Spreadsheets in Education (eJSiE)

Volume 2 | Issue 3 Article 4

4-11-2008

Revisiting Polya's summation techniques using a spreadsheet: from addition tables to Bernoulli polynomials

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Recommended Citation

Abramovich, Sergei and Sugden, Stephen J. (2008) Revisiting Polya's summation techniques using a spreadsheet: from addition tables to Bernoulli polynomials, *Spreadsheets in Education (eJSiE)*: Vol. 2: Iss. 3, Article 4.
Available at: http://epublications.bond.edu.au/ejsie/vol2/iss3/4

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Abstract
Recurrence relations in mathematics form a very powerful and compact way of looking at a wide range of relationships. Traditionally, the concept of recurrence has often been a difficult one for the secondary teacher to convey to students. Closely related to the powerful proof technique of mathematical induction, recurrences are able to capture many relationships in formulas much simpler than so-called direct or closed formulas. In computer science, recursive coding often has a similar compactness property, and, perhaps not surprisingly, suffers from similar problems in the classroom as recurrences: the students often find both the basic concepts and practicalities elusive. Using models designed to illuminate the relevant principles for the students, we offer a range of examples which use the modern spreadsheet environment to powerfully illustrate the great expressive and computational power of recurrences.

Keywords
recurrence, spreadsheet, induction, Bernoulli polynomials

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April 10, 2008

Abstract

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Submitted December 2007, revised and accepted April 2008.

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1 Introduction

Advocated by Polya as tools for learning and teaching problem solving [13], the closely-related concepts of recurrences, recursion, finite-differences, mathematical induction, and difference equations are both fundamental and powerful. They are *fundamental* in the sense that recursion and induction are woven into the very structure of the natural numbers [7], and *powerful* in at least two senses:

1. enabling concise expression of far-reaching principles or relationships
2. forming the basis of very compact and efficient algorithms for computation of many combinatorial objects

eJSiE **2**(3): 299–315

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We are interested in conveying the elegance and power of recursive concepts and methods in the teaching of mathematics through technology. To recognize and efficiently apply recursive methods using traditional facilities such as pencil, paper and algebra requires significant mathematical maturity. Such maturity is found only comparatively rarely in high-school students or beginning tertiary students, at least at the authors' institutions. In this paper, we argue that the modern spreadsheet environment enhances the presentation of Polya's ideas about the summation of perfect powers by offering new opportunities for the teacher to illustrate the simplicity, yet great power of recursive approaches. Such approaches are mathematically rigorous, yet quite accessible for the students, even for those whose algebraic background is very modest.

Throughout the paper, we examine a series of mathematical objects and their recursive definitions. It is shown that the spreadsheet offers a friendly and illustrative environment for investigation of the properties of recursively-defined concepts. Each such concept will be motivated by a combination of a concrete problem and its physical representation.

2 Addition and multiplication tables

As mentioned in [16], the program of axiomatization of arithmetic was originally undertaken by Grassmann [7] who, proceeding from the recursive definition of natural numbers, introduced the operations of addition and multiplication through such a definition also. In developing spreadsheet-based addition and multiplication tables, one can use this classic approach to the rigorization of arithmetic by using the software facility of recurrent counting. To clarify, consider

Problem 1. There are x red and y yellow counters on the desk. How many counters in both colors are there?

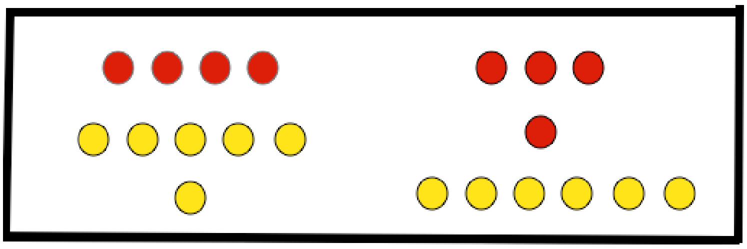

Figure 1: Counting counters recursively

The left and right parts of Figure 1 (in which $x = 4$ and $y = 6$) show, respectively, that the total number of counters can be counted through the equalities $4+6 = 4+(6-1)+1$ and $4+6 = (4-1)+1+6$. These equalities can be generalized to the identities $x+y = x+(y-1)+1$ and $x+y = (x-1)+1+y$. In a more formalized notation, and setting $A(x, y) = x + y$, these identities can be expressed through what may be referred to as partial difference equations in two variables [8]

	A	B	C	D	E	F	G	H	I	J	K
1	y \ x	1	2	3	4	5	6	7	8	9	10
2	1	2	3	4	5	6	7	8	9	10	11
3	2	3	4	5	6	7	8	9	10	11	12
4	3	4	5	6	7	8	9	10	11	12	13
5	4	5	6	7	8	9	10	11	12	13	14
6	5	6	7	8	9	10	11	12	13	14	15
7	6	7	8	9	10	11	12	13	14	15	16
8	7	8	9	10	11	12	13	14	15	16	17
9	8	9	10	11	12	13	14	15	16	17	18
10	9	10	11	12	13	14	15	16	17	18	19
11	10	11	12	13	14	15	16	17	18	19	20

Figure 2: Addition table

$A(x,y) = A(x,y-1)+1$ or $A(x,y) = A(x-1,y)+1$, satisfying boundary conditions $A(1,y) = y+1$ and $A(x,1) = x+1$, respectively. The spreadsheet pictured in Figure 2 generates an addition table by using one of the above partial difference equations.

2.1 Multiplication table as recursive structure

An effective way to develop a mathematical model that describes a new concept is to pose a problem that allows for the construction of a physical model to serve as a situational referent for the new concept. With this in mind, consider the following example.

Problem 2. On a parade, x people are marching in each of y columns. How many people are marching?

Figure 3 (where $x = 4, y = 7$) shows two ways of solving this problem recursively. Indeed, the bottom and top parts of Figure 3 can be described, respectively, through the equalities $4 \times 7 = (3 \times 7) + 7$ and $4 \times 7 = (4 \times 6) + 4$, which can be generalized to the identities $xy = (x-1)\,y + y$ and $xy = x\,(y-1) + x$.

In this case, setting $P(x,y) = xy$, we have, $P(x,y) = P(x-1,y) + y$ or $P(x,y) = P(x,y-1)+x$, satisfying boundary conditions $P(1,y) = y$ and $P(x,1) = x$, respectively. The spreadsheet pictured in Figure 4 generates a multiplication table by using one of these partial difference equations. Further discussion may be found in [2].

3 Power table

In this section, we consider the cardinality of a set of functions as a motivation for the development of a power table through a recursive definition. Each function has the same

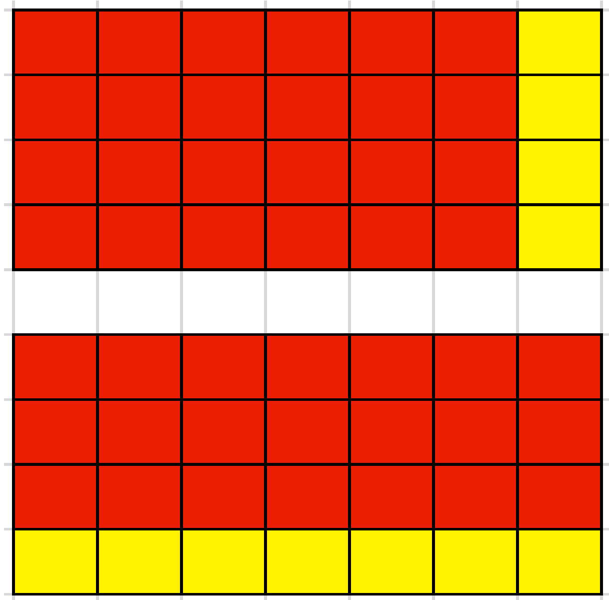

Figure 3: Representing a product of two numbers recursively

finite domain and the same finite codomain. Suppose that the domain has cardinality y and the codomain has cardinality x.

Problem 3. In how many ways can y objects be placed in x boxes? We consider an example in which $y = 3$ and $x = 2$. Suppose we have three objects, denoted p, q, r. Each object must be placed in one of two separate boxes, which we denote by 0 and 1. Thus, a box may be empty or include more than one object. Since no object can be in both boxes at the same time, we have the set of eight possibilities shown in Figure 5.

There are several ways to view this example, which is intended to illustrate the cardinality of a set of functions. Suppose that $f : \{p, q, r\} \to \{0, 1\}$. Then, the set of pairs for f must be of the form $\{(p, b_1), (q, b_2), (r, b_3)\}$, where each $b_i \in \{0, 1\}$. To see how many such sets are possible, we first observe that each b_i is independent of the others, and has 2 possible values. Thus, by the multiplication rule, we have 2^3 possible sets of pairs.

We have illustrated this principle in Figure 5 by using the traditional *arrow diagram*. The main point to notice here is that we must always have three arrows pointing from the domain. This is required since we are dealing with functions and not relations. Where do these arrows go? To use an archery or military analogy, each must hit a *target*. The next question is: "how many possible targets are there?". Here, we have just 2 targets—these are the elements of the codomain. Since each arrow must hit a target, each function may be represented by an ordered binary triple, or *bitstring* of length 3. In such a triple, the domain element, or pre-image is implicit, as we always assume the order p, q, r. The corresponding codomain element, or image, is just a bit. For example,

	A	B	C	D	E	F	G	H	I	J	K
1	y \ x	1	2	3	4	5	6	7	8	9	10
2	1	1	2	3	4	5	6	7	8	9	10
3	2	2	4	6	8	10	12	14	16	18	20
4	3	3	6	9	12	15	18	21	24	27	30
5	4	4	8	12	16	20	24	28	32	36	40
6	5	5	10	15	20	25	30	35	40	45	50
7	6	6	12	18	24	30	36	42	48	54	60
8	7	7	14	21	28	35	42	49	56	63	70
9	8	8	16	24	32	40	48	56	64	72	80
10	9	9	18	27	36	45	54	63	72	81	90
11	10	10	20	30	40	50	60	70	80	90	100

Figure 4: Multiplication table

the function $\{(p,0),(q,0),(r,1)\}$ may simply be represented by the bitstring 001. The number of possible functions, then, is simply the number of bitstrings of length 3, and this is well-known to be 2^3. In a typical arrow diagram, each of the mandatory three arrows can be imagined to represent a bit: that of its target, or image point. Figure 6 illustrates this idea for the function "001", that is, $f(p)=0, f(q)=0$, and $f(r)=1$.

In terms of recursion, that is to show that $2^4=(2^3)\times 2$, note that, as we have two boxes, object 4 can be added to each of the arrangements in two ways: either by being added to box 0 (empty or not) or to box 1 (empty or not).

In general, the transition from x boxes and y objects to x boxes and $y+1$ objects always results in x different ways to choose a box for the $(y+1)$-st object. Then setting $E(x,y)=x^y$, one can introduce the equation $E(x,y)=E(x,y-1)x$ subject to the boundary condition $E(x,1)=x$. The results of spreadsheet modelling of this partial difference equation are shown in Figure 7.

4 Modeling the sums of perfect powers

Investigation of sums of positive integer powers of the natural numbers has a long history [10]. Finding a formula for

$$S(n,m)=1^m+2^m+3^m+\cdots+n^m=\sum_{k=1}^{n}k^m$$

has interested mathematicians for more than 300 years since the time of Jakob Bernoulli (1654-1705) [17]. General power sums also arise commonly in statistics [9]. It is known

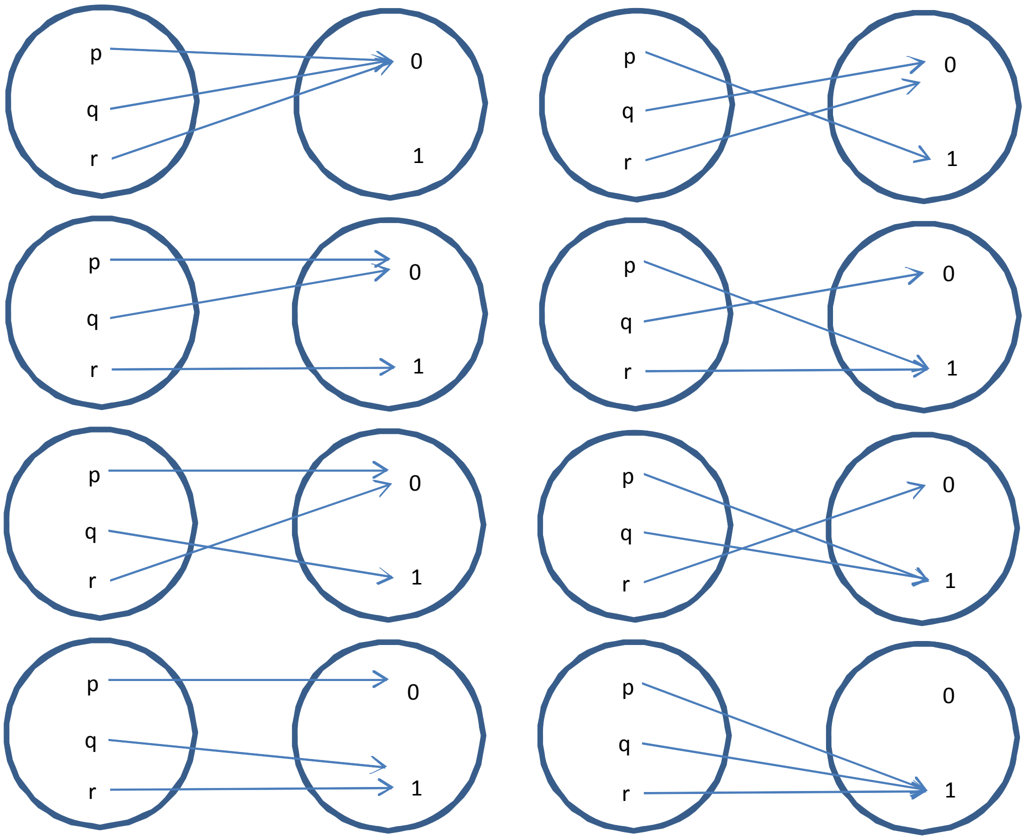

Figure 5: All functions $f : \{p, q, r\} \to \{0, 1\}$

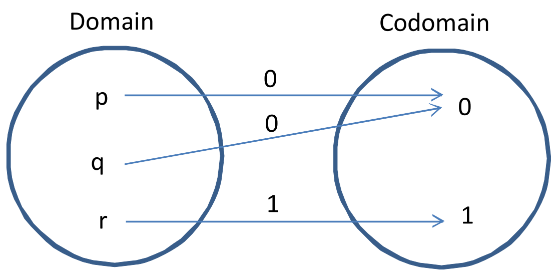

Figure 6: The function "001"

	A	B	C	D	E	F	G	H	I
1	y \ x	1	2	3	4	5	6	7	8
2	1	1	2	3	4	5	6	7	8
3	2	1	4	9	16	25	36	49	64
4	3	1	8	27	64	125	216	343	512
5	4	1	16	81	256	625	1296	2401	4096
6	5	1	32	243	1024	3125	7776	16807	32768
7	6	1	64	729	4096	15625	46656	117649	262144
8	7	1	128	2187	16384	78125	279936	823543	2097152

Figure 7: Power table

that a closed form expression—a polynomial in n of degree $m+1$ —represents this sum for any $m \geq 1$. We list the first few of these below.

$$S(n,1) = \sum_{k=1}^{n} k = \frac{1}{2}n\,(n+1)$$

$$S(n,2) = \sum_{k=1}^{n} k^2 = \frac{1}{6}n\,(n+1)\,(2n+1)$$

$$S(n,3) = \sum_{k=1}^{n} k^3 = \frac{1}{4}n^2\,(n+1)^2$$

$$S(n,4) = \sum_{k=1}^{n} k^4 = \frac{1}{30}n\,(n+1)\,(2n+1)\left(3n^2+3n-1\right)$$

$$S(n,5) = \sum_{k=1}^{n} k^5 = \frac{1}{12}n^2\,(n+1)^2\left(2n+2n^2-1\right)$$

$$S(n,6) = \sum_{k=1}^{n} k^6 = \frac{n}{42}\,(n+1)\,(2n+1)\left(3n^4+6n^3-3n+1\right)$$

Such polynomials have many interesting properties. For example, for each one, the sum of the coefficients is unity, and $n\,(n+1)\,/2$ is a factor.

4.1 Combinatorial motivation of the sums of perfect powers

To begin, consider the following combinatorial problematic situation used elsewhere [1] to introduce the use of a spreadsheet in teaching topics in discrete mathematics.

Problem 4. There are n types of objects with an unlimited number of each type, or, in other words, any object after being chosen is replaced in the stock. One makes all possible arrangements of m such objects (m–samples). In making up the arrangements, objects of the same type can be used and therefore each place in the m–sample can be filled in n different ways, because the stock of objects is unaltered by any choice. Two arrangements are regarded as different if they contain different numbers of elements of a certain type or if their elements are ordered differently. Arrangements of this type are called m–samples with repetitions of elements of n–types. By the general counting principle, known as the rule of product [15], the number of m–samples with repetitions of n objects equals n^m —the perfect mth power of an integer n.

For example, one can make up 10^9 different nine-digit ID strings using the ten digits $0 \dots 9$ as characters. Since we are dealing with strings here (rather than numbers) leading zeroes would be retained. Indeed, each ID string formed in this way is a nine-sample made of ten objects (digits) that may be repeated. In general, computing the number of all possible m–samples with repetitions from $1, 2, \dots, n$ types leads to the sum $1^m + 2^m + 3^m + \dots + n^m$.

4.1.1 Counting problems in geometry

There are counting problems in geometry that lead to the sums of perfect powers thereby giving geometry a combinatorial flavor.

1. **Counting segments within a segment.** The number of all segments within a line segment divided into n segments of equal length equals the sum $1 + 2 + 3 + \dots. + n$.

2. **Counting squares within a square.** The number of all squares on an $n \times n$–checkerboard equals the sum $1^2 + 2^2 + 3^2 + \dots + n^2$.

3. **Counting cubes within a cube.** The number of all cubes within an $n \times n \times n$–cube equals the sum $1^3 + 2^3 + 3^3 + \dots + n^3$.

4. **Counting hypercubes within a hypercube.** The total number of m–cubes within an n^m–cube equals the sum $1^m + 2^m + 3^m + \dots + n^m$.

4.1.2 A partial difference equation

Analyzing the development of the sums of perfect powers in the above geometric situations and setting $S(n,m) = 1^m + 2^m + 3^m + \dots + n^m$, results in the partial difference equation

$$S(n,m) = S(n-1,m) + n^m \tag{1}$$

subject to the boundary condition

$$S(1,m) = 1, m \geq 1 \tag{2}$$

The results of modeling equation (1) under condition (2) are shown in Figure 8.

	A	B	C	D	E	F	G	H	I
1	r \ n	1	2	3	4	5	6	7	8
2	1	1	3	6	10	15	21	28	36
3	2	1	5	14	30	55	91	140	204
4	3	1	9	36	100	225	441	784	1,296
5	4	1	17	98	354	979	2,275	4,676	8,772
6	5	1	33	276	1,300	4,425	12,201	29,008	61,776
7	6	1	65	794	4,890	20,515	67,171	184,820	446,964
8	7	1	129	2,316	18,700	96,825	376,761	1,200,304	3,297,456
9	8	1	257	6,818	72,354	462,979	2,142,595	7,907,396	24,684,612

Figure 8: Modelling the sums $S(n, m)$

4.2 Triangular numbers

When $m = 1$, equation 1 generates the sums of consecutive counting numbers starting from one. These sums are commonly referred to as triangular numbers [5]. An interesting fact, mentioned by Polya [13] is that the formula (3)

$$S(n, 1) = \frac{n(n+1)}{2} \tag{3}$$

can be used to derive formulas for $S(n,\ 2)$ and $S(n,\ 3)$. In what follows, the results of spreadsheet modeling of the sums $S(n, m)$ will be used to enhance the presentation of this idea by Polya. Furthermore, this idea will be extended to include the sums of higher powers. Such an extension would require an additional use of a spreadsheet that goes beyond numerical modeling of equation (2).

To begin, consider two sets of numbers, $S(n, 1)$ and $S(n, 2)$, displayed in rows 2 and 3, respectively, of the spreadsheet pictured in Figure 8. The ratios $S(n, 2)/S(n, 1)$ for $n = 1, 2, 3, \ldots$ form the following sequence

$$1, \frac{5}{3}, \frac{7}{3}, 3, \frac{11}{3}, \frac{13}{3}, \ldots$$

What do all these numbers have in common? When multiplied by three, they become consecutive odd numbers starting from three. In more general form, these ratios are

$$\frac{2n+1}{3}$$

and therefore one can come up with the following computationally driven conjecture

Proposition 1

$$S(n, 2) = \frac{2n+1}{3} S(n, 1) \tag{4}$$

Formula (4) can be proved by the method of mathematical induction, and this is discussed below. Another interesting observation is

Proposition 2

$$S(n,3) = S^2(n,1) \tag{5}$$

5 Modeling the sums $S(n,4)$ and $S(n,5)$

The idea of the sum $S(n,\, m)$ being a multiple of $S(n,1)$ and the fact that $S(n,\, m)$ is a polynomial of degree $m+1$ in n prompts the following spreadsheet-based approach to finding the sums $S(n,\, m)$ for $m > 3$. Consider $S(n,4)$, the sum of n consecutive fourth powers of natural numbers. Assuming that $S(n,1)$ is a factor of $S(n,4)$ brings about the formula

$$S\left(n,4\right) = \frac{n\left(n+1\right)}{2}\left(an^3 + bn^2 + cn + d\right)$$

where coefficients a, b, c and d are yet to be found. To this end, by using modeling data generated by the spreadsheet of Figure 8, one can develop the following system of simultaneous equations in four variables

$$\begin{aligned} S(1,4) &= 1\left(a+b+c+d\right) \\ S(2,4) &= 3\left(8a+4b+2c+d\right) \\ S(3,4) &= 6\left(27a+9b+3c+d\right) \\ S(4,4) &= 10\left(64a+16b+4c+d\right) \end{aligned} \tag{6}$$

which can be simplified to the form

$$\begin{aligned} a+b+c+d &= 1 \\ 24a+12b+6c+3d &= 17 \\ 81a+27b+9c+3d &= 49 \\ 320a+80b+20c+5d &= 177 \end{aligned} \tag{7}$$

One can use a spreadsheet function `MINVERSE` to find the values of coefficients a, b, c, and d. The corresponding spreadsheet is depicted in Figure 9, where the range `A1:D4` is entered with coefficients of the system of equations, the range `A6:D9` computes the inverse matrix of these coefficients, the range `F1:F4` is entered with the right-hand sides of the equations, and the range `F6:F9` is filled with the values of a, b, c, and d.

As a result, the following formula can be obtained

$$S\left(n,4\right) = \frac{n\left(n+1\right)\left(6n^3 + 9n^2 + n - 1\right)}{30}$$

The correctness of this formula can be verified within a spreadsheet by computing its values for different values of n and comparing the results with the numbers found for

	A	B	C	D	E	F
1	1	1	1	1		1
2	24	12	6	3		17
3	81	27	9	3		49
4	320	80	20	5		177
5						
6	- 1/6	1/6	- 1/6	1/30		2/5
7	1 1/2	-1 1/3	1 1/6	- 1/5		3/5
8	-4 1/3	3 1/6	-2 1/3	11/30		1/15
9	4	-2	1 1/3	- 1/5		- 1/15

Figure 9: Solving a system of four linear equations

$S(n,4)$ by a spreadsheet based on a partial difference equation (1). Note that fractional formatting (Figure 9, range `F6:F9`) has been used to clearly show the values of the rational coefficients a, b, d and d. Finally, this formula can be proved by the method of mathematical induction.

As an aside, it is interesting to note that, according to Polya [13, p79], the sum $S(n,4)$ can further be represented through eq (8).

$$S\left(n,4\right)=S\left(n,2\right)\frac{6S\left(n,1\right)-1}{5} \tag{8}$$

Proposition 3

$$S\left(n,4\right)=S\left(n,1\right)\frac{\left(6n^3+9n^2+n-1\right)}{15} \tag{9}$$

In much the same way, one can conjecture that

$$S\left(n,5\right)=\frac{n\left(n+1\right)}{2}\left(an^4+bn^3+cn^2+dn+e\right)$$

and then, using modeling data presented in Figure 9, find the solution to the system of simultaneous linear equations in five variables

$$\begin{aligned}
a+b+c+d+e&=1\\
16a+8b+4c+2d+e&=11\\
81a+27b+9c+3d+e&=46\\
256a+64b+16c+4d+e&=130\\
625a+125b+25c+5d+e&=295
\end{aligned}$$

As a result, the following formula can be obtained

Proposition 4

$$S\left(n,5\right)=\frac{n\left(n+1\right)}{12}\left(2n^4+4n^3+n^2-n\right) \tag{10}$$

	A	B	C	D	E	F	G
1	1	1	1	1	1		1
2	16	8	4	2	1		11
3	81	27	9	3	1		46
4	256	64	16	4	1		130
5	625	125	25	5	1		295
6							
7							
8	1/24	- 1/6	1/4	- 1/6	1/24		1/3
9	- 7/12	2 1/6	-3	1 5/6	- 5/12		2/3
10	2 23/24	-9 5/6	12 1/4	-6 5/6	1 11/24		1/6
11	-6 5/12	17 5/6	-19 1/2	10 1/6	-2 1/12		- 1/6
12	5	-10	10	-5	1		0

Figure 10: Solving a system of five linear equations

The above polynomial of the fourth degree can be factored to represent $S(n, 5)$ in the form mentioned in section 4.

Figure 10 shows the results of using the `MINVERSE` function in solving the above system of five linear equations in five variables.

To conclude this section, we note that a number of other, similar investigations may be made. For example, by first partitioning the collection of sums of powers into *sums of even powers* and *sums of odd powers*, we may seek the "patterns" given by eqs (11) and (12) respectively. Note the common factor $n^{2m-2} + O\left(n^{2m-3}\right)$.

$$S\left(n, 2m\right) = \frac{n\left(n+1\right)\left(2n+1\right)}{2\left(2m+1\right)}\left(n^{2m-2} + O\left(n^{2m-3}\right)\right) \qquad \text{for} \quad m \geq 1 \tag{11}$$

$$S\left(n, 2m+1\right) = \frac{n^2\left(n+1\right)^2}{2m+2}\left(n^{2m-2} + O\left(n^{2m-3}\right)\right) \qquad \text{for} \quad m \geq 1 \tag{12}$$

6 From conjecturing to proving

A computational environment of Figure 8 was used to develop a number of conjectures regarding the fundamental role of triangular numbers in the construction of the sums $S(n, m)$ for $m > 1$. Those conjectures, being technology-driven, were developed through inductive reasoning. In order to motivate formal justification of these conjectures, students can be given examples of incorrect generalizations resulting from inductive reasoning [11], [18]. Therefore, the next logical step of computer-enhanced mathematical activities is to provide students with experience in mathematical induction to prove their conjectures. For further examples where the spreadsheet environment is used to derive inductive hypotheses, see [2], [3], [14].

6.1 Proof of proposition 1

Proposition 1 (given by formula (4)) can be proved by the method of mathematical induction, referred to by Polya [12] as "the demonstrative phase" (p. 110). The first step of this method is to show that formula (4) is true for $n = 1$. Indeed, by definition, $S(1, 2) = 1$ and $S(1, 1) = 1$. A didactical importance of this elementary demonstration is that it allows one to see how the abstract form of Proposition 1, original (closed) definition of $S(n, m)$ and its numerical realization within a spreadsheet are connected.

The second step if the demonstrative phase is to test the transition from k to $k + 1$ [12]. Assuming that formula (4) is true for $n = k$, one has to prove that

$$S(k+1,2) = \frac{2k+3}{3}S(k+1,1) \tag{13}$$

Indeed, according to equation (1) and assumption (4)

$$S(k+1,2) = S(k,2) + (k+1)^2 = \frac{2k+1}{3}S(k,1) + (k+1)^2$$

Therefore, one has to prove that

$$\frac{2k+1}{3}S(k,1) + (k+1)^2 = \frac{2k+3}{3}S(k+1,1)$$

The last equality can be rewritten in the form

$$\frac{2k}{3}(S(k+1,1) - S(k,1)) + S(k+1,1) - \frac{1}{3}S(k,1) = (k+1)^2 \tag{14}$$

It follows from formula (14) that $S(k + 1, 1) - S(k, 1) = k + 1$. Thus equality (14) can be replaced by

$$\frac{2k}{3}(k+1) + \frac{2}{3}S(k,1) + k + 1 = (k+1)^2$$

Finally, applying formula (14) to the last equality one gets

$$\frac{2k(k+1)}{3} + \frac{k(k+1)}{3} + k + 1 = (k+1)^2$$

thus proving equality (13).

6.2 Proof of proposition 2

Again, we employ the method of mathematical induction. We have $S(1, 3) = 1$ and $S(1, 1) = 1$. Proceeding in a similar manner to the proof of Proposition 1, we assume that $S(k, 3) = S(k, 1)^2$. One may then write:

$$
\begin{aligned}
S(k+1,3) &= S(k,3) + (k+1)^3 \\
&= S^2(k,3) + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \left(\frac{k+1}{2}\right)^2 \left(k^2+4k+4\right) \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 \\
&= S^2(k+1,1)
\end{aligned}
$$

6.3 Proof of proposition 3

Again, we proceed inductively. When $n=1$, both sides of the eq (9) are equal to unity. Assuming that

$$S(k,4) = S(k,1)\,\frac{6k^3+9k^2+k-1}{15}$$

one has to show that

$$S(k+1,4) = S(k+1,1)\,\frac{6(k+1)^3+9(k+1)^2+k}{15}$$

Indeed,

$$
\begin{aligned}
S(k+1,4) &= S(k,4) + (k+1)^4 \\
&= S(k,1)\,\frac{\left(6k^3+9k^2+k-1\right)}{15} + (k+1)^4 \\
&= \frac{k(k+1)}{2} \times \frac{\left(6k^3+9k^2+k-1\right)}{15} + (k+1)^4 \\
&= \frac{k+1}{30}\left(6k^4+9k^3+k^2-k\right) + \frac{30(k+1)^4}{30} \\
&= \frac{k+1}{30}\left(\left(6k^4+9k^3+k^2-k\right) + 30(k+1)^3\right) \\
&= \frac{(k+1)\left(6k^4+39k^3+91k^2+89k+30\right)}{30}
\end{aligned}
$$

On the other hand,

$$S(k+1,4) = \frac{(k+1)(k+2)}{2} \times \frac{6(k+1)^3+9(k+1)^2+k}{15}$$

It remains to be shown that $(k+2)\left(6(k+1)^3+9(k+1)^2+k\right) = 6k^4+39k^3+91k^2+$

$89k + 30$. A few lines of algebra achieves this. In much the same way, a similar formula for $S(n, 5)$ can be proved. However, the algebra starts to become quite tedious.

7 The Bernoulli connection

In section 4, we considered sums of powers and defined the quantity $S(n, m)$. It is well-known that sums of positive integer powers of the natural numbers, i.e., $S(n, m)$, are closely related to the Bernoulli family of polynomials. Denote by $B_m(x)$ the *Bernoulli polynomial* of degree m. Then eq (15) gives us the connection [4, p. 804].

$$S(n, m) = \frac{B_{m+1}(n+1) - B_{m+1}(0)}{m+1} \tag{15}$$

The *Bernoulli numbers* are the constant terms in the Bernoulli polynomials. Denoting the mth Bernoulli number by b_m, we have $b_m = B_m(0)$. It is of interest to investigate whether a reasonably simple spreadsheet model may be used to determine the Bernoulli polynomials from sums of powers. However, we first briefly describe a model to generate the Bernoulli numbers, via the recurrence of eq (16). We start with $b_0 = 1$, and then recursively apply eq (16) for $m \geq 1$. Other methods are possible, but this is perhaps the simplest.

$$\sum_{k=0}^{m} \binom{m+1}{j} b_j = 0 \qquad \text{with } m \geq 1 \tag{16}$$

The Bernoulli polynomials $B_n(x)$ may then be computed by a simple sequence of antiderivatives. We have $B_0(x) = 1$, and then recursive application of (17) for $n \geq 1$ yields the sequence of polynomials.

$$B_n(x) = b_n + \int_0^x nB_{n-1}(t)\,dt \tag{17}$$

Equations (16) and (17) are the basis of the spreadsheet model shown in Figure 11.

8 Conclusion and possible further investigations

We have considered several examples of the pervasive, powerful and related concepts of recursion and induction. In a variety of contexts, the cognitive frameworks of these fundamental mathematical principles are extremely useful. They provide not only the certainty of rigorous proof, but also compactness of expression and much mathematical insight. They also offer many fruitful opportunities for exploration by both teacher and student. In many instances, this is particularly well illustrated by casting the example under discussion into the spreadsheet environment. While not as complete as some may wish, the spreadsheet has reasonable support for recurrences. Indeed, the ease with which the final, important inductive step of the implementation a simple recurrence may be achieved with a double-click of the mouse is rather deceptive. It is very important, then, that the teacher be able to help the students bridge the considerable semantic gap

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
1			COMBIN(m, j) transpose																					
2	B[m]	m \ j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
3	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	- 1/2	1		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
5	1/6	2			1	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153	171	190	210
6	0	3				1	4	10	20	35	56	84	120	165	220	286	364	455	560	680	816	969	1140	1330
7	- 1/30	4					1	5	15	35	70	126	210	330	495	715	1001	1365	1820	2380	3060	3876	4845	5985
8	-0	5						1	6	21	56	126	252	462	792	1287	2002	3003	4368	6188	8568	11628	15504	20349
9	1/42	6							1	7	28	84	210	462	924	1716	3003	5005	8008	12376	18564	27132	38760	54264
10	0	7								1	8	36	120	330	792	1716	3432	6435	11440	19448	31824	50388	77520	116280
11	- 1/30	8									1	9	45	165	495	1287	3003	6435	12870	24310	43758	75582	125970	203490
12	0	9										1	10	55	220	715	2002	5005	11440	24310	48620	92378	167960	293930
13	5/66	10											1	11	66	286	1001	3003	8008	19448	43758	92378	184756	352716
14	-0	11												1	12	78	364	1365	4368	12376	31824	75582	167960	352716
15	- 20/79	12													1	13	91	455	1820	6188	18564	50388	125970	293930
16	0	13														1	14	105	560	2380	8568	27132	77520	203490
17	1 1/6	14															1	15	120	680	3060	11628	38760	116280
18	0	15																1	16	136	816	3876	15504	54264
19	-7 7/76	16																	1	17	153	969	4845	20349
20	0	17																		1	18	171	1140	5985
21	54 34/35	18																			1	19	190	1330
22	-0	19																				1	20	210
23	-529 1/8	20																					1	21
24		21																						1

Figure 11: Bernoulli numbers in Excel

between spreadsheet representation, and mathematical abstraction. This point has been discussed in greater detail in [14].

The authors' teaching experience indicates that students learn abstract ideas best when these ideas are supported by concrete problems that have lucid and easy-to-understand visual representations. As this paper has demonstrated, the idea of recursion, associated originally with counting techniques that are offered as alternatives to direct counting, can be communicated through making reference to physical models involving concrete objects such as counters, grids, boxes, combination locks, etc. In turn, the spreadsheet facility of recurrent counting can be used in generating numerical data as a setting for one's engagement in pattern recognition and conjecturing. In that way, such high level mathematical activity as a formal proof of a computationally-driven conjecture can be preceded by a computational experiment grounded in a real life situation. This approach to the teaching of mathematics can be presented through the following sequence: from concrete problem to its physical model to a mathematical model to computational modeling to technology-enabled conjecturing and finally to a formal mathematical proof. Adding a new dimension to Polya's views on teaching and learning problem solving, this general pedagogical idea was applied in this paper to the concept of recursion using a spreadsheet as a digital medium.

References

[1] Abramovich, S. (1995). Technology-Motivated Teaching of Advanced Topics in Discrete Mathematics. *Journal of Computers in Mathematics and Science Teaching*,**14**(3), 391–418.

[2] Abramovich, S. (2007). Uncovering Hidden Mathematics of the Multiplication Table Using Spreadsheets, *Spreadsheets in Education*, **2**(2): 158–176.

[3] Abramovich, S. and Sugden, S. (2008). Diophantine Equations as a Context for Technology-enhanced Training in Conjecturing and Proving. *PRIMUS*, **18**(3), 1–19.

[4] Abramowitz, M and Stegun, I.A. (1965). *Handbook of Mathematical Functions*, Dover.

[5] Beiler, A. (1966). *Recreations in the Theory of Numbers*. New York: Dover.

[6] Conference Board of the Mathematical Sciences. (2001). *The Mathematical Education of Teachers*. Washington, D.C.: Mathematical Association of America.

[7] Grassmann, H. (1861). *Lehrbuch der Arithmetik*. Berlin: Th. Chr. Fr. Enslin.

[8] Heins, A.E. (1941). On the solution of partial difference equations. *American Journal of Mathematics*, **63**(2): 435–442.

[9] Weisstein, E. W. "k-Statistic." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/k-Statistic.html. Accessed 15 Nov 2007.

[10] Kotiah, T. C. T. (1993). Sums of powers of integers – A review. I*nternational Journal of Mathematical Education in Science and Technology*, **24**(6), 863–847.

[11] Lutzer, C.V. (2005). The role of proof in mathematics and mathematics education. *PRIMUS*, **15**(4), 374–384.

[12] Polya, G. (1954). *Induction and Analogy in Mathematics*. Princeton, NJ: Princeton University Press.

[13] Polya, G. (1962), *Mathematical Discovery, v. 1*. New York: John Wiley and Sons.

[14] Sugden, S.J. (2001). Bits, Binary, Binomials and Recursion: Helping IT Students Understand Mathematical Induction. *Quaestiones mathematicae*, Journal of The South African Mathematical Society, Supplement #1, 133–140.

[15] Vilenkin, N.Y. (1971). *Combinatorics*. New York: Academic Press.

[16] Wang, H. (1957). The Axiomatization of Arithmetic. *Journal of Symbolic Logic*, **22**(2), 145–158.

[17] http://scienceworld.wolfram.com/biography/BernoulliJakob.html. Accessed 15 Nov 2007.

[18] Wu, H. (1999). Professional development of mathematics teachers. *Notices of the American Mathematical Society*, **46**(5), 535–542.