic that their teachers received themselves, this issue was addressed only to the point that the role of the teacher and the importance of context and technology were acknowledged as being legitimate directions of research in mathematics education [6]. This paper is written to address the issue of preparing high school teachers (hereafter referred to as teachers) in the use of inequalities as problem-solving tools in computational environments in accord with standards for teaching and recommendations for teachers in North America [8], [15], [16].

Inequalities

The Principles and Standards for School Mathematics [16] recommend that in grades 9–12 all students should "understand the meaning of equivalent forms of expressions, equations, inequalities and relations; write equivalent forms of equations, inequalities, and systems of equations and solve them with fluency ... using technology in all cases" (p. 296). In addition, "instructional programs ... should enable all students to select and use various types of reasoning and methods of proof" (p. 342). These ambitious expectations for high school mathematics curricula and teaching undoubtedly raise the level of professional standards for the teachers from their current position.

Indeed, the recommendations for high school teacher preparation in algebra and number theory provided by the Conference Board of the Mathematical Sciences [8] include the need for the teachers to understand "the ways that basic ideas of number theory and algebraic structures underlie rules for operations on expressions, equations, and inequalities" (p. 40) and the importance of courses within which the teachers "could examine the crucial role of algebra in use of computer tools like spreadsheets and the ways that [technology] might be useful in exploring algebraic ideas" (p. 41). In addition, the National Council for Accreditation of Teacher Education [15] recommends that future teachers of mathematics be given an opportunity to learn ways of using numerical computation and estimation techniques in applications, and extend such techniques to algebraic expressions. Most recently, it has been argued that standards for teaching and recommendations for teachers should be supported by rich problems that "convey important aspects of mathematical thinking [and] the distinctive cohesiveness of mathematics" [19, p869]. All these pedagogical ideas point at the important role that training in the use of inequalities should play in the preparation of teachers.

This paper suggests using spreadsheet modeling as a milieu for the teachers' training in the use of inequalities and associated proof techniques. In some cases, a spreadsheet will be used as a generator of problems on computational efficiency leading to the use of inequalities. In other cases, a context within which computational environments were created will be extended to allow for inequalities to be used as problem-solving tools. Thus the focus of this paper is shifted from a traditional pedagogy of utilizing technology for solving inequalities to using inequalities as problem-solving tools in computing applications. As mentioned elsewhere [1], in a traditional learning environment of a ready-made undergraduate mathematics, prospective teachers are missing a context in which problems involving inequalities arise and receive little or no training in the use of inequalities. As a result, high school students are given almost no support that would enable them, as they continue mathematical studies at the tertiary level, to handle basic techniques of calculus including "epsilon-delta" definitions and convergence tests for infinite series and improper integrals. While calculus is based on a systematic use of inequalities and finding approximations to infinite structures expressed in terms of inequalities, undergraduates are not familiar with these important tools of mathematics. Some level of familiarity with the use of inequalities, including various methods of proof, is important for high school students' long-term development [20]; that is, for their subsequent success in undergraduate mathematics where inequalities serve as important tools in understanding basic concepts of calculus. All this begins with the preparation

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of teachers.

2 Inequalities arising through computational modeling of linear equations

Many problems found across K–12 curriculum can be reduced to linear equations in two variables. The case of Diophantine equations is of particular importance for spreadsheet-enabled mathematics – using a spreadsheet one can generate solutions to such equations by computing values of linear combinations of pairs of whole numbers and comparing these values to the right-hand side of equation in question. As an example, consider the following problem, different variations of which can be found in educational literature around the world [11], [14], [17], [22].

Problem 1. *A pet store sold only birds and cats. The store's owner asked his clerk to count how many animals there were in the store. The clerk counted 18 legs. How many cats and birds might there have been?*

In designing a spreadsheet-based environment for numerical modeling of the pet store problem the following problem-solving situation arises: Given the total number of legs among animals, determine the maximum number of each type of animal that might have been in the store. In a decontextualized form, the problem is to find the greatest values of variables x and y which satisfy the Diophantine equation

$$ax + by = n \tag{1}$$

($n = 18$, $a = 2$, and $b = 4$ in the case of Problem 1). Knowing such values of x and y (that is, the largest total for each type of animal) enables one to generate solutions within tables that do not include unnecessary computations. It is through resolving such a computational problem that pedagogically useful activities involving appropriate use of inequalities and associated proof techniques can come into play.

To begin, note that rough upper estimates for an x-range and y-range are quite apparent (provided that one uses context as a support system): $x \leq n$ and $y \leq n$. In mathematics, however, even apparent statements require formal demonstration. Therefore, these simple inequalities can be used as a springboard into content-specific proof techniques. One such technique is based on reasoning known as proof by contradiction — where, for the sake of argument, one makes an assumption contrary to what has to be proved, arrives at an absurd result, and then concludes that the original assumption must have been wrong, since it led to this result. This type of argument (sometimes referred to as reduction to an absurdity) makes use of the so-called law of excluded middle — a statement which cannot be false, must then be true. Indeed, if, on the contrary, $x > n$, then for any $y \geq 0$ it follows that $n = ax + by > an \geq n$. This contradiction (i.e., the false inequality $n > n$) suggests that x cannot be greater than n, thus $x \leq n$. Likewise, assuming that $y > n$ for any $x \geq 0$ yields $n = ax + by > bn \geq n$. It might be helpful to repeat this argument using the pet store context so that each mathematical sentence would be supported by a situational referent.

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	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y
1	n=	18	◀	▶	Total number of legs								y\x	0	1	2	3	4	5	6	7	8	9		
2													0										18		
3	a=	2	◀	▶	Number of legs on animal A								1								18				
4													2						18						
5	b=	4	◀	▶	Number of legs on animal B								3				18								
6													4		18										
7																									
8																									
9																									

Figure 1: Five solutions displayed.

Being intuitively apparent in contextual terms and thus didactically useful, the above upper estimates of ranges for variables x and y in equation (1) can be significantly improved. The need for such an improvement can be computationally-driven. Indeed, by generating solutions to Problem 1 within an 18×18 table one can see that the number of cells within which computations have to be carried out may be reduced ninefold. Using a combination of formal and contextual arguments, one can prove (by contradiction) the inequality $x \leq n/a$. Indeed, the formal assumption $x > n/a$ for any $y > 0$ results in the following contradictory conclusion: $n = ax + by > a(n/a) = n$. A measurement model for division may be helpful for the contextual representation of this argument.

Furthermore, taking into account that x is an integer variable and using the function INT (described in a spreadsheet environment as a tool that rounds a number down to the nearest integer), yield an even stronger inequality $x \leq INT(n/a)$. Similarly, the inequality $y \leq INT(n/b)$ can be used as an upper estimate for the variable y. As an application of the last two inequalities to modeling equation (1), spreadsheets pictured in Figures 1 and 2 (see Appendix for programming details) generate computationally efficient x- and y-ranges for different values of n.

Note that the knowledge of lower estimates for variables x and y can be used to further improve computational efficiency of the environment in question. Indeed, the inequality $y \geq k$, where k is a positive integer less than n enables for the improvement of the x-range found. To this end, one can write $x = (n - by)/a \leq (n - bk)/a < n/a$. Thus the inequality $x \leq (n - bk)/a$ as an upper estimate for x is an improvement over an earlier found estimate $x \leq n/a$. Finally, taking into account that x is an integer variable results in a stronger inequality $x \leq INT[(n - bk)/a]$. Likewise, the assumption $x \geq k$, $0 < k < n$, and the fact that y is an integer variable can be exploited in refining the y-range found earlier.

Spreadsheets in which information about a lower estimate for one variable is used to improve an upper estimate for another variable are pictured in Figures 3 and 4 (see Appendix for programming details). They show that the larger the lower bound for the variables y and x, respectively, the smaller the upper bound for those variables. This computationally-driven statement can be interpreted in the following contextual

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	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y
1	n=	20	◄	▶	Total number of legs								y\x	0	1	2	3	4	5	6	7	8	9	10	
2													0											20	
3	a=	2	◄	▶	Number of legs on animal A								1									20			
4													2							20					
5	b=	4	◄	▶	Number of legs on animal B								3					20							
6													4			20									
7													5	20											
8																									
9																									

Figure 2: Six solutions displayed.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y
1	n=	18	◄	▶	Total number of legs								y\x	0	1	2	3	4	5						
2													2						18						
3	a=	2	◄	▶	Number of legs on animal A								3				18								
4													4		18										
5	b=	4	◄	▶	Number of legs on animal B																				
6																									
7	Lower bound for x		0		◄	▶																			
8																									
9	Lower bound for y		2		◄	▶																			
10																									

Figure 3: The larger y_0, the smaller x-range.

terms: the more animals of one type the store has, the fewer animals of another type are there. In that way, the meaning of inequalities can be communicated to future teachers of mathematics in three different ways: contextually, computationally, and mathematically.

A similar didactical approach will be used in the next section allowing for new inequalities and associated proof techniques to be discussed. To conclude this section, note that teachers can be involved in other pedagogically useful activities on proving inequalities in the context of designing a spreadsheet as a partitioner of a whole number into three specified summands [3].

3 Inequalities as tools in modeling non-linear problems

Interesting activities on the use of inequalities and associated proof techniques can be carried out in the context of spreadsheet modeling of non-linear equations that represent unit fractions as the sum of two or more like fractions. There are problems both within and outside mathematics in which such representations are important. There has been recent increase of interest, both mathematical and pedagogical, in problems involving unit fractions [2], [4], [9], [13]. Furthermore, as recorded by Plutarch [21], unit fractions

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	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y
1	n=	22	◄	►	Total number of legs								y\x	4	5	6	7	8	9	10	11				
2													0								22				
3	a=	2	◄	►	Number of legs on animal A								1						22						
4													2				22								
5	b=	4	◄	►	Number of legs on animal B								3		22										
6																									
7	Lower bound for x			4	◄	►																			
8																									
9	Lower bound for y			0	◄	►																			
10																									

Figure 4: The larger x_0, the smaller y-range.

(often referred to as Egyptian fractions) can be found in the context of an ancient geometry problem of finding rectangles with area numerically equal to perimeter. As an extension of this problem, consider

Problem 2. *Find all rectangles of integer sides whose area, numerically, is n times as much as its semi-perimeter.*

Setting x and y to be integer dimensions of a rectangle, this problem can be reduced to a non-linear algebraic equation

$$\frac{1}{n} = \frac{1}{x} + \frac{1}{y} \tag{2}$$

As in the case of equation (1), in designing a spreadsheet-based environment for modeling solutions to equation (2) the following context-bounded inquiry into its structure can be raised: Given n (the ratio of area to semi-perimeter of a rectangle), determine the largest values for each of the variables x and y (dimensions of the rectangle). This time, however, unlike equation (1), in which rough (upper) estimates for x and y-ranges could be found almost intuitively, equation (2), where x, y, and n relate to each other in a non-linear way, does not allow for such an intuitive approach. A first step in finding upper estimates for x and y in equation (2) could be to use the identity

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)} \tag{3}$$

which represents $1/n$ as the sum of the largest and the smallest possible fractions (or, alternatively, as will be shown below, through the dimensions of the rectangle with the largest perimeter) satisfying equation (2). This observation brings about preliminary upper and lower estimates for x and y in the form of the inequalities $n+1 \leq x \leq n(n+1)$ and $n+1 \leq y \leq n(n+1)$.

Having experience with equation (1), the teachers may wonder: Could these inequalities be improved? With this in mind, note that if $n > 1$, then $n(n+1) > 2n$. Indeed, dividing both sides of the last inequality by n yields $n+1 > 2$, a true inequality. Now

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one can show that at most one denominator (or, alternatively, rectangle's side) may be greater than $2n$. Invoking proof by contradiction, that is, assuming $y \geq x > 2n$, one can arrive at the following absurd conclusion: $1/n = 1/x + 1/y > 1/2n + 1/2n = 1/n$. Therefore, if $x \leq y$, then $n + 1 \leq x \leq 2n$ and $n + 1 \leq y \leq n(n+1)$. Likewise, at most one fraction in the right-hand side of equation (2) may have a denominator smaller than $2n$ (or, alternatively, only one side of a rectangle may be smaller than $2n$). In this way, it appears that the inequalities

$$n + 1 \leq x \leq 2n \leq y \leq n(n+1) \tag{4}$$

when compared to those previously found, enable for the design of a more computationally efficient spreadsheet-based environment for modeling solutions to equation (2). A spreadsheet that incorporates inequalities (4) in the case $n = 6$ is pictured in Figure 5 (`D2:I2` – the x-range; `C3:C33` – the y-range). The results of computations suggest that there are five ways to partition $1/6$ into the sum of two unit fractions.

It should be noted that by modeling solutions to equation (2) for different values of n, one can observe the following pattern: the solution associated with identity (3) is separated from other solutions by a wide gap on a spreadsheet template. Indeed, as Figure 5 shows, this gap spans from row 26 to row 42; that is, 40% of the template generates no solution. This observation enables one to further improve computational efficiency of the environment. To this end, note that, in general, because identity (3) provides a solution to equation (2) for any integer n, an upper estimate for y can be improved in comparison with $n(n+1)$. Let k be the smallest number greater than one such that

$$\frac{1}{n} = \frac{1}{n+k} + \frac{k}{n(n+k)}$$

The fraction $k/(n(n+k))$ is a unit fraction provided that k divides n, $k \geq 2$. Therefore, assuming that the pair $(x, y) = (1/(n+1), 1/n(n+1))$ is always a solution to equation (2), the inequalities

$$n + 2 \leq x \leq 2n \leq y \leq n(n+2)/2 \tag{5}$$

enable one to further improve computational efficiency of the spreadsheet environment (see Appendix for programming details). One can see that $\frac{n(n+1)}{n(n+2)/2} = 2\left(1 - \frac{1}{n+2}\right) \geq \frac{3}{2}$; thus the old y-range is at least 1.5 times larger than the new (improved) one. Visually, this difference becomes apparent by comparing Figure 5 to Figure 6, as the latter shows a spreadsheet built on inequalities (5). The next section will show how in addition to computationally-driven use of inequalities, new ideas and techniques can be introduced through the advancement of contextual arguments.

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	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	n=	6													
2				2	3	4	5	6	7	8	9	10	11	12	
3			2												
4			3												
5			4												
6			5												
7			6												
8			7												
9			8												
10			9												
11			10												
12			11												
13			12											6	
14			13												
15			14												
16			15									6			
17			16												
18			17												
19			18								6				
20			19												
21			20												
22			21												
23			22												
24			23												
25			24							6					
26			25												
27			26												
28			27												
29			28												
30			29												
31			30												
32			31												
33			32												
34			33												
35			34												
36			35												
37			36												
38			37												
39			38												
40			39												
41			40												
42			41												
43			42						6						

Figure 5: Using inequalities (4).

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	A	B	C	D	E	F	G	H	I	J
1			Note: This table does not							
2			include the known solution x = 7 and y = 42							
3	n=	6								
4				8	9	10	11	12		
5			12					6		
6			13							
7			14							
8			15		6					
9			16							
10			17							
11			18	6						
12			19							
13			20							
14			21							
15			22							
16			23							
17			24	6						

Figure 6: Using inequalities (5).

4 Geometric context as a springboard into new uses of inequalities

The unity of context, computing, and mathematics is a useful pedagogical triad for it enables the introduction of new techniques associated with the use of inequalities. As was mentioned above in passing and with no justification, identity (3) can be interpreted as a solution to equation (2) that corresponds to the rectangle with the largest perimeter. In what follows, it will be demonstrated how one can advance this comment to the status of being a rigorously proved mathematical proposition. It is through such advancement that new ideas and proof techniques associated with inequalities will be discussed. An important aspect of this discussion is the value of geometric roots of algebraic propositions; in other words, the value of context in acquiring new mathematical knowledge.

In order to find a rectangle with the largest perimeter that satisfies the conditions of Problem 2, one has to define its perimeter as a function of one of the dimensions, say y, with n being a parameter. To this end, note that, as follows from equation (2), the other dimension $x = ny/(y - n)$. Assuming $x \leq y$ yields $2n \leq y \leq n(n + 1)$. The sum of two dimensions $y + ny/(y - n)$ can be simplified to the form $y^2/(y - n)$ enabling one to define the perimeter function $P(y) = 2y^2/(y - n)$ on the segment $[2n, n(n + 1)]$. At this point the teachers can be reminded of the extreme value theorem (studied in a calculus course) which provides a theoretical foundation for finding the largest (and the smallest) perimeter. In that way, following a recommendation by the Conference Board

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of the Mathematical Sciences [8], an explicit connection between high school and college mathematics curricula can be established.

The next step is to show that $P(y)$ monotonically increases as the integer variable y increases so that $P(y+1) > P(y)$ for all $y \in [2n, n(n+1)]$. In other words, one has to prove that the inequality

$$\frac{P(y+1)}{P(y)} > 1 \tag{6}$$

holds true on the segment $[2n, n(n+1)]$. In that way, geometric context associated with equation (2) brings about the need for the use of inequalities by the teachers. Furthermore, another type of argument based on a straightforward combination of earlier established facts and commonly referred to as direct proof can be used in proving inequality (6).

To this end, one can start with using simple rules for operations on algebraic expressions to show that

$$\frac{P(y+1)}{P(y)} = \frac{(y+1)^2 (y-n)}{y^2 (y-n+1)} = \frac{(1+1/y)^2}{1+1/(y-n)} \tag{7}$$

In order to proceed, two new inequalities have to be established. The first one results from the relationships $(1+1/y)^2 = 1 + 2/y + 1/y^2 > 1 + 2/y$. In fact, the inequality

$$\left(1 + \frac{1}{y}\right)^2 > 1 + \frac{2}{y} \tag{8}$$

is a special case of a more general statement (known as *Bernoulli's inequality* [10]): $(1+y)^r \geq 1 + ry$, $y > -1$, $r > 1$ or $r < 0$. The proof of this can be found, for example, in a classic (high school-oriented) treatise by Korovkin [12]. The second inequality

$$\frac{2}{y} \geq \frac{1}{y-n} \tag{9}$$

can be easily proved under the assumption $y \geq 2n$ by using basic rules of algebra. Finally, one can complete the proof of inequality (6) by applying inequalities (8) and (9) in evaluating the far-right fraction in (7) as follows:

$$\frac{\left(1+\frac{1}{y}\right)^2}{1+\frac{1}{y-n}} > \frac{1+\frac{2}{y}}{1+\frac{1}{y-n}} \geq \frac{1+\frac{1}{y-n}}{1+\frac{1}{y-n}} = 1$$

Therefore, $P(y) \leq P(n(n+1)) = 2n^2 (n+1)^2 / n^2 = 2(n+1)^2$. In other words, identity (3), indeed, determines the rectangle with the largest perimeter and area being n times as much as half of this perimeter.

The extreme value theorem mentioned above guarantees that the function $P(y)$ has both the global maximum and minimum on the segment $[2n, n(n+1)]$; in other words, in the context of Problem 2 there exists a rectangle with the smallest perimeter also. In
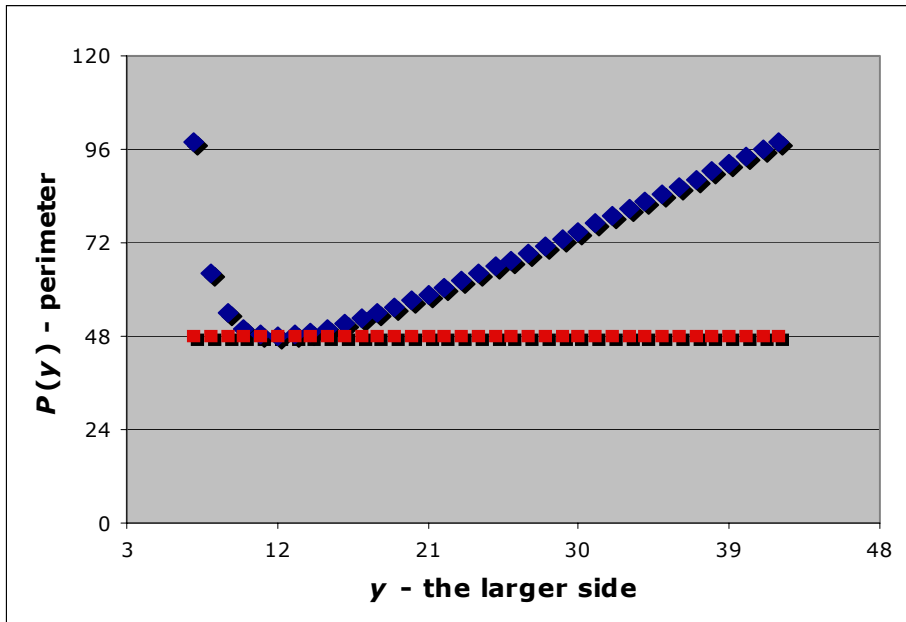
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Figure 7: Graph of perimeter function $P(y) = 2y^2/(y-n)$ for $n+1 \leq y \leq n(n+1)$, $n = 6$.

order to find such a rectangle, one can make use of a famous analytic inequality (with profound geometric meaning) known as the arithmetic mean–geometric mean inequality:

$$\frac{u+v}{2} \geq \sqrt{uv}, \text{ for } u \geq 0, v \geq 0 \qquad (10)$$

with equality taking place when $u = v$. Inequality (10), whose three-dimensional analogue and general form are discussed, respectively, in section 6 of this paper and [12], can be proved directly by demonstrating that the difference between its left and right-hand sides is non-negative. The use of inequality (10) in estimating the function $P(y)$ is not a straightforward one, though. It requires a specific representation of this function based on the identity $y^2/(y-n) = y - n + n^2/(y-n) + 2n$. This makes it possible to apply inequality (10) to the sum and estimate the function $P(y)$ as follows:

$$P(y) = 4\left(\frac{y - n + n^2/(y-n)}{2} + n\right) \geq 4\left(\sqrt{(y-n)\frac{n^2}{y-n}} + n\right) = 8n$$

Thus $P(y) \geq 8n$ with equality taking place when $y - n = n^2/(y-n)$; that is, when $y = 2n$. In other words, the rectangle with the smallest perimeter and area being n times as much as half of this perimeter is a square with side equal to $2n$.

Note that results of this section can be developed using direct proof methods based on geometric reasoning. One such method is to graph the function $P(y)$ using the

spreadsheet graphing capability (Figure 7) and observe that $\max_{2n \leq y \leq n(n+1)} P(y) = P[n(n+1)] = 2(n+1)^2$ and $\min_{2n \leq y \leq n(n+1)} P(y) = P(2n) = 8n$. Another method is based on the fact that the only local extremum of the function $P(y)$ exists at the point $y = 2n$. Indeed, the equation $2y^2/(y-n) = k$ has a single solution only when $k = 8n$ and $y = 2n$. This fact, supported by the graph in Figure 7, can be proved algebraically by equating the discriminant of this (quadratic) equation to zero. Finally, one can use all three techniques, including the one used to prove inequality (6), to show that the function $P(y)$ monotonically increases on the segment $[2n, n(n+1)]$ thus having $P[n(n+1)] = 2(n+1)^2$ as its maximum.

5   Transition to three-dimensional modeling

Although a spreadsheet is commonly used in mathematics education as a modeling tool for problems with at most two variables [5], several computational methods enable its use beyond two dimensions. In the case of modeling equations in whole numbers, these include the method of virtual three-dimensional computing based on the use of circular references and iterations [3] and that of constructing level lines of integer values. To illustrate the latter method, consider the following problem that leads to a three-dimensional analogue of equation (2).

   Problem 3. *Find the total number of right rectangular prisms of integer sides whose volume, numerically, is n times as much as the half of its surface area.*

   Similar to Problem 2, setting x, y, and z to be integer dimensions of a right rectangular prism, results in the equation

$$\frac{1}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \tag{11}$$

Rewriting equation (11) in the form of

$$z = \frac{nxy}{xy - n(x+y)} \tag{12}$$

makes it possible to use the two-dimensional computational capacity of a spreadsheet in numerical modeling of the level lines $z = \text{const}$ of integer values; that is, given n, to find those integer pairs (x, y) for which z is an integer also (see Appendix for programming details). As before, a computational problem of finding the largest and the smallest values for each of the variables x and y satisfying equation (12) gives rise to interesting activities on the use of inequalities.

   The symmetric nature of equation (11) suggests that $n < x \leq y \leq z$ need only be considered. Next, it might be helpful to start with generating integer values of z defined by equation (12) without regard to the computational efficiency of the environment. In that way, by modeling equation (11) for, say, $n \leq 3$, one can get some intuitive ideas as to what estimates for x and y might prove to be helpful in generating solutions for $n > 3$. In doing so, one can conjecture, that at most two denominators in the right-hand

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side of equation (11) may be greater than $3n$, and confirm this formally using proof by contradiction. Indeed, the assumption $3n < x \leq y \leq z$ results in the false conclusion

$$\frac{1}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{1}{3n} + \frac{1}{3n} + \frac{1}{3n} = \frac{1}{n}$$

This suggests that

$$n + 1 \leq x \leq 3n \tag{13}$$

Another computationally-driven conjecture is that at most one of x, y and z may be greater than $2n(n+1)$. In order to prove this conjecture, one can use an indirect proof – a combination of proof by contradiction and constructive proof where contradiction is constructed by calculating an appropriate example (a classic demonstration of an indirect proof can be found in [18]). To this end, one can choose $k > 1$, $y = 2n(n+1)+1$, $z = 2n(n+1) + k$, and then construct a contradictory inequality for such values of k, y and z. Indeed, we have:

$$\frac{1}{n} = \frac{1}{x} + \frac{1}{2n(n+1)+1} + \frac{1}{2n(n+1)+k} < \frac{1}{x} + \frac{2}{2n(n+1)+1} = \frac{1}{x} + \frac{1}{n(n+1)+0.5}$$

Therefore,

$$\frac{1}{x} > \frac{1}{n} - \frac{1}{n(n+1)+0.5} = \frac{n^2 + 0.5}{n(n^2 + n + 0.5)}$$

whence $x < n + n/\left(n^2 + 0.5\right) < n + 1$. This conclusion contradicts the inequality $x \geq n + 1$; thus $y \leq 2n(n+1)$.

In order to find a lower estimate for y, once again, spreadsheet calculations without regard to their efficiency can be helpful in conjecturing that $y \geq 2n + 1$. To prove this conjecture, one can use an indirect proof. To this end, one may assume that for a certain integer $k \geq 1$, there exists an integer m, $k < m < n + 1$, such that for $x = n + k$ and $y = n + m$, the value of y in equation (11), contrary to the above computationally-driven conjecture, satisfies the inequality $y < 2n + 1$. These values of x and y yield

$$\frac{1}{z} = \frac{1}{n} - \frac{1}{n+k} - \frac{1}{n+m} = \frac{n^2 + 2kn + km}{n(n+k)(n+m)}$$

whence

$$z = n\left[1 + \frac{n(m-k)}{n^2 + 2kn + km}\right] = n\left(1 + \frac{m-k}{n + 2k + km/n}\right) < 3n$$

This, however, contradicts the inequality $z \geq 3n$ which, in turn, follows from the simple fact that if $x \leq y \leq z < 3n$, then $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} > \frac{1}{n}$. Therefore, the inequalities

$$2n + 1 \leq y \leq 2n(n+1) \tag{14}$$

determine a segment within which the variable y in equation (11) varies.

Finally, one can prove that no denominator may be greater than $n(n+1)(n^2+n+1)$. To this end, identity (3) can be applied to itself to get

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n^2+n+1} + \frac{1}{n(n+1)(n^2+n+1)} \tag{15}$$

Thus

$$z \le n(n+1)(n^2+n+1) \tag{16}$$

To conclude this section, note that, due to inequalities (13), the variable x varies over the set $\{n+i \mid i = 1, 2, \ldots, 2n\}$. Therefore, following Polya's strategy of using a simpler problem as a means of understanding a more difficult one [18], equation (11) can be reduced to the following $2n$ equations

$$\frac{i}{n(n+i)} = \frac{1}{y} + \frac{1}{z}, \quad i = 1, 2, \ldots, 2n \tag{17}$$

some of which can be simplified to have the form of equation (2) studied earlier. Such a reduction enables one to solve equation (17) for each i which divides $n(n+i)$ and, in doing so, to estimate from below the number of solutions of equation (11). For example, when $n = 3$, only two values of i turn equation (17) into equation (2) yielding, through such an approach, 13 solutions of equation (11) out of 21 total (see Figure 8). However, when $n = 2$, the reduction approach gives the same number of solutions to equation (11) as one can find through its direct spreadsheet modeling. This discussion leads to the problem of representing a non-unit fraction as the sum of two unit fractions, something that is beyond the scope of this paper.

6   Discussion of modeling data and its alternative interpretation

The spreadsheet pictured in Figure 8 (see Appendix for programming details) is built on the use of inequalities (13), (14) and (16) in the case $n = 3$. It shows the results of generating all partitions of $1/3$ into three unit fractions. For example, using the contents of the cells B2, E1, D8, and E8 as denominators of unit fractions, yields the partition $1/3 = 1/4 + 1/13 + 1/156$ with the largest product of the right-hand side's denominators among all 21 partitions so generated. Another partition, $1/3 = 1/9 + 1/9 + 1/9$, is associated with the smallest such product. In the next sections these statements will be supported by a combination of mathematical and computational arguments, involving the use of inequalities. By changing the content of cell B2 (value of n), representations of other unit fractions as the sum of three like fractions (dimensions of corresponding rectangular prisms) can be generated and triples with specific properties can be identified. Note that under certain conditions, the variables in equation (11) can be given another geometric interpretation; namely, if n is the radius of a circle inscribed in a triangle with the heights

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	A	B	C	D	E	F	G	H	I	J	K
1		◄ ►		y\x	4	5	6	7	8	9	
2	n=	3		7			42	21			
3	xmax=	9		8		120	24		12		
4	ymax=	24		9		45	18			9	
5	zmax=	156		10		30	15				
6				11							
7				12		20	12				
8				13	156						
9				14	84						
10				15	60	15					
11				16	48						
12				17							
13				18	36						
14				19							
15				20	30						
16				21	28						
17				22							
18				23							
19				24	24						
20											

Figure 8: At most one denominator in each partition of 1/3 is greater than 24.

x, y, and z, then the quadruple (n, x, y, z) satisfies equation (11). Indeed, if a, b and c are the sides of such a triangle then its area, A, is given by eq (18).

$$A = n\left(\frac{a+b+c}{2}\right) = \frac{ax}{2} = \frac{by}{2} = \frac{cz}{2} \tag{18}$$

whence equation (11). However, not every integer solution of equation (11) corresponds to a triangle in which a circle of radius n is inscribed. In this regard, one can be reminded about the classic triangle inequality – the sum of any two sides of a triangle is greater than the third side. It is in that sense that one may consider Heron's formula, eq (19), for area of triangle in terms of its sides.

$$A = \sqrt{p\,(p-a)\,(p-b)\,(p-c)} \tag{19}$$

and the semi-perimeter, p, is defined for "all" a, b, and c. Using equalities (18) and applying simple algebraic transformations to formula (19) yield the modification of Heron's formula given by eq (20).

$$A = \frac{1}{\sqrt{\frac{1}{n}\left(\frac{1}{n}-\frac{2}{x}\right)\left(\frac{1}{n}-\frac{2}{y}\right)\left(\frac{1}{n}-\frac{2}{z}\right)}} \tag{20}$$

The right-hand side of eq (20) is defined for $z \geq y \geq x > 2n$. Therefore, one can see that only three out of 21 triples generated by the spreadsheet of Figure 8 satisfy the last inequality; namely $(7, 7, 21)$, $(8, 8, 12)$ and $(9, 9, 9)$. In the next section both plane and solid geometry contexts will be extended to allow for the use of arithmetic mean-geometric mean inequality in solving three dimensional extremum problems. In particular, through a combination of formal (the use of inequalities) and informal (the use of computing) arguments, it will be shown that the triples of heights $(9, 9, 9)$ and $(7, 7, 21)$ correspond, respectively, to triangles with the smallest and the largest area being circumscribed about a circle of radius three linear units.

7   Formal and informal approaches to solving three-dimensional problems

One can use inequalities in solving problems on minimum and maximum in the context of three-dimensional modeling. For example, a natural extension of activities described in the last section is to find rectangular prisms with the smallest and the largest surface area satisfying the condition of Problem 3. In order to find such a prism with the smallest surface area, one can use the three-dimensional case of the arithmetic-mean-geometric-mean inequality of eq (21).

$$\frac{u+v+w}{3} \geq \sqrt[3]{uvw}, \quad u \geq 0, v \geq 0, w \geq 0 \tag{21}$$

A direct proof of inequality (21) may consist in reducing it to the form $p^3 + q^3 + r^3 \geq 3pqr$, where $p = \sqrt[3]{u}$, $q = \sqrt[3]{v}$, and $r = \sqrt[3]{w}$, demonstrating that the difference between the

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left and right-hand sides of the last inequality can be expressed as a sum in which every term is obviously non-negative (a special case of what in [10, p55] is referred to as the identity of Hurwitz and Muirhead).

Note that half of the surface area of the rectangular prism with dimensions x, y, and z satisfies the equality $xy + yz + xz = xyz/n$. Therefore, those values of the variables that provide the smallest value for xyz (the volume of rectangular prism) provide the smallest value for its surface area also. Substituting $x = 1/u$, $y = 1/v$, and $z = 1/w$ in (21) results in the inequality

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3\sqrt[3]{\frac{1}{xyz}}$$

whence

$$xyz \geq \left(\frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}\right)^3 = 27n^3$$

with equality taking place when $x = y = z = 3n$. Thus a cube with side $3n$ has the smallest surface area among all rectangular prisms whose volume is n times as much as half of this surface area. In particular, when $n = 3$ the triple $(9, 9, 9)$ represents such a cube.

Inequality (21) does not allow one to find the largest surface area among rectangular prisms satisfying the condition of Problem 3. In order to find such area, one can first find it numerically within a spreadsheet. In doing so, one can discover that denominators in the right-hand side of identity (15) provide the largest value for the product xyz; that is, these denominators are the dimensions of the corresponding rectangular prism. Mathematically this fact can be established through demonstrating that the function $P(n, k) = (n + k)^2 \left[\frac{n}{k}(n + k) + 1\right]^2 \frac{n}{k}$, where $1 \leq k \leq 2n$ is chosen to satisfy the identity

$$\frac{1}{n} = \frac{1}{n + k} + \frac{1}{\frac{n}{k}(n + k) + 1} + \frac{1}{\frac{n}{k}(n + k)\left[\frac{n}{k}(n + k) + 1\right]}$$

monotonically decreases so that $P(n, k) \leq P(n, 1) = n(n + 1)^2[n(n + 1) + 1]^2$ for all $1 \leq k \leq 2n$. In particular, when $n = 3$ the triple $(4, 13, 156)$ represents dimensions of a rectangular prism with the largest surface area satisfying the condition of Problem 3. The graph of the function $P(n, k)$ for $n = 3$ is shown in Figure 9. It provides visual demonstration of monotonic descent of the function $P(3, k)$ for $1 \leq k \leq 6$. A similar problem on minimum and maximum can be posed in the context of the above-mentioned plane geometry interpretation of equation (11). Indeed, among all triangles of integer heights circumscribed around a circle of radius n, one triangle has the smallest area and one triangle has the largest area. In order, to find the triangle with the smallest area one can set $u = p - a$, $v = p - b$, $w = p - c$ and apply inequality (21) to the right-hand side of formula (19) as follows

$$A \leq \sqrt{p\left(\frac{p - a + p - b + p - c}{3}\right)^3} = \sqrt{\frac{p^4}{27}} = \frac{p^2}{3\sqrt{3}} \qquad (22)$$

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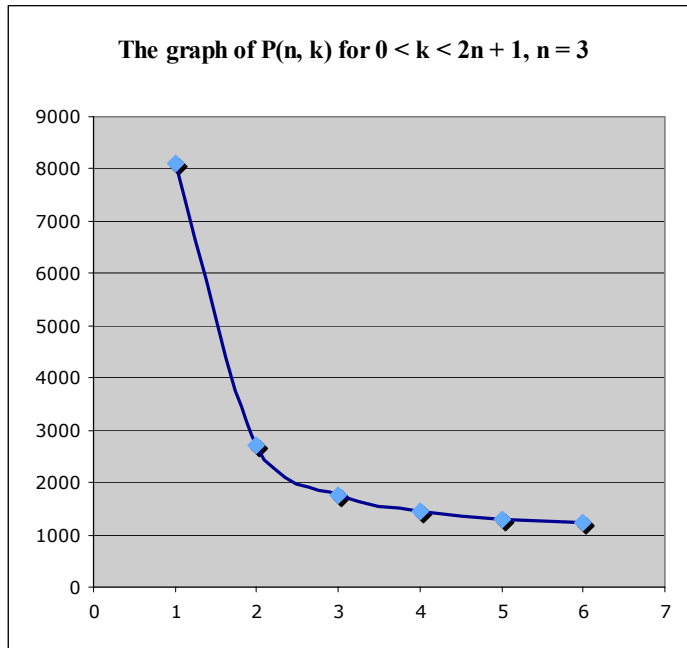**The graph of P(n, k) for 0 < k < 2n + 1, n = 3**

Figure 9: Visualizing the equality $\max P(n, k) = P(n, 1)$ for $n = 3$.

Inequality (22) becomes an equality when $a = b = c$, a case of an equilateral triangle circumscribed about a circle with radius n. Equalities (18) imply that $A = pn$; thus, substituting A/n for p in the far-right part of (22) yields $A \geq 3\sqrt{3}n^2$ with equality taking place for an equilateral triangle with height and side equal, respectively, to $3n$ and $2\sqrt{3}n$. In particular, when $n = 3$ one gets $(x, y, z) = (9, 9, 9)$ – one of the two triples mentioned at the end of section 6.

To find a circumscribed triangle with the maximum area, one can use a spreadsheet. Indeed, through spreadsheet modeling one can conjecture that the area of triangle in question attains the greatest value when $x = 2n + 1$ and $y = 2n + 1$, $z = n(2n + 1)$. Using formula (20) results in

$$A = n^2 (2n + 1) \sqrt{\frac{2n + 1}{2n - 1}}$$

for these values of x, y and z. In particular, when $n = 3$ one gets $(x, y, z) = (7, 7, 21)$ – another triple mentioned at the end of section 6; one can find the area of a triangle with this triple of heights using the Pythagorean theorem. A formal demonstration of the last formula for area requires the use of multivariable differential calculus that is beyond the scope of this paper.

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8 Summary

This paper has demonstrated how a spreadsheet can be used as a milieu for high school teachers' training in the use of inequalities. In this regard, several didactical aspects of this milieu are worth emphasizing. First, the approach suggested presents inequalities not as abstract artifacts disconnected from real life but rather as emerging tools in computing applications leading to the discussion of the effectiveness of spreadsheet as a computational medium. It appears that such application-oriented focus in the use of inequalities enhances one's learning of associated proof techniques including proof by contradiction, proof by construction, direct and indirect proofs.

Second, the approach showed how one can make a transition from novice to expert practice in spreadsheet modeling so that results obtained in the context of novice practice can facilitate this transition. Indeed, it is an outcome of a novice practice in modeling solutions to equation (2) that enabled visualization of their quasi-symmetrical density on a spreadsheet template and thus prompted the idea of using a refined inequality towards improving computational efficiency of the medium. Furthermore, the approach showed how spreadsheet modeling could be used for computationally driven conjecturing of inequalities that would be difficult to formulate otherwise.

Third, the approach has situated computing activities in the context of authentic problem solving. This combination of context, computing, and mathematics connects inequalities to their situational referents as illustrated in the case of modeling Problem 1. Furthermore, the focus on context in using inequalities for modeling non-linear problematic situations made it possible for a natural extension of problem solving activities so that such famous tools of mathematics as the arithmetic mean-geometric mean inequality and Bernoulli's inequality were introduced.

Finally, the notion of electronic spreadsheet being a milieu for teachers' training in the use of inequalities may include a possibility of incorporating numerical/graphical representations of analytical properties of functions formulated in terms of inequalities. This implies that while inequalities discussed in this paper are indeed powerful problem-solving tools, there are problems in mathematics that require the development of new methods applicable to the analysis of those processes for which inequality-oriented techniques become ineffective. The study of such methods is the focus of tertiary mathematics curricula and students' success at that level depends on their proficiency with pre-calculus concepts, inequalities included. Thus it is important that mathematics education courses provide future teachers with opportunities for developing a sound command of using inequalities and associated proof techniques in the context of secondary mathematics curricula.

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Appendix

The purpose of this section is to provide details of spreadsheet programming and, in doing so, to stress the computational aspects of inequalities discussed in this paper. To this end, the notation `(A1)=` will be used to present a formula defined in cell `A1`. It should be noted that syntactic versatility of spreadsheets enables one to construct both visually and computationally identical environments using syntactically different formulas. Therefore, in what follows, the focus is on the inequality-related aspects of programming rather than on a specific syntactic structure of the environments involved. In general, using inequalities as conditional arguments of the spreadsheet conditional function `IF` enables the tool to identify computationally efficient ranges of variables involved thus making more intelligent the trial-and-error approach to solving equations that model Problems 1 through 3 respectively.

1. Spreadsheet formulas for modeling Problem 1 (equation (1)).

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(a) Figures 1 and 2.
(N1)= =0; (M2)= =0; (O1)= =IF(N1=" "," ",IF(N1<INT(n/a),N1+1," "))
– replicated across row 1; (M3)= =IF(M2=" "," ",IF(M2<INT(n/b),M2+1,"
")) – replicated down column M; (N2)= =IF(OR(N$1=" ",$M2=" ")," ",
IF(n=a*N$1+b*$M2, n," ")) – replicated across rows and down columns.
Note that using the inequality M2<INT(n/b) instead of M2<n/b is, indeed,
a computational improvement. For example, the pair $(n, b) = (18, 4)$ yields
INT(18/4)=4 – the corresponding y-range generated by the spreadsheet is
$[0, 4]$; without rounding $18/4$ down to 4, the spreadsheet would display $[0, 5]$
as a y-range.

(b) Figures 3 and 4.
(N1)= =xmin; (M2)= =ymin; (O1)= =IF(N1=" "," ",
IF(N1<INT(n−ymin*b)/a,N1+1," ")) – replicated across row 1;
(M3)= =IF(M2=" "," ",IF(M2<INT((n−xmin*a)/b),M2+1," ")) – replicated
down column M; (N2)= =IF(OR(N$1=" ",$M2=" ")," ",
IF(n=a*N$1+b*$M2, n," ")) – replicated across all rows and down all columns
of the table.

2. Spreadsheet formulas for modeling Problem 2 (equation (2), Figure 6).
Due to inequalities (5), x- and y-ranges can be generated using the following
spreadsheet formulas: (C4)= =n+2; (D4)= =IF(C4<2*n,C4+1," ") - replicated
across row 4; (B5)= =2*n; (B6)= =IF(B5<n*(n+2)/2,B5+1," ") – replicated
down column B. Finally, (C5)= =IF(OR(C$4=" ",$B5=" ")," ",
IF(1/n=1/C$4+1/$B5,n," ")) – replicated across all rows and down all columns
of the table; this formula generates all solutions except one (known) solution dis-
played automatically at the top of the spreadsheet in Figure 6.

3. Spreadsheet formulas for modeling Problem 3 (equation (11), Figure 8).
Due to inequalities (13) and (14), x- and y-ranges in equation (11) can be generated
using the following spreadsheet formulas: (E1)= =n+1; (F1)= =IF(E1<3*n,E1+1,"
") – replicated across row 1; (D2)= =2*n+1; (D3)= =IF(E1<2*n*(n+1),E1+1,"
") – replicated down column D. Finally, (E2)=
=IF(OR(E$1=" ",$D2=" ")," ",IF(E$1*$D2=n*(E$1+$D2)," ",

IF(AND(E$1<=$D2,INT(n*E$1*$D2/(E$1*$D2−n*(E$1+$D2)))
=n*E$1*$D2/(E$1*$D2−n*(E$1+$D2)),
n*E$1*$D2/(E$1*$D2−n*(E$1+$D2))>=$D2),
n*E$1*$D2/(E$1*$D2−n*(E$1+$D2))," "))) – replicated across all rows and down
all columns of the table; this formula generates all solutions to Problem 3 for different
values of n through numerical modeling of the level lines of integer values defined by
formula (12).

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