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# A Simple Spreadsheet-Based Exposition of the Markowitz Critical Line Method for Portfolio Selection*

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**Abstract**

The critical line method for mean-variance portfolio selection, developed by Harry Markowitz over a half a century ago, is an important analytical tool for modern portfolio management. The method in its original form is a sophisticated algorithm for portfolio optimization under general linear constraints. Therefore, a challenge for instructors of investment courses is how to explain the method to business students who are unfamiliar with advanced mathematical and programming tools. This study illustrates pedagogically that, under practically relevant constraints including investment limits on individual securities and disallowance of short sales, the method can still be covered in investment courses where only general algebraic skills and statistical concepts are required. The use of electronic spreadsheets for portfolio construction not only significantly reduces the computational burden, but also makes the analytical materials involved less abstract for business students. This study, which provides spreadsheet-based illustrations of the required computations, is intended to make the Markowitz analysis more accessible to business students.

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## 1 Introduction

Mean-variance portfolio theory, which is a key component of the modern finance curriculum, originates from the pioneering work of Markowitz [8], a 1990 Nobel Laureate. We

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learn from that study the importance of covariances of returns between risky securities, such as stocks, in the determination of portfolio risk, as measured in variance or standard deviation of returns. For a given set of securities for portfolio investments, the lower the covariances, the better is the achievable risk-return trade-off. Efficient portfolio selection is about constrained optimization of such a trade-off.

The constraints for efficient portfolio selection in Markowitz [8] are confined to disallowance of short sales. As short sales are about selling borrowed securities to others, disallowance of short sales is the condition that the proportions of investment funds as allocated to individual securities, commonly called portfolio weights, be non-negative. For investment funds to be fully allocated among the securities considered, an implicit constraint is that portfolio weights sum to one. A major task, therefore, is to determine which of the securities considered are to have positive portfolio weights (i.e., to be selected) for each efficient portfolio. From a geometric illustration in Markowitz [8], we learn that each efficient portfolio can be captured by a point on some connected line segments, called critical lines, in a multi-dimensional space of portfolio weights.

In contrast, efficient portfolio construction in Markowitz [9,10,11] pertains to a general case where the constraints — including equality and inequality constraints — can be any linear combinations of portfolio weights. Examples include upper and lower investment limits on individual securities or some combinations of securities. The constraints being so general, the corresponding solution method, called the critical line method, is a sophisticated quadratic programming algorithm. It is worth noting that, although the method was developed more than half a century ago, it is still an important analytical tool for practical mean-variance portfolio selection today.[1] The recent extension of the method by Jacobs, Levy, and Markowitz [5] to select portfolios with realistic short sales further illustrates its versatility as an investment tool.

The work of Markowitz, which has had a profound impact on the finance profession, is highly regarded by finance academics and practitioners alike. As Rubinstein [14] observes, while commemorating the 50-year anniversary of modern portfolio theory, "Markowitz's approach is now commonplace among institutional portfolio managers who use it both to structure their portfolios and measure their performance... Indeed, the ideas in his 1952 paper have become so interwoven into financial economics that they can no longer be disentangled." (p.1044). However, Sharpe [15] — another 1990 Nobel Laureate — comments that the work of Markowitz is "often cited, less often read (at least completely)" (p.xiii) because of the formal mathematical materials involved. Not surprisingly, the analytical details of the critical line method for portfolio selection, though important in practice, are considered by most instructors of investment courses to be well beyond the standard finance curriculum.

Algebraic simplicity being the norm for pedagogic purposes in business education, efficient portfolio selection problems covered in investment courses often allow frictionless short sales of securities. Under the assumption of frictionless short sales, which

[1] Various software products based on the Markowitz analysis are currently available to investors. Efficient Solutions Inc. of Ridgefield, CT, Wagner Math Finance, Daniel H. Wagner Associates, of Malvern, PA, and Zephyr Associates Inc. of Zephyr Cove, NV are among the U.S. producers.

is unrealistic in practice, the short seller not only provides no collateral for any borrowed securities, but also has immediate access to the cash proceeds from the short-sale transactions. As there are no restrictions on any portfolio weights, including their signs, except that they must sum to one, efficient portfolios can easily be constructed. The corresponding portfolio weights can be determined directly from some algebraic formulas, such as those in Elton, Gruber, Brown, and Goetzmann [3, Chapter 6] and Merton [12].

Efficient portfolio selection without short sales, while practically relevant, is more complicated. To simplify the analytical task, one approach is to characterize the covariance structure of security returns with some specific models. They include, among many competing models, the single index model, where a linear relationship between the return of each security and the return of a market index is assumed, and the constant correlation model, where a constant correlation of returns between any two different securities is assumed. The corresponding analytical details and references to the literature can be found in Elton, Gruber, Brown, and Goetzmann [3, Chapter 9].

An alternative approach is numerical in nature. The availability of electronic spreadsheet tools, such as Microsoft Excel *Solver*®, allows many portfolio selection problems to be solved numerically, thus bypassing the analytical details that are potentially challenging to business students. Examples of the numerical approach can be found in Benninga [1, Chapter 11], Carter, Dare, and Elliott [2], Kwan [6], and Pace [13]. As no simplification is made to the original covariance structure of security returns, the numerical approach is able to produce the same numerical results as those solved analytically. From a pedagogic perspective, however, the reliance on numerical solutions alone inevitably leave unanswered questions as to how the critical line method actually works and how the mean-variance efficient frontier is related to the corresponding critical lines.

Although the analytical details of efficient portfolio selection (without simplifying the covariance structure) under a general set of linear constraints are well beyond the scope of the standard finance curriculum, to illustrate pedagogically the concept of critical lines in some settings is still possible. For example, Haugen [4, Chapter 5] provides a geometric exposition of critical lines for a three-security case, with and without frictionless short sales. Kwan and Yuan [7] consider, from a pedagogic perspective, efficient portfolio selection without short sales, by applying the critical line method to a revised analytical setting, where multivariate differential calculus and constrained optimization tools are required.

Also from a pedagogic perspective, the present study illustrates the critical line method (without altering the original analytical setting) by using algebraic and statistical tools that are generally familiar to business students. Like standard textbook coverage of mean-variance portfolio theory, the required statistical concepts here are confined to expected values, variances, and covariances of random variables and their linear combinations. The required algebraic skills include working with linear and quadratic expressions where there are symbols with subscripts, as well as simple matrix operations for solving linear equations. No prior knowledge of multivariate differential calculus and constrained optimization tools is necessary.

This study considers efficient portfolio selection disallowing short sales and then also having investment limits (implicitly, upper limits) on individual securities. Both are practically relevant constraints, especially for institutional investors. In each case, once the corresponding portfolio selection problem is formulated for a given set of input data, the use of electronic spreadsheets to perform the analysis involved allows the critical line method to be implemented without the usual computational distractions. The spreadsheet-based approach here is intended to make the Markowitz analysis more accessible to business students. From the classroom experience of the author of this study, who has been teaching the critical line method in undergraduate and M.B.A.-level investment courses for many years, the availability of spreadsheet tools for carrying out the algorithmic details, in addition to the required matrix operations, does make the method less abstract for students. Indeed, spreadsheet-based exercises allow students to improve significantly their understanding of the Markowitz analysis, its underlying concept, and its various implications.[2]

The rest of the paper is organized as follows: In order to explain the idea of critical lines, relate them to the efficient frontier, and facilitate the analysis that follows, section 2 considers a simple case where frictionless short sales are allowed. Section 3 performs portfolio analysis without short sales by using the critical line method. The method is illustrated with a spreadsheet example there. Section 4 extends the analysis in Section 3 by considering also investment limits on individual securities. The same spreadsheet example is extended as well. Section 5 provides some concluding remarks.

## 2 Portfolio selection with frictionless short sales

For a given set of $n$ risky securities, labeled as $i = 1, 2, \ldots, n$, let $R_i$, $\mu_i$, and $\sigma_i^2 = Var(R)$ be the random return, the expected return, and the variance of returns of security $i$, respectively. Let $\sigma_{ij} = Cov(R_i, R_j)$ be the covariance of returns between securities $i$ and $j$, with $\sigma_{ij} = \sigma_{ji}$ and $\sigma_{ii} = \sigma_i^2$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. For a portfolio $p$ based on the $n$ securities, let $x_i$ be the proportion of available investment funds as allocated to security $i$ satisfying the condition that[3]

$$x_1 + x_2 + \cdots + x_n = 1. \tag{1}$$

This condition ensures that the investment funds be fully allocated among the $n$ securities considered. The random and expected portfolio returns, being weighted averages of

[2] The use of spreadsheets and, in particular, Excel for illustrating the Markowitz analysis serves the pedagogic objectives of this study much better than does the use of a different computing environment, such as MATLAB® or GAUSS®. Given the popularity of Excel in the business world, business students tend to be already familiar with basic Excel operations by the time they enroll in investment courses. Their Excel training is acquired either formally from an academic course or informally from assignments and projects for various courses that require the use of spreadsheets.

[3] Illustrations with $n$ being some specific integers, such as 2, 3 or 4, may be necessary for students who are unfamiliar with algebraic expressions of series. Such illustrations will make the algebraic material below less abstract for them.

random returns and expected returns of individual securities, respectively, are

$$R_p = x_1 R_1 + x_2 R_2 + \cdots + x_n R_n \tag{2}$$

and

$$\mu_p = x_1 \mu_1 + x_2 \mu_2 + \cdots + x_n \mu_n. \tag{3}$$

The variance of portfolio returns, $\sigma_p^2 = Var(R_p)$, is the sum of $n^2$ terms, with each term being of the form $x_i x_j \sigma_{ij}$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.[4]

Efficient portfolio selection is about allocating investment funds to achieve the lowest risk for a specified expected return or to achieve the highest expected return for a specified risk, where risk is stated in terms of variance of returns. The specified expected return or risk depends on the investor's attitude toward risk. Letting $\lambda$ be a non-negative parameter that quantifies the investor's risk tolerance (or, equivalently, letting $1/\lambda$ capture the investor's risk aversion), the portfolio's certainty equivalent return is $\mu_p - \sigma_p^2/\lambda$. That is, the investor is indifferent between a risky outcome — as characterized by an expected return of $\mu_p$ and a variance of returns of $\sigma_p^2$ — and a certain return of $\mu_p - \sigma_p^2/\lambda$, which is less than $\mu_p$. For a highly risk tolerant investor, risk is not a concern; the investor seeks to maximize the portfolio's expected return. The less risk tolerant the investor, the lower is the portfolio's certainty equivalent return.

To construct an efficient portfolio for a given value of $\lambda$ is to find the set of portfolio weights $x_1, x_2, \ldots, x_n$ that maximizes $\mu_p - \sigma_p^2/\lambda$ or, equivalently, minimizes $\sigma_p^2 - \lambda\mu_p$. The efficient frontier is a collection of efficient portfolios for different values of $\lambda$, as plotted on the plane of expected return and standard deviation of returns. As the only constraint is the one that equation (1) provides, we can minimize instead the Lagrangian

$$L = \sigma_p^2 - \lambda\mu_p - \theta\left(x_1 + x_2 + \cdots + x_n - 1\right), \tag{4}$$

where $\theta$, often called a Lagrange multiplier, is an additional unknown variable.

To explain the Lagrangian approach intuitively to students who are unfamiliar with it, we can start with assessing the idea of finding the values of $x_1, x_2, \ldots, x_n$ that minimize

[4] For students who are familiar with summation signs, we can write each of the above equations more compactly, and express $\sigma_p^2$ as $\sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij}$. However, for students who are unfamiliar with series expressions, we may have to provide some illustrative examples. To find $\sigma_p^2$ where $n = 3$, for example, we first arrange all the covariances and portfolio weights in the following manner:

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $x_1$ | $\sigma_{11}$ | $\sigma_{12}$ | $\sigma_{13}$ |
| $x_2$ | $\sigma_{21}$ | $\sigma_{22}$ | $\sigma_{23}$ |
| $x_3$ | $\sigma_{31}$ | $\sigma_{32}$ | $\sigma_{33}$ |

Once we multiply each covariance term by the portfolio weights in the same row and in the same column, we have the following:

| | | |
|---|---|---|
| $x_1x_1\sigma_{11}$ | $x_1x_2\sigma_{12}$ | $x_1x_3\sigma_{13}$ |
| $x_2x_1\sigma_{21}$ | $x_2x_2\sigma_{22}$ | $x_2x_3\sigma_{23}$ |
| $x_3x_1\sigma_{31}$ | $x_3x_2\sigma_{32}$ | $x_3x_3\sigma_{33}$ |

The sum of these 9 terms is $\sigma_p^2 = \sum_{i=1}^3 \sum_{j=1}^3 x_i x_j \sigma_{ij}$.

$\sigma_p^2 - \lambda\mu_p$ without any attention to equation (1). For these $n$ unknown variables to be solved, we require $n$ equations to be deduced from an explicit expression of $\sigma_p^2 - \lambda\mu_p$ in terms of the same variables. Even if the $n$ equations are known, there is no guarantee that the solved values of $x_1, x_2, \ldots, x_n$ also satisfy equation (1). A good example is the case of $\lambda = 0$, where the set of $x_1, x_2, \ldots, x_n$ that minimizes $\sigma_p^2 - \lambda\mu_p$ $(= \sigma_p^2)$ is a set of zeros. Obviously, equation (1) is violated in this case. Adding equation (1) directly to the same set of $n$ equations is not a valid remedy because the addition will result in a total of $n+1$ equations for the $n$ unknown variables $x_1, x_2, \ldots, x_n$. A viable remedy, therefore, requires the presence of an extra unknown variable, such as the $\theta$ in equation (4).

To see whether a minimized $L$ corresponds to a minimized $\sigma_p^2 - \lambda\mu_p$ satisfying equation (1), let us consider a set of values of $x_1, x_2, \ldots, x_n$, and $\theta$, labeled as $\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n$, and $\widehat{\theta}$, respectively, that can potentially provide the lowest $L$. Intuitively, this set of values cannot be the best choice if a change to any of them leads to a lower $L$. Suppose for now that

$$\widehat{x}_1 + \widehat{x}_2 + \cdots + \widehat{x}_n - 1 \neq 0. \tag{5}$$

If an increase (decrease) in $\widehat{\theta}$ causes $L$ to increase, then a decrease (increase) in $\widehat{\theta}$ will cause $L$ to decrease instead. This decrease in $L$ inevitably violates the idea that the set of $\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n$, and $\widehat{\theta}$ gives us the lowest $L$. Thus, to ensure the attainment of the lowest $L$, we must have

$$\widehat{x}_1 + \widehat{x}_2 + \cdots + \widehat{x}_n - 1 = 0. \tag{6}$$

That is, equation (1) must be satisfied. Accordingly, the set of portfolio weights that minimizes $L$ must also minimize $\sigma_p^2 - \lambda\mu_p$.

## 2.1 An algebraic solution

Students who are familiar with multivariate differential calculus can recognize that, with $x_1, x_2, \ldots, x_n$, and $\theta$ being the decision variables, the portfolio solution corresponding to each given value of $\lambda$ is from the set of $n+1$ equations based on $\partial L/\partial x_i = 0$, for $i = 1, 2, \ldots, n$, and $\partial L/\partial \theta = 0$. The following is an algebraic approach to reach the same $n+1$ equations:

To find the set of portfolio weights that minimizes $L$, let us consider an arbitrary security $i$ among the $n$ securities considered. Specifically, let us write the random return of portfolio $p$ in equation (2) equivalently as

$$R_p = x_i R_i + R_i^*, \tag{7}$$

where $R_i^*$ is $x_1R_1 + x_2R_2 + \cdots + x_nR_n$ with the term $x_iR_i$ removed. The expected return of portfolio $p$ in equation (3) is then

$$\mu_p = x_i\mu_i + \mu_i^*, \tag{8}$$

where $\mu_i^*$ is the expected value of $R_i^*$. For example, in a three-security case, $R_2^*$ is $x_1R_1 + x_3R_3$ and $\mu_2^*$ is $x_1\mu_1 + x_3\mu_3$.

Given equations (7) and (8), the portfolio's variance of returns can be written as

$$\sigma_p^2 = x_i^2 Var(R_i) + 2x_i Cov(R_i, R_i^*) + Var(R_i^*). \tag{9}$$

Thus, equation (4) becomes

$$L = x_i^2 Var(R_i) + x_i \left[2Cov(R_i, R_i^*) - \lambda\mu_i - \theta\right] + \text{ terms not containing } x_i, \tag{10}$$

which is a quadratic expression of $x_i$. By completing the square for all terms that contain $x_i^2$ and $x_i$, we have

$$L = \frac{1}{Var(R_i)}\left[x_i Var(R_i) + Cov(R_i, R_i^*) - \frac{1}{2}(\lambda\mu_i + \theta)\right]^2 + \text{ terms not containing } x_i. \tag{11}$$

Suppose that, for a given $\lambda$, we have, as a tentative solution to this portfolio selection problem, a set of values of $x_1, x_2, \ldots, x_n$, and $\theta$. If we change the value of $x_i$ alone in an attempt to improve the solution, the best possible improvement requires an $x_i$ that makes the first term on the right hand side of equation (11) vanish. This requires

$$x_i Var(R_i) + Cov(R_i, R_i^*) - \frac{1}{2}(\lambda\mu_i + \theta) = 0. \tag{12}$$

Given that

$$x_i Var(R_i) = Cov(R_i, x_i R_i) \tag{13}$$

and

$$Cov(R_i, x_i R_i) + Cov(R_i, R_i^*) = Cov(R_i, x_1 R_1 + x_2 R_2 + \cdots + x_n R_n), \tag{14}$$

equation (12) can be written equivalently as

$$2\sigma_{i1} x_1 + 2\sigma_{i2} x_2 + \cdots + 2\sigma_{in} x_n - \theta = \lambda\mu_i. \tag{15}$$

Here, $\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{in}$ are $Cov(R_i, R_1), Cov(R_i, R_2), \ldots, Cov(R_i, R_n)$, respectively.

Notice that the security $i$ considered above can be any of the $n$ securities. Thus, by letting $i = 1, 2, \ldots, n$, equation (15) actually represents a set of $n$ linear equations with the unknown variables being $x_1, x_2, \ldots, x_n$, and $\theta$. As indicated earlier, the values of $x_1, x_2, \ldots, x_n$, and $\theta$ that minimize $L$ require equation (1) to hold. Including also equation (1), we now have a total of $n + 1$ linear equations, as required to determine the values of these $n + 1$ variables in terms of $\lambda$. Notice also that equation (15) covers the results of $\partial L/\partial x_i = 0$, for $i = 1, 2, \ldots, n$, and that equation (1) follows directly from $\partial L/\partial\theta = 0$.

### 2.2 The mean-variance efficient frontier and the corresponding critical line in a multi-dimensional space of portfolio weights

For computational convenience, the $n + 1$ linear equations that equations (1) and (15) represent can be written compactly in matrix notation as[5]

$$\underline{W}\,\underline{Z} = \underline{H} + \lambda\underline{K}, \tag{16}$$

[5] In order to help students with no prior knowledge of matrix algebra understand the material here, it may be necessary for us to illustrate first how a small set of linear equations can be written in a



 

where

$$\underline{W} = \begin{bmatrix} 2\sigma_{11} & 2\sigma_{12} & \cdots & 2\sigma_{1n} & -1 \\ 2\sigma_{21} & 2\sigma_{22} & \cdots & 2\sigma_{2n} & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2\sigma_{n1} & 2\sigma_{n2} & \cdots & 2\sigma_{nn} & -1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}, \tag{17}$$

$$\underline{Z} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \theta \end{bmatrix}, \tag{18}$$

$$\underline{H} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{19}$$

and

$$\underline{K} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \\ 0 \end{bmatrix}. \tag{20}$$

For a given value of $\lambda$, the corresponding efficient portfolio weights can be obtained from the matrix equation

$$\underline{Z} = \underline{W}^{-1}\underline{H} + \lambda \underline{W}^{-1}\underline{K}, \tag{21}$$

where $\underline{W}^{-1}$ is the inverse of $\underline{W}$. Given the input data for portfolio analysis, as $\underline{W}$, $\underline{H}$, and $\underline{K}$ are known, to find $\underline{Z}$ from equation (21) on computers using available spreadsheet functions, such as MINVERSE and MMULT in Microsoft Excel (for matrix inversion and multiplication, respectively), is straightforward.

Letting $a_i$ and $b_i$ be the $i$-th elements of the $(n+1)$-element column vectors that $\underline{W}^{-1}\underline{H}$ and $\underline{W}^{-1}\underline{K}$ represent, respectively, we have from equation (21)

$$x_i = a_i + b_i\lambda, \text{ for } i = 1, 2, \ldots, n, \tag{22}$$

and

$$\theta = a_{n+1} + b_{n+1}\lambda. \tag{23}$$

matrix form. Doing so will allow these students to establish the equivalence of the solutions with and without using matrix algebra. The use of Excel functions for the illustration, such as MINVERSE and MMULT (for matrix inverse and multiplication, respectively), will make the corresponding matrix operations more intuitive to these students.

With $a_1, a_2, \ldots, a_{n+1}$ and $b_1, b_2, \ldots, b_{n+1}$ determined, each specific value of the parameter $\lambda$ corresponds to a set of efficient portfolio weights $x_1, x_2, \ldots, x_n$ and an associated value of the Lagrange multiplier $\theta$.[6]

Equation (22) is the parametric form of a line, called the critical line, in a multi-dimensional space of portfolio weights, with $\lambda$ being the parameter. With portfolio weights always summed to one, each efficient portfolio based on $n$ risky securities can be captured as a point on a line in an $(n-1)$-dimensional space of portfolio weights. For example, in a three-security case, the corresponding critical line can be plotted on the plane of any two of the three portfolio weights. As the efficient frontier (on the plane of expected return and standard deviation of returns) for a given set of risky securities is a collection of efficient portfolios for investors with different levels of risk tolerance, the critical line that equation (22) represents is a collection of the corresponding efficient portfolio weights for these investors. Each efficient portfolio corresponds to a point on the critical line.

## 3 Portfolio selection without short sales

If short sales are disallowed, any negative value of $x_i$, for $i = 1, 2, \ldots, n$, is unacceptable. Students who are familiar with constrained optimization can recognize that solving a portfolio selection problem with $x_1, x_2, \ldots, x_n$, and $\theta$ being the decision variables involves the Kuhn-Tucker conditions. Formally, these conditions include the first-order conditions (in a set of $n+1$ equations) and the complementarity conditions to establish, for each $x_i$, when to use a corresponding slack variable to avoid an unacceptable outcome. The following is an algebraic approach to derive these conditions:

To avoid a negative $x_i$ for all values of $\lambda \geq 0$ in an $n$-security case, let us return to the algebraic expression of $L$ in equation (11). Intuitively, the lower the value of the first term (a non-negative term) on the right hand side of equation (11), the lower is the corresponding value of $L$. If a negative value of $x_i$ is required to make this term vanish, the value of $Cov(R_i, R_i^*) - \frac{1}{2}(\lambda\mu_i + \theta)$ must be positive. If so, we simply let $x_i = 0$ because a positive value of $x_i$ inevitably worsens the portfolio solution. With $x_i = 0$, we have

$$x_i Var(R_i) + Cov(R_i, R_i^*) - \frac{1}{2}(\lambda\mu_i + \theta) > 0. \tag{24}$$

Before a portfolio selection problem is solved, however, it is unknown what value of $x_i$ is required to make the first term on the right hand side of equation (11) vanish. If

[6] The same algebraic results can also be reached by applying Cramer's rule to the set of $n+1$ equations that equation (16) represents. Each of $a_1, a_2, \ldots, a_{n+1}$ and $b_1, b_2, \ldots, b_{n+1}$ is simply a ratio of two determinants. The denominator in each ratio is the determinant of $\underline{W}$, a square matrix consisting of all the coefficients of the $n+1$ unknown variables. In the case of $a_i$, for $i = 1, 2, \ldots, n+1$, the numerator is the determinant of a revised matrix, which is $\underline{W}$ with its $i$-th column substituted by the column vector $\underline{H}$. In the case of $b_i$, for $i = 1, 2, \ldots, n+1$, the numerator is the determinant of another matrix, which is $\underline{W}$ with its $i$-th column substituted by the column vector $\underline{K}$ instead. To use Cramer's rule to reach equations (22) and (23) on computers, we can use an available spreadsheet function, such as MDETERM in Microsoft Excel, to find the determinants involved.

$x_i$ turns out to be positive, equation (12) is still valid. To accommodate the potential outcome of a negative $x_i$, which must be replaced by a zero, let us combine equation (12) and inequality (24) into

$$x_i Var\ (R_i) + Cov(R_i, R_i^*) - \frac{1}{2}(\lambda\mu_i + \theta + \delta_i) = 0, \tag{25}$$

where $\delta_i$ is a non-negative slack variable with the following properties: If $x_i$ is positive, $\delta_i$ is zero and equation (25) reduces to equation (12). If $x_i$ is zero instead, $\delta_i$ is positive and equation (25) is equivalent to inequality (24). Strictly speaking, if $x_i$ is zero, $\delta_i$ can still be zero, but this requires $Cov(R_i, R_i^*) - \frac{1}{2}(\lambda\mu_i + \theta)$ to be zero as well.

Just like the frictionless short sale case, which provides equation (15), we can write equation (25) equivalently as

$$2\sigma_{i1}x_1 + 2\sigma_{i2}x_2 + \cdots + 2\sigma_{in}x_n - \theta - \delta_i = \lambda\mu_i. \tag{26}$$

Since this equation holds for $i = 1, 2, \ldots, n$, we have, including equation (1), a set of $n + 1$ linear equations for the $2n + 1$ unknown variables $x_1, x_2, \ldots, x_n$, $\delta_1, \delta_2, \ldots, \delta_n$, and $\theta$ (all to be solved in terms of $\lambda$). Students who are familiar with constrained optimization can recognize that these $n+1$ linear equations are the first-order conditions for minimizing $\sigma_p^2 - \lambda\mu_p$, subject to the conditions that portfolio weights sum to one and no negative portfolio weights be allowed. They can also recognize the complementarity conditions that one of $x_i$ and $\delta_i$ be zero (often stated as $x_i \geq 0$, $\delta_i \geq 0$, and $x_i\delta_i = 0$, for $i = 1, 2, \ldots, n$) in this constrained optimization problem. For a given value of $\lambda$, if we know which of the $n$ securities are to be selected to the corresponding efficient portfolio, then we still have an exact number of equations to solve the unknown variables because, between each pair of variables $x_i$ and $\delta_i$, one of them must be zero. The challenge, however, is to identify the selected securities for each non-negative value of $\lambda$.

### 3.1 A simple description of the critical line method for portfolio selection

For ease of exposition, let us consider a security to be *in* (*out*) if the security is included in (excluded from) an efficient portfolio without short sales. For each security $i$ that is *out*, as $x_i = 0$, the unknown variable pertaining to the security is $\delta_i$. For example, if it is known that, for a given value of $\lambda$ in a three-security case, securities 1 and 3 are *in* and security 2 is *out*, as $\delta_1 = x_2 = \delta_3 = 0$, the three equations that equation (26) represents, along with equation (1), provide the following set of four linear equations, from which the remaining unknown variables ($x_1, \delta_2, x_3$, and $\theta$) can be solved:

$$\begin{aligned} 2\sigma_{11}x_1 + 2\sigma_{13}x_3 - \theta &= \lambda\mu_1, \\ 2\sigma_{21}x_1 - \delta_2 + 2\sigma_{23}x_3 - \theta &= \lambda\mu_2, \\ 2\sigma_{31}x_1 + 2\sigma_{33}x_3 - \theta &= \lambda\mu_3, \\ x_1 + x_3 &= 1. \end{aligned} \tag{27}$$

Equivalently, in matrix notation, we have

$$\begin{bmatrix} 2\sigma_{11} & 0 & 2\sigma_{13} & -1 \\ 2\sigma_{21} & -1 & 2\sigma_{23} & -1 \\ 2\sigma_{31} & 0 & 2\sigma_{33} & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \delta_2 \\ x_3 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ 0 \end{bmatrix}. \tag{28}$$

This is a three-security case of matrix equation (16), where the second element of $\underline{Z}$ is substituted by $\delta_2$ and the second column of $\underline{W}$ is substituted by the corresponding column of the negative of a $4 \times 4$ identity matrix.

As this example illustrates, once the *in*/*out* status of each of the $n$ securities considered is known, we can use equation (21) in its revised form to determine the efficient portfolio weights. Specifically, we substitute, for each security $i$ that is *out*, the $i$-th element of $\underline{Z}$ with a $\delta_i$ and column $i$ of $\underline{W}$ with the corresponding column of the negative of an $(n+1) \times (n+1)$ identity matrix. Denoting the $i$-th element of $\underline{Z}$ as $z_i$, we have $z_i = x_i$ if security $i$ is *in* and $z_i = \delta_i$ if security $i$ is *out*, for $i = 1, 2, \ldots, n$, as well as $z_{n+1} = \theta$. Analogous to equations (22) and (23) for the frictionless short-sale case, we have the linear relationships

$$z_i = a_i + b_i\lambda, \text{ for } i = 1, 2, \ldots, n+1, \tag{29}$$

where $a_i$ and $b_i$ are, respectively, the $i$-th elements of the $(n+1)$-element column vectors that $\underline{W}^{-1}\underline{H}$ and $\underline{W}^{-1}\underline{K}$ represent. For a set of portfolio weights corresponding to a given value of $\lambda$ to be acceptable, we must have $z_i \geq 0$, for $i = 1, 2, \ldots, n$. The Markowitz critical line method, as described in the following, allows us to identify the *in*/*out* status of securities for different values of $\lambda$.

Intuitively, if $\lambda$ is infinitely high, risk is not a concern and thus the corresponding efficient portfolio consists only of the security with the highest expected return among the $n$ securities considered. In case of a tie, we can break it by arbitrarily adding an infinitesimal value to one of the expected returns in question so that any distortions to the portfolio results are inconsequential. Given the initial *in*/*out* status of each security, we revise $\underline{W}$ and $\underline{Z}$ in equations (17) and (18) accordingly. Equation (21) allows us to write $x_i$ for each security $i$ that is *in*, as well as $\delta_i$ for each security $i$ that is *out*, in the form of $a_i + b_i\lambda$. If we decrease $\lambda$ from this initial value, a critical value of $\lambda$ will eventually be encountered. This is the value of $\lambda$ below which the conditions of $x_i \geq 0$ and $\delta_i \geq 0$, for $i = 1, 2, \ldots, n$, do not hold for all $n$ securities. All it takes is a violation by one security, which is the security where $-a_i/b_i$ is the highest among all cases of $b_i > 0$. The reason is that, if $b_i > 0$, $a_i + b_i\lambda \geq 0$ implies $\lambda \geq -a_i/b_i$ and thus, as $\lambda$ decreases from its initial value, a violation occurs as soon as $\lambda$ is marginally below the highest value of $-a_i/b_i$ among such cases.

To allow $\lambda$ to decrease below this critical value, we must revise the *in*/*out* status of the security that makes the portfolio solution unacceptable. The status change leads to corresponding revisions in $\underline{W}$ and $\underline{Z}$ as well. With a new set of $a_i$ and $b_i$, for $i = 1, 2, \ldots, n$, from equation (21), we can determine the next critical value of $\lambda$, which is also the highest $-a_i/b_i$ among all cases of $b_i > 0$. Upon identifying the security that makes the portfolio solution unacceptable and changing its *in*/*out* status, the same procedure to revise $\underline{Z}$ and $\underline{W}$ continues. The iterative procedure, which generates a series of critical values of $\lambda$, stops when no more positive critical values can be found.

From a pedagogic perspective, the way equation (16) is revised above (to allow the unknown variables among $x_1, x_2, \ldots, x_n$, $\delta_1, \delta_2, \ldots, \delta_n$, and $\theta$ to be solved in terms of $\lambda$) is an improvement over the Markowitz [10] version. In Markowitz [10], the slack variable $\delta_i$ for any security $i$ that is *out* does not appear in equation (16). Instead, while the $i$-th element of $\underline{Z}$ remains as $x_i$, the $i$-th row and the $i$-th column of $\underline{W}$ are substituted by a "unit cross," whose elements are all zeros, except for the $(i, i)$-element, which is a one. With the $i$-th element of $\underline{K}$ substituted by a zero, equation (16) ensures that $x_i = 0$. Equation (16) without the slack variables still allows the unknown portfolio weights and $\theta$ to be solved in terms of $\lambda$. However, the search for the next critical value of $\lambda$ in each iterative step requires a comparison of the various values of $\lambda$ obtained from the intersections of the current critical line and all potential critical lines. In contrast, our approach, which determines the next critical value of $\lambda$ by simply finding the highest $-a_i/b_i$ where $b_i > 0$ among $i = 1, 2, \ldots, n$, is more direct.

The efficient portfolio corresponding to each critical value of $\lambda$ is a corner portfolio. The *in*/*out* status of each security in all efficient portfolios between two adjacent corner portfolios remains the same. Analogous to the frictionless short-sale case, as portfolio weights sum to one, each efficient portfolio without short sales based on $n$ securities can be captured by a point on the corresponding critical line in an $(n-1)$-dimensional space of portfolio weights. Further, a movement on the efficient frontier between two corner portfolios, on the plane of expected return and standard deviation of returns, can be captured by a movement on the corresponding critical line. As a corner portfolio is where two critical lines meet, a movement on the efficient frontier, from $\lambda = \infty$ to $\lambda = 0$, corresponds to a movement on the connected critical lines, from one end to the other end of these line segments.

## 3.2 A spreadsheet-based illustration

To illustrate the critical line method using Excel, let us consider a three-security case with following input data: $\mu_1 = 0.05$, $\mu_2 = 0.08$, $\mu_3 = 0.12$, $\sigma_1 = 0.02$, $\sigma_2 = 0.05$, $\sigma_3 = 0.10$, $\sigma_{12} = \sigma_{21} = 0.0004$, $\sigma_{13} = \sigma_{31} = 0.0002$, and $\sigma_{23} = \sigma_{32} = 0.0010$. Given equations (1) and (26), the four linear equations as required to construct the efficient frontier without short sales are as follows:

$$\begin{aligned}
(0.0008x_1 - \delta_1) + 0.0008x_2 + 0.0004x_3 - \theta &= 0.05\lambda \\
0.0008x_1 + (0.0050x_2 - \delta_2) + 0.0020x_3 - \theta &= 0.08\lambda \\
0.0004x_1 + 0.0020x_2 + (0.0200x_3 - \delta_3) - \theta &= 0.12\lambda \\
x_1 + x_2 + x_3 &= 1
\end{aligned} \tag{30}$$

Figures 1 and 2 show an Excel worksheet for the search of corner portfolios without short sales based on these input data. Only four steps are required here. The initial portfolio in Step 1 consists only of security 3, the security with the highest expected return among the three securities considered. As both security 1 and 2 are *out*, we revise the $4 \times 4$ matrix $\underline{W}$ in B9:E12 — a three-security case of equation (17) — so that its first two columns are now the corresponding columns of the negative of a $4 \times 4$ identity matrix in B14:E17. The resulting $\underline{W}$ and $\underline{W}^{-1}$ are displayed in B23:E26 and B29:E32, respectively. The latter is produced by using the worksheet function MINVERSE (for matrix inversion). The computed results of $\underline{W}^{-1}\underline{H}$ and $\underline{W}^{-1}\underline{K}$, labeled as A (containing $a_1, a_2, a_3$, and $a_4$) and B (containing $b_1, b_2, b_3$, and $b_4$), are displayed in F29:F32 and G29:G32, respectively. These computations require the worksheet function MMULT (for matrix multiplication). The three cells in H29:H31, labeled as Ratio, shows $-a_i/b_i$ if $b_i > 0$, for $i = 1, 2$, and 3. The highest of these values, 0.45, labeled as Lambda, is displayed in H35. This is a critical value of $\lambda$, implying that, for $\lambda$ marginally less than 0.45, security 2 ought to be selected for the portfolio as well.

The corner portfolio results corresponding to $\lambda = 0.45$, including the three portfolio weights, the expected portfolio return, and the standard deviation of portfolio returns, are displayed in B35:F35. The required formulas for the various computations to complete Step 1 are displayed in rows 38 to 52 of the worksheet. Notice that, although the results of Step 1 in B35:F35 for a portfolio consisting of a single security are so obvious that the corresponding computations seem unnecessary, the formulas used are intended to be replicated for each of the subsequent steps. Given the results in Step 1, the simplest way to perform Step 2 is to make a single change to C21 (from *out* to *in*) in a separate worksheet, which is an exact copy of the first 35 rows of the original worksheet. Likewise, Step 3 (Step 4) requires a status change of the security identified in Step 2 (Step 3) to a separate worksheet, which is an exact copy of the first 35 rows of the worksheet for Step 2 (Step 3). In order to reduce the number of pages for Figures 1 and 2, however, we put the four steps on the same worksheet instead. The required changes to various fixed-cell locations for Steps 2-4, which can be deduced from the corresponding formulas in Step 1, are not noted in Figure 1.

Notice also that, instead of performing multi-step calculations leading to the expected portfolio return ($\mu_p$) and the standard deviation of portfolio returns ($\sigma_p$) for each corner portfolio ($p$), we use direct formulas involving matrix multiplications and, in the case of $\sigma_p$, also the worksheet function SQRT (for square root). Of course, we can still find $\mu_p$ by computing the three individual products $x_i \mu_i$, for $i = 1, 2$, and 3, and then summing the results with simple Excel operations. Likewise, in the case of $\sigma_p$, we can start with finding the nine individual products $x_i x_j \sigma_{ij}$, for $i = 1, 2$, and 3 and $j = 1, 2$, and 3, and summing the results to reach $\sigma_p^2$ afterwards. However, by writing $\mu_p$ as the product of a row vector of expected returns ($\mu_1, \mu_2$, and $\mu_3$) and a column vector of portfolio weights ($x_1, x_2$, and $x_3$), as shown in E35, we are able to reach $\mu_p$ directly. In the case of $\sigma_p$, the computational simplicity that matrix operations provide is even more obvious. By writing the sum of the nine individual cases of $x_i x_j \sigma_{ij}$, for $i = 1, 2$, and 3 and $j = 1, 2$, and 3, as the product of a 3-element row vector of portfolio weights, a $3 \times 3$ covariance matrix, and a 3-element column vector of the same portfolio weights and then taking the square root of the sum, as is done in F35, we are able to reach $\sigma_p$ directly as well.

In Step 2, where only security 1 is *out*, the only column in $\underline{W}$ requiring substitution



 

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | Sec.1 | Sec.2 | Sec.3 | | | | | | |
| 2 | | | | | | | | | | |
| 3 | Exp.Ret. | 0.05 | 0.08 | 0.12 | | | | | | |
| 4 | | | | | | | | | | |
| 5 | Cov.Mat. | 0.0004 | 0.0004 | 0.0002 | | | | | | |
| 6 | | 0.0004 | 0.0025 | 0.001 | | | | | | |
| 7 | | 0.0002 | 0.001 | 0.01 | | | | | | |
| 8 | | | | | | H | K | | | |
| 9 | W | 0.0008 | 0.0008 | 0.0004 | -1 | 0 | 0.05 | | Sec.1 | |
| 10 | | 0.0008 | 0.005 | 0.002 | -1 | 0 | 0.08 | | Sec.2 | |
| 11 | | 0.0004 | 0.002 | 0.02 | -1 | 0 | 0.12 | | Sec.3 | |
| 12 | | 1 | 1 | 1 | 0 | 1 | 0 | | | |
| 13 | | | | | | | | | | |
| 14 | Neg.Iden. | -1 | 0 | 0 | 0 | | | | | |
| 15 | | 0 | -1 | 0 | 0 | | | | | |
| 16 | | 0 | 0 | -1 | 0 | | | | | |
| 17 | | 0 | 0 | 0 | -1 | | | | | |
| 18 | | | | | | | | | | |
| 19 | | | | | | | | | | |
| 20 | **Step 1** | Sec.1 | Sec.2 | Sec.3 | | | | | | |
| 21 | | out | out | in | | | | | | |
| 22 | | | | | | | | | | |
| 23 | W | -1 | 0 | 0.0004 | -1 | | Sec.1 | out | | |
| 24 | | 0 | -1 | 0.002 | -1 | | Sec.2 | out | | |
| 25 | | 0 | 0 | 0.02 | -1 | | Sec.3 | in | | |
| 26 | | 0 | 0 | 1 | 0 | | | | | |
| 27 | | | | | | | | | Revised | Corn.Port. |
| 28 | | | | | | A | B | Ratio | Status | X |
| 29 | Inv. of W | -1 | 0 | 1 | -0.0196 | -0.0196 | 0.07 | 0.28 | | 0 |
| 30 | | 0 | -1 | 1 | -0.018 | -0.018 | 0.04 | 0.45 | in | 0 |
| 31 | | 0 | 0 | 0 | 1 | 1 | 0 | | | 1 |
| 32 | | 0 | 0 | -1 | 0.02 | 0.02 | -0.12 | | | |
| 33 | | | | | | | | | | |
| 34 | | X1 | X2 | X3 | Exp.Ret. | St.Dev. | | Lambda | | |
| 35 | Corn.Port. | 0 | 0 | 1 | 0.12 | 0.1 | | 0.45 | | |
| 36 | | | | | | | | | | |
| 37 | Formulas: | | | | | | | | | |
| 38 | 1. W (in B9:E12): twice the covariance matrix in B5:D7 augmented by extra row and column as shown. | | | | | | | | | |
| 39 | 2. First 3 elements of K (in G9:G12): {=TRANSPOSE(B3:D3)}. | | | | | | | | | |
| 40 | 3. W (in B23:E26): copy and paste B23=IF(B$21="in",B9,B14) to B23:D26, and E23=E9 to E23:E26. | | | | | | | | | |
| 41 | 4. Security status (in G23:H25): {=TRANSPOSE(B20:D21)}. | | | | | | | | | |
| 42 | 5. Inverse of W (in B29:E32): {=MINVERSE(B23:E26)}. | | | | | | | | | |
| 43 | 6. A (in F29:F32): {=MMULT(B29:E32,F$9:F$12)}. | | | | | | | | | |
| 44 | 7. B (in G29:G32): {=MMULT(B29:E32,G$9:G$12)}. | | | | | | | | | |
| 45 | 8. Ratio (in H29:H31): copy and paste H29=IF(G29>0,-F29/G29,"") to H29:H31. | | | | | | | | | |
| 46 | 9. Lambda (in H35): =MAX(0,H29:H31). | | | | | | | | | |
| 47 | 10. Revised status (in I29:I31): copy and paste I29=IF(AND(H$35>0,H29=H$35),IF(H23="in","out","in"),"") | | | | | | | | | |
| 48 | to I29:I31. | | | | | | | | | |
| 49 | 11. X (in J29:J31): copy and paste J29=IF(H23="in",F29+G29*H$35,0) to J29:J31. | | | | | | | | | |
| 50 | 12. Corner portfolio weights X1, X2, and X3 (in B35:D35): {=TRANSPOSE(J29:J31)}. | | | | | | | | | |
| 51 | 13. Expected return of corner portfolio (in E35): =MMULT(B$3:D$3,J29:J31). | | | | | | | | | |
| 52 | 14. Standard deviation of corner portfolio (in F35): =SQRT(MMULT(B35:D35,MMULT(B$5:D$7,J29:J31))). | | | | | | | | | |

Figure 1: Excel example (no short sales).

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 54 | **Step 2** | Sec.1 | Sec.2 | Sec.3 | | | | | | |
| 55 | | out | in | in | | | | | | |
| 56 | | | | | | | | | | |
| 57 | W | -1 | 0.0008 | 0.0004 | -1 | | Sec.1 | out | | |
| 58 | | 0 | 0.005 | 0.002 | -1 | | Sec.2 | in | | |
| 59 | | 0 | 0.002 | 0.02 | -1 | | Sec.3 | in | | |
| 60 | | 0 | 1 | 1 | 0 | | | | | |
| 61 | | | | | | | | | Revised | Corn.Port. |
| 62 | | | | | | A | B | Ratio | Status | X |
| 63 | Inv. of W | -1 | 0.87619 | 0.12381 | -0.00383 | -0.00383 | 0.034952 | 0.109537 | in | 0 |
| 64 | | 0 | 47.61905 | -47.619 | 0.857143 | 0.857143 | -1.90476 | | | 0.648501 |
| 65 | | 0 | -47.619 | 47.61905 | 0.142857 | 0.142857 | 1.904762 | -0.075 | | 0.351499 |
| 66 | | 0 | -0.85714 | -0.14286 | 0.004571 | 0.004571 | -0.08571 | | | |
| 67 | | | | | | | | | | |
| 68 | | X1 | X2 | X3 | Exp.Ret. | St.Dev. | | Lambda | | |
| 69 | Corn.Port. | 0 | 0.648501 | 0.351499 | 0.09406 | 0.052372 | | 0.109537 | | |
| 70 | | | | | | | | | | |
| 71 | **Step 3** | Sec.1 | Sec.2 | Sec.3 | | | | | | |
| 72 | | in | in | in | | | | | | |
| 73 | | | | | | | | | | |
| 74 | W | 0.0008 | 0.0008 | 0.0004 | -1 | | Sec.1 | in | | |
| 75 | | 0.0008 | 0.005 | 0.002 | -1 | | Sec.2 | in | | |
| 76 | | 0.0004 | 0.002 | 0.02 | -1 | | Sec.3 | in | | |
| 77 | | 1 | 1 | 1 | 0 | | | | | |
| 78 | | | | | | | | | Revised | Corn.Port. |
| 79 | | | | | | A | B | Ratio | Status | X |
| 80 | Inv. of W | 257.8585 | -225.933 | -31.9253 | 0.98723 | 0.98723 | -9.01277 | | | 0.97541 |
| 81 | | -225.933 | 245.5796 | -19.6464 | -0.00786 | -0.00786 | 5.992141 | 0.001311 | out | 0 |
| 82 | | -31.9253 | -19.6464 | 51.57171 | 0.020629 | 0.020629 | 3.020629 | -0.00683 | | 0.02459 |
| 83 | | -0.98723 | 0.007859 | -0.02063 | 0.000792 | 0.000792 | -0.05121 | | | |
| 84 | | | | | | | | | | |
| 85 | | X1 | X2 | X3 | Exp.Ret. | St.Dev. | | Lambda | | |
| 86 | Corn.Port. | 0.97541 | 0 | 0.02459 | 0.051721 | 0.019905 | | 0.001311 | | |
| 87 | | | | | | | | | | |
| 88 | **Step 4** | Sec.1 | Sec.2 | Sec.3 | | | | | | |
| 89 | | in | out | in | | | | | | |
| 90 | | | | | | | | | | |
| 91 | W | 0.0008 | 0 | 0.0004 | -1 | | Sec.1 | in | | |
| 92 | | 0.0008 | -1 | 0.002 | -1 | | Sec.2 | out | | |
| 93 | | 0.0004 | 0 | 0.02 | -1 | | Sec.3 | in | | |
| 94 | | 1 | 0 | 1 | 0 | | | | | |
| 95 | | | | | | | | | Revised | Corn.Port. |
| 96 | | | | | | A | B | Ratio | Status | X |
| 97 | Inv. of W | 50 | 0 | -50 | 0.98 | 0.98 | -3.5 | | | 0.98 |
| 98 | | 0.92 | -1 | 0.08 | 0.000032 | 0.000032 | -0.0244 | | | 0 |
| 99 | | -50 | 0 | 50 | 0.02 | 0.02 | 3.5 | -0.00571 | | 0.02 |
| 100 | | -0.98 | 0 | -0.02 | 0.000792 | 0.000792 | -0.0514 | | | |
| 101 | | | | | | | | | | |
| 102 | | X1 | X2 | X3 | Exp.Ret. | St.Dev. | | Lambda | | |
| 103 | Corn.Port. | 0.98 | 0 | 0.02 | 0.0514 | 0.0199 | | 0 | | |
| 104 | | | | | | | | | | |

Figure 2: Excel example (no short sales), continued

by the corresponding column of the negative of an identity matrix is column 1. The resulting $\underline{W}$ is displayed in `B57:E60`. As the critical value of $\lambda = 0.109537$ identified in Step 2 and shown in `H69` pertains to the status change of security 1, the efficient portfolios in Step 3 consists all three securities. Thus, in Step 3, as shown in `B74:E77`, the original $\underline{W}$ is retained. The critical value of $\lambda = 0.001311$ that this step provides corresponds to the status change of security 2 and, accordingly, securities 1 and 3 are *in* and security 2 is *out* for Step 4. Thus, as shown in `B91:E94`, the only column of the original $\underline{W}$ requiring substitution by the corresponding column of the negative of an identity matrix is column 2. As the only displayed cell among `H97:H99`, labeled as Ratio, shows a negative value, but no critical value of $\lambda$ can be negative, the iterative process to search for corner portfolios is now completed. The portfolio results corresponding to $\lambda = 0$ are shown in `B103:F103`.[7]

In view of the results in Figures 1 and 2, the efficient portfolio weights, expressed in terms of the risk tolerance parameter $\lambda$, are as follows:

Step 1:

$$x_1 = x_2 = 0,\ x_3 = 1,\ \text{for } 0.45000 \leq \lambda; \tag{31}$$

Step 2:

$$\begin{aligned} x_1 &= 0, \\ x_2 &= 0.85714 - 1.90476\lambda, \\ x_3 &= 0.14286 + 1.90476\lambda,\ \text{for } 0.10954 \leq \lambda \leq 0.45000; \end{aligned} \tag{32}$$

Step 3:

$$\begin{aligned} x_1 &= 0.98723 - 9.01277\lambda, \\ x_2 &= -0.00786 + 5.99214\lambda, \\ x_3 &= 0.02063 + 3.02063\lambda,\ \text{for } 0.00131 \leq \lambda \leq 0.10954; \end{aligned} \tag{33}$$

Step 4:

$$\begin{aligned} x_1 &= 0.98000 - 3.50000\lambda, \\ x_2 &= 0, \\ x_3 &= 0.020000 + 3.50000\lambda,\ \text{for } 0 \leq \lambda \leq 0.00131. \end{aligned} \tag{34}$$

Thus, by specifying a value of $\lambda$, we can find the corresponding portfolio weights. With the portfolio weights known, the corresponding expected portfolio return and standard deviation of portfolio returns can be determined as well.

[7] Notice that, as the number of securities considered for efficient portfolio selection changes, the entire worksheet as shown in Figures 1 and 2 has to be revised accordingly. In order to ensure that students understand the example in Figures 1 and 2, it is useful to ask them to perform the same analysis for a case where $n = 4$ or $n = 5$ with some given input data. To perform this exercise properly, students must know both the iterative procedure and the required Excel operations in each step. This exercise will enable students to appreciate more fully the intuition underlying the critical line method. A similar exercise is also useful for the example in Figures 4 and 5, as shown later in the paper.

As each of the second, third, and fourth sets of linear relationships above is in a parametric form, we can eliminate the parameter $\lambda$ from each set to obtain a corresponding line on a plane — a critical line — where the two perpendicular axes are any two of $x_1$, $x_2$, and $x_3$. Figure 3 shows, as an illustration, the case involving the $(x_1, x_2)$-plane. Here, the efficient portfolio in Step 1 is the point $(0, 0)$. All efficient portfolios in Step 2 are on the line segment connecting $(0, 0)$ and $(0, 0.6485)$. The line segment from $(0, 0.6485)$ to $(0.9754, 0)$ covers all efficient portfolios in Step 3. Finally, in Step 4, the line segment from $(0.9754, 0)$ to $(0.9800, 0)$ captures the remaining efficient portfolios, with the point $(0.9800, 0)$ representing the global minimum variance portfolio where $\lambda = 0$. Given that a point on these connected line segments — critical lines — represents an efficient portfolio, a movement of the point there captures the corresponding movement along the efficient frontier on the plane of expected return and standard deviation of returns. Each corner portfolio, which is the intersecting point of two critical lines, is where the status change of a security occurs. The three corner portfolios in Figure 3 are the points $(0, 0)$, $(0, 0.6485)$, and $(0.9754, 0)$. As the choice of the two perpendicular axes is arbitrary, the critical lines on the $(x_1, x_3)$-plane or the $(x_2, x_3)$-plane also convey the same information.

## 4 Portfolio selection with investment limits and disallowance of short sales

Suppose that there are also investment limits on individual securities besides disallowance of short sales. In an $n$-security case, let us impose the conditions of $0 \leq x_i \leq c_i$, where $c_i$ is a pre-determined investment limit on security $i$, for $i = 1, 2, \ldots, n$. The sum of the individual investment limits, $c_1 + c_2 + \cdots + c_n$, must be no less than one. Otherwise, full allocation of investment funds becomes impossible. Analytically, the imposition of investment limits will result in $n$ additional slack variables in the Kuhn-Tucker conditions for optimality to avoid over-investments in any securities. The following is an algebraic approach to introduce these slack variables to the optimality conditions:

To search for an appropriate set of equations to construct the efficient frontier with investment limits and disallowance of short sales, let us return to the expression of $L$ in equation (11). Again, the lower the value of the first term (a non-negative term) on the right hand side of equation (11), the lower is the achievable value of $L$. If a value of $x_i > c_i$ is required to make this term vanish, the value of $x_i Var(R_i) + Cov(R_i, R_i^*) - \frac{1}{2}(\lambda\mu_i + \theta)$, where $x_i = c_i$ (which minimizes this term without violating the condition of $0 \leq x_i \leq c_i$), must be negative.

In order to use this algebraic feature to revise equation (26), let us first clarify the status of each security in the portfolio. With both investment limits and disallowance of short sales, the status of security $i$ must be one of the following three cases: *out*, *in*, and *up*, corresponding to $x_i = 0$, $0 < x_i < c_i$, and $x_i = c_i$, respectively. However, before the portfolio selection problem is solved, the status of the security is unknown. Suppose for now that the selection of security $i$ to the portfolio is assured, but what remains unknown is whether the security is *in* or *up*. To accommodate both cases, we



 

| | X1 | X2 | X3 |
|---|---|---|---|
| Corn.Port. | 0 | 0 | 1 |
| Corn.Port. | 0 | 0.648501 | 0.351499 |
| Corn.Port. | 0.97541 | 0 | 0.02459 |
| Corn.Port. | 0.98 | 0 | 0.02 |

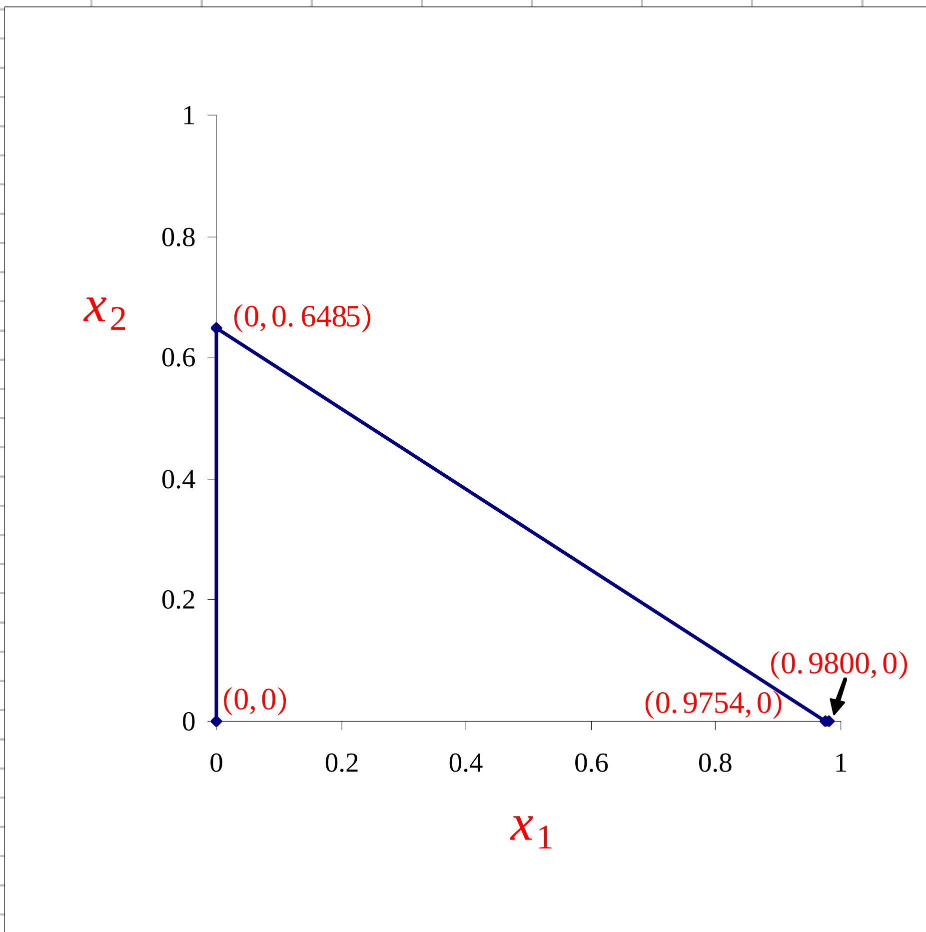


Figure 3: Critical Lines on the $(x_1, x_2)$-plane for the efficient frontier from the example in figures 1 and 2

write equation (12) as

$$x_i Var(R_i) + Cov(R_i, R_i^*) - \frac{1}{2}(\lambda\mu_i + \theta - \phi_i) = 0, \tag{35}$$

by adding a non-negative slack variable $\phi_i$. This additional variable has the following properties: If security $i$ is *in*, $\phi_i$ is zero, allowing equation (12) to be retained. If security $i$ is *up* instead, $\phi_i$ is positive, implying that $x_i Var(R_i) + Cov(R_i, R_i^*) - \frac{1}{2}(\lambda\mu_i + \theta)$, where $x_i = c_i$, is negative. Strictly speaking, if security $i$ is *up*, $\phi_i$ can still be zero. This requires $x_i = c_i$ to be the result of equation (12); that is, the attainment of $x_i = c_i$ does not involve any forced reduction of $x_i$ from a higher value in order to satisfy the condition of $x_i \leq c_i$.

To accommodate also the case where security $i$ is *out*, we revise equation (35) as

$$x_i Var(R_i) + Cov(R_i, R_i^*) - \frac{1}{2}(\lambda\mu_i + \theta + \delta_i - \phi_i) = 0. \tag{36}$$

Here, both $\delta_i$ and $\phi_i$ are non-negative slack variables with following properties: If security $i$ is *in* or *up*, $\delta_i$ is zero; if security $i$ is *in* or *out*, $\phi_i$ is zero. Similar to the case involving equations (12) and (15), as well as the case involving equations (25) and (26), we can write equation (36) as

$$2x_1\sigma_{i1} + 2x_2\sigma_{i2} + \cdots + 2x_n\sigma_{in} - \theta - \delta_i + \phi_i = \lambda\mu_i. \tag{37}$$

Equation (37) holds for $i = 1, 2, \ldots, n$. Along with equation (1), we have $n+1$ equations. Students who are familiar with constrained optimization can recognize that this set of $n+1$ equations provides the first-order conditions. They can also recognize that the complementarity conditions are $0 \leq x_i \leq c_i$, $\delta_i \geq 0$, $\phi_i \geq 0$, $x_i\delta_i = 0$, and $(c_i - x_i)\phi_i = 0$, for $i = 1, 2, \ldots, n$. The complementarity conditions confirm our algebraic results that, if security $i$ is *out*, $x_i = \phi_i = 0$; if security $i$ is *in*, $\delta_i = \phi_i = 0$; and if security $i$ is *up*, $x_i = c_i$ and $\delta_i = 0$. Thus, as long as the status of each security is known, although there are $3n+1$ unknown variables (i.e., $x_1, x_2, \ldots, x_n$, $\delta_1, \delta_2, \ldots, \delta_n$, $\phi_1, \phi_2, \ldots, \phi_n$, and $\theta$), we do have enough equations to solve these variables in terms of $\lambda$.

To illustrate how the set of $n+1$ equations allows us to construct efficient portfolios, let us consider a five-security case where securities 1 and 2 are *up*, securities 3 and 4 are *in*, and security 5 is *out*. In this case, we have the following set of linear equations:

$$\begin{aligned}
\phi_1 + 2\sigma_{13}x_3 + 2\sigma_{14}x_4 - \theta &= -(2c_1\sigma_{11} + 2c_2\sigma_{12}) + \lambda\mu_1 \\
\phi_2 + 2\sigma_{23}x_3 + 2\sigma_{24}x_4 - \theta &= -(2c_1\sigma_{21} + 2c_2\sigma_{22}) + \lambda\mu_2 \\
+2\sigma_{33}x_3 + 2\sigma_{34}x_4 - \theta &= -(2c_1\sigma_{31} + 2c_2\sigma_{32}) + \lambda\mu_3 \\
+2\sigma_{43}x_3 + 2\sigma_{44}x_4 - \theta &= -(2c_1\sigma_{41} + 2c_2\sigma_{42}) + \lambda\mu_4 \\
+2\sigma_{53}x_3 + 2\sigma_{54}x_4 - \delta_5 - \theta &= -(2c_1\sigma_{51} + 2c_2\sigma_{52}) + \lambda\mu_5 \\
x_3 + x_4 &= 1 - (c_1 + c_2)
\end{aligned} \tag{38}$$

Its equivalent matrix equation,

$$\begin{bmatrix} 1 & 0 & 2\sigma_{13} & 2\sigma_{14} & 0 & -1 \\ 0 & 1 & 2\sigma_{23} & 2\sigma_{24} & 0 & -1 \\ 0 & 0 & 2\sigma_{33} & 2\sigma_{34} & 0 & -1 \\ 0 & 0 & 2\sigma_{43} & 2\sigma_{44} & 0 & -1 \\ 0 & 0 & 2\sigma_{53} & 2\sigma_{54} & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ x_3 \\ x_4 \\ \delta_5 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - c_1 \begin{bmatrix} 2\sigma_{11} \\ 2\sigma_{21} \\ 2\sigma_{31} \\ 2\sigma_{41} \\ 2\sigma_{51} \\ 1 \end{bmatrix} - c_2 \begin{bmatrix} 2\sigma_{12} \\ 2\sigma_{22} \\ 2\sigma_{32} \\ 2\sigma_{42} \\ 2\sigma_{52} \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ 0 \end{bmatrix}, \tag{39}$$

retains the same algebraic form as equation (16). As equation (39) illustrates, for each security $i$ in an $n$-security case that is *out*, we still substitute the $i$-th element of the column vector $\underline{Z}$ with $\delta_i$ and substitute column $i$ of $\underline{W}$ with the corresponding column of the negative of an $(n+1) \times (n+1)$ identity matrix. For each security $i$ that is *up*, however, we substitute the $i$-th element $\underline{Z}$ with $\phi_i$, substitute column $i$ of $\underline{W}$ with the corresponding column of an $(n+1) \times (n+1)$ identity matrix, and subtract from column vector $\underline{H}$ column $i$ of $c_i\underline{W}$ (which is proportion $c_i$ of the original $\underline{W}$). The more securities that are *up*, the more subtractions from $\underline{H}$ are required. Once $\underline{W}, \underline{Z}$, and $\underline{H}$ are revised in this manner, we can still use equation (21) and the same critical line method to construct the efficient frontier.

Notice that, just like the portfolio selection problem without short sales considered earlier in this study, the way equation (16) is revised to accommodate investment limits on individual securities is also a novel approach. Its novelty is in how the algebraic form of equation (16) is retained in order to simplify the analysis that follows. If the Markowitz [9] method, as intended for general linear constraints, were applied directly to this specific portfolio selection problem, the required computations, which include intersecting potential critical lines during each iterative step, would be much too complicated for business students to follow.

## 4.1 A spreadsheet-based illustration

Let us return to the same three-security Excel example and impose also a common 70% investment limit on each security, i.e., $c = c_1 = c_2 = c_3 = 0.7$. The set of linear equations for portfolio construction according to equations (1) and (37) is as follows:

$$\begin{aligned} (0.0008x_1 - \delta_1 + \phi_1) + 0.0008x_2 + 0.0004x_3 - \theta &= 0.05\lambda \\ 0.0008x_1 + (0.0050x_2 - \delta_2 + \phi_2) + 0.0020x_3 - \theta &= 0.08\lambda \\ 0.0004x_1 + 0.0020x_2 + (0.0200x_3 - \delta_3 + \phi_3) - \theta &= 0.12\lambda \\ x_1 + x_2 + x_3 &= 1 \end{aligned} \tag{40}$$

Figures 4 and 5 shows the corresponding Excel worksheet, where the search for corner portfolios requires four steps. The initial portfolio in Step 1, for an infinitely high $\lambda$, consists of securities 2 and 3, the two securities with the highest expected returns among the three securities considered. To achieve the highest possible expected portfolio return, security 3 is assigned a 70% portfolio weight. Thus, in Step 1, security 1 is *out*, security

2 is *in*, and security 3 is *up*, as indicated in `B25:D25`. The revised $\underline{W}$ in `B28:E31` for this step is the original $\underline{W}$ in `B9:E12` with its first and third columns substituted by the corresponding columns of the negative of the identity matrix in `B19:E22` and of the identity matrix in `B14:E17`, respectively. The column in the original $\underline{W}$ to be used for revising $\underline{H}$ for this step is indicated in `B26:D26`. With security 3 being *up* (coded as 1 for computational convenience), the revised $\underline{H}$ in `F28:F31` is the original $\underline{H}$ in `F9:F12` minus 70% of the third column of the original $\underline{W}$, which is `D9:D12`.

The computations of $\underline{W}^{-1}$ in `B34:E37` and, subsequently, A and B in `F34:G37` are the same as those in Figure 1. Likewise, the computations of Ratio (1) in `H34:H36` — $-a_i/b_i$, for $i = 1, 2$, and 3, if $b_i$ is positive — are the same as those for Ratio in Figure 1. Ratio (1) provides, for each security $i$, the value of $\lambda$ below which the variable in equation (40) pertaining to the security (i.e., $x_i, \delta_i$, or $\phi_i$) would become negative. Ratio (2) in `I34:I36` — $(c - a_i)/b_i$, for $i = 1, 2$, and 3, if security $i$ is *in* and $b_i$ is negative — provides, for each security $i$ that is *in*, the value of $\lambda$ below which the security would exceed the investment limit $c$. By allowing $\lambda$ to decrease from an infinitely high initial value, the maximum among the computed values in Ratio (1) and Ratio (2) is the first critical value we encounter, as a marginally lower value of $\lambda$ would render the portfolio infeasible. This critical value of $\lambda = 0.2925$, labeled as Lambda, is shown in `I40`. The corner portfolio weights, expected portfolio return, and standard deviation of portfolio returns, in `B40:F40`, corresponding to this critical value of $\lambda$, are obtained in the same manner as those in Figure 1. As this critical value of $\lambda$ is provided by security 3, we change the status of the security from *up* to *in* for Step 2.

The required formulas for the first 40 rows beyond those already described in Figure 1 are listed in rows 43 to 54 of the worksheet. We can easily perform the remaining steps by revising `B25:D25` only on copies (to separate worksheets) of the first 40 rows of the worksheet. However, in Figures 4 and 5, where all four steps are performed on a common worksheet to minimize the number of pages for the display, some changes to various formulas in Step 1 for the subsequent steps are required. As such changes can be deduced from those provided for Step 1, they are not explicitly noted in Figure 4.

For Step 2, as no security is *up*, the revised $\underline{H}$ in `F60:F63` retains the same values as the original $\underline{H}$ in `F9:F12`. The critical value of $\lambda = 0.10954$ from this step, in `I72`, is caused by security 1 and thus, in Step 3, all three securities are *in*. Again, with no security being *up*, the revised $\underline{H}$ in `F78:F81` retains its original values. However, as the critical value of $\lambda = 0.03187$ from this step, in `I90`, is caused by a violation of investment limit on security 1, Step 4 is the case where security 1 is *up* and the remaining two securities are *in*. Accordingly, the revised $\underline{H}$ in `F96:F99` is the original $\underline{H}$ in `F9:F12` minus 70% of the first column of the original $\underline{W}$, which is `B9:B12`. As no more non-negative critical value of $\lambda$ can be found in Step 4, the search is completed.

The results in Figures 4 and 5 allow us to express the efficient portfolio weights in terms of the risk tolerance parameter $\lambda$ as follows:

Step 1:

$$x_1 = 0,\ x_2 = 0.3,\ x_3 = 0.7,\ \text{for } 0.29250 \leq \lambda; \tag{41}$$

| | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | Sec.1 | Sec.2 | Sec.3 | | | | | | | |
| 2 | | | | | | Investment Limit c | | | | | |
| 3 | Exp.Ret. | 0.05 | 0.08 | 0.12 | | 0.7 | | | | | |
| 4 | | | | | | | | | | | |
| 5 | Cov.Mat. | 0.0004 | 0.0004 | 0.0002 | | | | | | | |
| 6 | | 0.0004 | 0.0025 | 0.001 | | | | | | | |
| 7 | | 0.0002 | 0.001 | 0.01 | | | | | | | |
| 8 | | | | | | H | K | | | | |
| 9 | W | 0.0008 | 0.0008 | 0.0004 | -1 | 0 | 0.05 | | Sec.1 | | |
| 10 | | 0.0008 | 0.005 | 0.002 | -1 | 0 | 0.08 | | Sec.2 | | |
| 11 | | 0.0004 | 0.002 | 0.02 | -1 | 0 | 0.12 | | Sec.3 | | |
| 12 | | 1 | 1 | 1 | 0 | 1 | 0 | | | | |
| 13 | | | | | | | | | | | |
| 14 | Iden. | 1 | 0 | 0 | 0 | | | | | | |
| 15 | | 0 | 1 | 0 | 0 | | | | | | |
| 16 | | 0 | 0 | 1 | 0 | | | | | | |
| 17 | | 0 | 0 | 0 | 1 | | | | | | |
| 18 | | | | | | | | | | | |
| 19 | Neg.Iden. | -1 | 0 | 0 | 0 | | | | | | |
| 20 | | 0 | -1 | 0 | 0 | | | | | | |
| 21 | | 0 | 0 | -1 | 0 | | | | | | |
| 22 | | 0 | 0 | 0 | -1 | | | | | | |
| 23 | | | | | | | | | | | |
| 24 | **Step 1** | Sec.1 | Sec.2 | Sec.3 | | | | | | | |
| 25 | | out | in | up | | | | | | | |
| 26 | UP Code | 0 | 0 | 1 | | | | | | | |
| 27 | | | | | | H | K | | | | |
| 28 | W | -1 | 0.0008 | 0 | -1 | -0.00028 | 0.05 | Sec.1 | out | | |
| 29 | | 0 | 0.005 | 0 | -1 | -0.0014 | 0.08 | Sec.2 | in | | |
| 30 | | 0 | 0.002 | 1 | -1 | -0.014 | 0.12 | Sec.3 | up | | |
| 31 | | 0 | 1 | 0 | 0 | 0.3 | 0 | | | | |
| 32 | | | | | | | | | | Revised | Corn.Port. |
| 33 | | | | | | A | B | Ratio (1) | Ratio (2) | Status | X |
| 34 | Inv. Of W | -1 | 1 | 0 | -0.0042 | -0.00238 | 0.03 | 0.07933 | | | 0 |
| 35 | | 0 | 0 | 0 | 1 | 0.3 | 0 | | | | 0.3 |
| 36 | | 0 | -1 | 1 | 0.003 | -0.0117 | 0.04 | 0.2925 | | in | 0.7 |
| 37 | | 0 | -1 | 0 | 0.005 | 0.0029 | -0.08 | | | | |
| 38 | | | | | | | | | | | |
| 39 | | X1 | X2 | X3 | Exp.Ret. | St.Dev. | | | Lambda | | |
| 40 | Corn.Port. | 0 | 0.3 | 0.7 | 0.108 | 0.07446 | | | 0.2925 | | |
| 41 | | | | | | | | | | | |
| 42 | Formulas (beyond those analogous formulas already described in Figure 1): | | | | | | | | | | |
| 43 | 1. UP Code (in B26:D26): copy and paste B26=IF(B25="up",1,0) to B26:D26. | | | | | | | | | | |
| 44 | 2. First 3 columns of W (in B28:D31): copy and paste B28=IF(B$25="in",B9,IF(B$25="out",B19,B14)) to B28:D31. | | | | | | | | | | |
| 45 | 3. H (in F28:F31): copy and paste F28=F9-F$3*(B$26*B9+C$26*C9+D$26*D9) to F28:F31. | | | | | | | | | | |
| 46 | 4. First three elements of K (in G28:G30): {=TRANSPOSE(B$3:D$3)}. | | | | | | | | | | |
| 47 | 5. A (in F34:F37): {=MMULT(B34:E37,F28:F31)}. | | | | | | | | | | |
| 48 | 6. B (in G34:G37): {=MMULT(B34:E37,G28:G31)}. | | | | | | | | | | |
| 49 | 6. Ratio (1) (in H34:H36): copy and paste H34=IF(G34>0,-F34/G34,"") to H34:H36. | | | | | | | | | | |
| 50 | 7. Ratio (2) (in I34:I36): copy and paste I34=IF(AND(I28="in",G34<0),(F$3-F34)/G34,"") to I34:I36. | | | | | | | | | | |
| 51 | 8. Lambda (in I44): =MAX(0,H38:I40). | | | | | | | | | | |
| 52 | 9. X (in K34:K36): copy and paste K34=IF(I28="in",F34+G34*I$40,IF(I28="out",0,F$3)) to K34:K36. | | | | | | | | | | |
| 53 | 10. Revised status (in J34:J36): copy and paste J34=IF(I$40<0,"",IF(H34=I$40,IF(OR(I28="out",I28="up"),"in", | | | | | | | | | | |
| 54 | "out"),IF(I34=I$40,"up",""))) to J34:J36. | | | | | | | | | | |

Figure 4: Excel example (with investment limits and no short sales)

| 55 | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 56 | **Step 2** | Sec.1 | Sec.2 | Sec.3 | | | | | | | |
| 57 | | out | in | in | | | | | | | |
| 58 | UP Code | 0 | 0 | 0 | | | | | | | |
| 59 | | | | | | H | K | | | | |
| 60 | W | -1 | 0.0008 | 0.0004 | -1 | 0 | 0.05 | Sec.1 | out | | |
| 61 | | 0 | 0.005 | 0.002 | -1 | 0 | 0.08 | Sec.2 | in | | |
| 62 | | 0 | 0.002 | 0.02 | -1 | 0 | 0.12 | Sec.3 | in | | |
| 63 | | 0 | 1 | 1 | 0 | 1 | 0 | | | | |
| 64 | | | | | | | | | | Revised | Corn.Port. |
| 65 | | | | | | A | B | Ratio (1) | Ratio (2) | Status | X |
| 66 | Inv. Of W | -1 | 0.87619 | 0.12381 | -0.00383 | -0.00383 | 0.03495 | 0.10954 | | in | 0 |
| 67 | | 0 | 47.619 | -47.619 | 0.85714 | 0.85714 | -1.90476 | | 0.0825 | | 0.64850136 |
| 68 | | 0 | -47.619 | 47.619 | 0.14286 | 0.14286 | 1.90476 | -0.075 | | | 0.35149864 |
| 69 | | 0 | -0.85714 | -0.14286 | 0.00457 | 0.00457 | -0.08571 | | | | |
| 70 | | | | | | | | | | | |
| 71 | | X1 | X2 | X3 | Exp.Ret. | St.Dev. | | | Lambda | | |
| 72 | Corn.Port. | 0 | 0.6485 | 0.3515 | 0.09406 | 0.05237 | | | 0.10954 | | |
| 73 | | | | | | | | | | | |
| 74 | **Step 3** | Sec.1 | Sec.2 | Sec.3 | | | | | | | |
| 75 | | in | in | in | | | | | | | |
| 76 | UP Code | 0 | 0 | 0 | | | | | | | |
| 77 | | | | | | H | K | | | | |
| 78 | W | 0.0008 | 0.0008 | 0.0004 | -1 | 0 | 0.05 | Sec.1 | in | | |
| 79 | | 0.0008 | 0.005 | 0.002 | -1 | 0 | 0.08 | Sec.2 | in | | |
| 80 | | 0.0004 | 0.002 | 0.02 | -1 | 0 | 0.12 | Sec.3 | in | | |
| 81 | | 1 | 1 | 1 | 0 | 1 | 0 | | | | |
| 82 | | | | | | | | | | Revised | Corn.Port. |
| 83 | | | | | | A | B | Ratio (1) | Ratio (2) | Status | X |
| 84 | Inv. Of W | 257.859 | -225.933 | -31.9253 | 0.98723 | 0.98723 | -9.01277 | | 0.03187 | up | 0.7 |
| 85 | | -225.933 | 245.58 | -19.6464 | -0.00786 | -0.00786 | 5.99214 | 0.00131 | | | 0.18310627 |
| 86 | | -31.9253 | -19.6464 | 51.5717 | 0.02063 | 0.02063 | 3.02063 | -0.00683 | | | 0.11689373 |
| 87 | | -0.98723 | 0.00786 | -0.02063 | 0.00079 | 0.00079 | -0.05121 | | | | |
| 88 | | | | | | | | | | | |
| 89 | | X1 | X2 | X3 | Exp.Ret. | St.Dev. | | | Lambda | | |
| 90 | Corn.Port. | 0.7 | 0.18311 | 0.11689 | 0.06368 | 0.02438 | | | 0.03187 | | |
| 91 | | | | | | | | | | | |
| 92 | **Step 4** | Sec.1 | Sec.2 | Sec.3 | | | | | | | |
| 93 | | up | in | in | | | | | | | |
| 94 | UP Code | 1 | 0 | 0 | | | | | | | |
| 95 | | | | | | H | K | | | | |
| 96 | W | 1 | 0.0008 | 0.0004 | -1 | -0.00056 | 0.05 | Sec.1 | up | | |
| 97 | | 0 | 0.005 | 0.002 | -1 | -0.00056 | 0.08 | Sec.2 | in | | |
| 98 | | 0 | 0.002 | 0.02 | -1 | -0.00028 | 0.12 | Sec.3 | in | | |
| 99 | | 0 | 1 | 1 | 0 | 0.3 | 0 | | | | |
| 100 | | | | | | | | | | Revised | Corn.Port. |
| 101 | | | | | | A | B | Ratio (1) | Ratio (2) | Status | X |
| 102 | Inv. Of W | 1 | -0.87619 | -0.12381 | 0.00383 | 0.00111 | -0.03495 | | | | 0.7 |
| 103 | | 0 | 47.619 | -47.619 | 0.85714 | 0.24381 | -1.90476 | | -0.2395 | | 0.24380952 |
| 104 | | 0 | -47.619 | 47.619 | 0.14286 | 0.05619 | 1.90476 | -0.0295 | | | 0.05619048 |
| 105 | | 0 | -0.85714 | -0.14286 | 0.00457 | 0.00189 | -0.08571 | | | | |
| 106 | | | | | | | | | | | |
| 107 | | X1 | X2 | X3 | Exp.Ret. | St.Dev. | | | Lambda | | |
| 108 | Corn.Port. | 0.7 | 0.24381 | 0.05619 | 0.06125 | 0.02358 | | | 0 | | |

Figure 5: Excel example (with investment limits and no short sales), continued

Step 2:

$$\begin{aligned} x_1 &= 0, \\ x_2 &= 0.85714 - 1.90476\lambda, \\ x_3 &= 0.14286 + 1.90476\lambda, \ \text{for } 0.10954 \leq \lambda \leq 0.29250; \end{aligned} \tag{42}$$

Step 3:

$$\begin{aligned} x_1 &= 0.98723 - 9.01277\lambda, \\ x_2 &= -0.00786 + 5.99214\lambda, \\ x_3 &= 0.02063 + 3.02063\lambda, \ \text{for } 0.03187 \leq \lambda \leq 0.10954; \end{aligned} \tag{43}$$

Step 4:

$$\begin{aligned} x_1 &= 0.7, \\ x_2 &= 0.24381 - 1.90476\lambda, \\ x_3 &= 0.05619 + 1.90476\lambda, \ \text{for } 0 \leq \lambda \leq 0.03187. \end{aligned} \tag{44}$$

These linear relationships also allow us to find, for any $\lambda \geq 0$, the corresponding portfolio weights. Then, the corresponding expected portfolio return and standard deviation of portfolio returns can be determined as well.

Once we eliminate the parameter $\lambda$ from each of the last three sets of linear relationships above, we can capture these relationships graphically on a plane, with the two perpendicular axes being any two of $x_1, x_2$, and $x_3$. For example, as shown in Figure 6, the efficient portfolio in Step 1 is the point $(0, 0.3)$ on the $(x_1, x_2)$-plane. All efficient portfolios in Step 2 are on the line segment — a critical line on the $(x_1, x_2)$-plane — between $(0, 0.3)$ and $(0, 0.6485)$. The line segment between $(0, 0.6485)$ and $(0.7, 0.1831)$ covers all efficient portfolios in Step 3. In Step 4, the line segment between $(0.7, 0.1831)$ and $(0.7, 0.2438)$ covers the remaining efficient portfolios, with the point $(0.7, 0.2438)$ being the global minimum variance portfolio where $\lambda = 0$. The point where two critical lines meet gives us a corner portfolio, where a status change of a security occurs. As Figure 6 shows, the three corner portfolios are the points $(0, 0.3)$, $(0, 0.6485)$, and $(0.7, 0.1831)$. A point on the efficient frontier, on the plane of expected return and standard deviation of returns, has a corresponding point on these connected line segments. Further, as we move along the efficient frontier, where $\lambda \geq 0$, the corresponding changes in portfolio weights are revealed by the movement of a point on these connected line segments, from one end to the other end.

Before concluding, it is worth noting that, although the spreadsheet illustrations in this study are for $n = 3$, where all the input data for the analysis are provided, the analysis can be made more practically relevant if larger-scaled portfolio selection problems with real-world data are attempted. To do so will involve the following components of an investment course: (1) empirical estimation of the required input data for mean-variance analysis on some publicly traded securities and (2) implementation of the analysis on the estimated data. The work, to be performed on spreadsheets, can

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| | X1 | X2 | X3 |
|---|---|---|---|
| Corn.Port. | 0 | 0.3 | 0.7 |
| Corn.Port. | 0 | 0.648501 | 0.351499 |
| Corn.Port. | 0.7 | 0.183106 | 0.116894 |
| Corn.Port. | 0.7 | 0.24381 | 0.05619 |

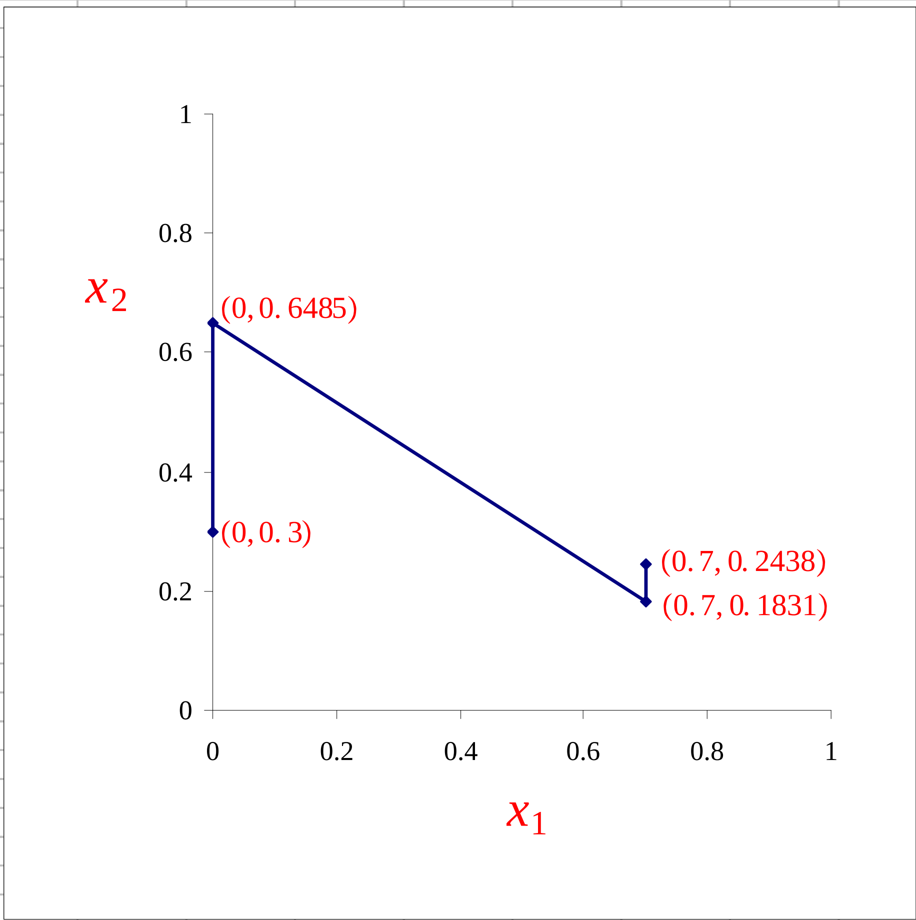


Figure 6: Critical Lines on the $(x_1, x_2)$-plane for the Efficient Frontier from the Example in Figures 4 and 5

either be divided into two separate assignments for students, as the course progresses, or treated as a single assignment upon the completion of both components of the course. By assigning different sets of securities with different risk-return characteristics to individual students (or groups of students), we can discuss in class various portfolio concepts based on the results obtained.

## 5 Concluding remarks

The Markowitz critical line method for portfolio selection under general linear constraints, though a significant advancement in the investment literature, is well beyond the scope of the standard finance curriculum because it is a sophisticated algorithm. Thus, for many decades since its original publication, the method has remained a mystery to many business students. This pedagogic study has provided a simple exposition of the method by considering two specific but practically relevant constraints, i.e., investment limits on individual securities and disallowance of short sales. This study has shown that the method requires only basic algebraic tools and statistical concepts, thus allowing the Markowitz analysis to be more accessible to business students.

This study is intended to complement other pedagogic studies on mean-variance portfolio selection, such as those using numerical approaches to bypass the analytical details and those simplifying the covariance structure of security returns to reduce the analytical burden. This study is able to reduce the analytical burden in portfolio construction while maintaining the original covariance structure. It also enables business students to appreciate more fully an important feature of the Markowitz analysis that there is a correspondence between the mean-variance efficient frontier (on the plane of expected return and standard deviation of returns) and the critical lines (in a multi-dimensional space of portfolio weights).

The use of electronic spreadsheets in this study allows business students to follow the Markowitz analysis under specific constraints without computational distractions. As the only time-consuming computations in the analysis are for solving simultaneous linear equations (if performed manually), the use of spreadsheet functions for basic matrix operations to solve these equations is able to reduce significantly the computational burden without masking any analytical features of the method. Spreadsheet-based matrix operations being easy to follow, their simplicity and computational advantage can easily be recognized by students with or without prior knowledge of matrix algebra. This computational simplification, in turn, enables students to pay more attention to the conceptual aspect of the Markowitz analysis.

For advanced students, including those who are already familiar with multivariate differential calculus and optimization tools, instructors can present the same optimization problems more formally. Once the optimality conditions — including those in equations (15), (26), and (37) — are reached, the same spreadsheet approach as described in this study can be applied directly. Instructors may find it useful to extend the analysis in this study to accommodate additional linear constraints. Constraints such as specific limits on the aggregate portfolio weights of subsets of securities are practically relevant. In cases where such limits are stated as equality constraints, the corresponding extensions are straightforward. Cases involving inequality constraints, however, are more complicated. They are suitable only for courses where students have some prior knowledge of multivariate optimization.

Finally, it is worth noting that, although this pedagogic study has its focus on the Markowitz critical line method for mean-variance portfolio analysis, the same idea of using simple algebraic tools can be extended to other types of decision problems as well. As long as a decision setting can be formulated as an optimization problem with a quadratic objective function subject to some specific linear constraints, a set of optimality conditions, which allows the solution to be reached, can still be established without explicitly using any multivariate differential calculus tools. Therefore, the idea as presented in this pedagogic study should also be of interest to instructors in other academic disciplines who wish to make their quadratic-programming materials accessible to more students with divergent mathematical backgrounds.

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