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On Mathematical Problem Posing by Elementary Pre-teachers: The Case of Spreadsheets

Sergei Abramovich
State University of New York at Potsdam, abramovs@potsdam.edu

Eun Kyeong Cho
University of New Hampshire, eunkyeong.cho@unh.edu

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Sergei Abramovich
State University of New York at Potsdam
abramovs@potsdam.edu

Eun Kyeong Cho
University of New Hampshire
eunkyeong.cho@unh.edu

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1 Introduction

Problem posing has long been recognized as an important pedagogical tool in the teaching of mathematics [10], [18], [19], [23], [24], [33], [35]. Influenced by research carried out in this area, the National Council of Teachers of Mathematics [27] has referred to problem posing as "an activity that is at the heart of doing mathematics" (p. 138) and, acknowledging the advent of technology into the classroom, noted "computer programs can engage students in posing and solving problems" (p. 76). Towards this end, already in the early 1990s, the advances in educational applications of software tools and the importance of problem posing for the development of mathematics have been linked together in the standards for teachers: "technology may be used to enhance and extend mathematics learning and teaching ... in the areas of problem posing and problem solving ... [allowing] students to design their own explorations and create their own mathematics" [28, p134]. Most commonly known examples of using technology for posing mathematical problems include the development of conjectures in dynamic geometry environments (e.g., [26], [45]). Yet nowadays, the appropriate use of spreadsheets

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enables other areas of pre-college mathematics to be explored from a problem-posing perspective.

At the elementary level, by using a spreadsheet, one can turn a routine arithmetical problem into a challenging mathematical investigation [1]. Through such an investigation, numbers used to pose a problem may become parameters that one can alter and test within the software and then choose (or reject) as data for a new problem. In other words, the availability of technology-enabled variation of the conceptual and syntactic structure of an existing problem statement provides a source of new problems [23].

At the secondary level, one's ability to interpret correctly the results of spreadsheet modeling of a single Diophantine equation in two or three variables can be used for posing a variety of word problems that lead to the systems of algebraic equations, both linear and non-linear, with friendly solutions [2]. When such use of spreadsheets is integrated into mathematics teacher education, prospective teachers can gain valuable research-like experience in taking intellectual risk by posing mathematically meaningful problems based on numerical and graphical patterns observed and then attempting to solve these problems. In that way, a spreadsheet becomes an agent of mathematical activities that require teachers to use higher order thinking and reasoning skills.

While there are many studies concerning mathematics teacher education and problem posing [13], [15], [21], [40], they do not deal with the use of a spreadsheet, or technology, in general. In the focus of this article is the authors' analysis of problems posed by prospective elementary teachers (referred to below as teachers) using a spreadsheet. In what follows, problem posing as a form of cognitive activity in mathematics teacher education is understood as the unity of "the creation of a new problem from a situation or experience" [38, p20] using technology and finding solution to the so created problem. The article concerns certain problem-posing activities that computer spreadsheet naturally affords, allowing teachers to avoid using algebra otherwise required for posing grade-appropriate problems with friendly data. It is the combination of elementary mathematics curriculum and spreadsheet's computational capability to support the curriculum that limits the scope of problem-posing situations discussed below to problems bounded by discrete elementary concepts.

It should also be noted that by using a spreadsheet as a modeling tool in an educational context, one can come across various challenging mathematical situations, which, otherwise, are unlikely to be encountered [3, 4]. Consequently, problems motivated by such use of a spreadsheet are unlikely to be posed. An example of such a problem that can be introduced to prospective teachers as a window on more sophisticated mathematical explorations (in the context of simple problem-posing experiments) is mentioned briefly at the end of the article as its discussion merits a separate study. In general, it is the emergence of computers that enabled new ways in which problems can be posed [9], sometimes opening up new areas of mathematics research [20].

The article reflects on a number of courses taught at SUNY Potsdam that provide opportunities for teachers in posing developmentally and culturally appropriate arithmetical word problems. These include problems the solutions of which are generated within a spreadsheet environment designed by the course instructor specifically for the purpose of problem posing. The authors assume that problems posed by teachers using a spreadsheet are free from conceptual flaws as the flawless use of mathematical concepts is integrated into a computational environment through its construction by 'a more knowledgeable other' [43]. Although the teachers were not introduced to the details of spreadsheet programming, the article includes Appendix explaining how to program the environments involved.

2 The didactic complexity of problem posing with spreadsheets

The use of a spreadsheet in problem posing can be characterized as a cultural support of the teachers' ability to develop new curriculum materials for a mathematics classroom. It is cultural in a sense that teachers are encouraged to make use of a spreadsheet, developed by advanced members of a technological culture for various practical and scholastic purposes [34] and retrofitted by others for mathematics education [7]. It is support in a sense that, in the specific context of problem posing, teachers learn how to put to work the spreadsheet's computational power in order to transform an input (data for a problem) into an output (the problem's solution). In other words, a spreadsheet generates solutions to a problem that is about to be posed. This implies that problem posing and problem solving are inherently linked to each other through the use of technology.

Introducing a spreadsheet as a cultural support of mathematical problem posing, one should note that the existence and availability of computational tools do not guarantee their appropriate application unless one examines the effects of support in the context of using the tools [12]. With this in mind, the ideas of this article resulted from the authors' analysis of problem-posing activities by teachers. As part of the activities, teachers were expected to critically reflect on their own problems and discuss the role of computational environments they used to develop the problems.

However, in order for a spreadsheet to have a positive effect on problem posing, one should not only know how to use the tool but, more important, how to interpret the results of spreadsheet modeling. This interpretation requires understanding of what may be called the didactical coherence of a problem. This notion is defined through the Venn diagram of Figure 1 as the intersection of three mutually non-exclusive components: numerical coherence, contextual coherence and pedagogical coherence.

While the three components will be explained in the context of spreadsheet use in detail below, here is their brief description. This description can be applied to a non-technological problem posing as well. Numerical coherence of a problem refers to its formal solvability within a given number system. Put another way, if a problem has a solution expressed by a number (or a set of numbers), it is numerically coherent. Contextual coherence of a problem comes into play when its solution should be interpreted in terms of a context within which problem posing occurs. Besides the need to understand the context of a problem statement, it requires one's appreciation of hidden assumptions grounded into one's real-life experience and cultural background. Pedagogical coherence

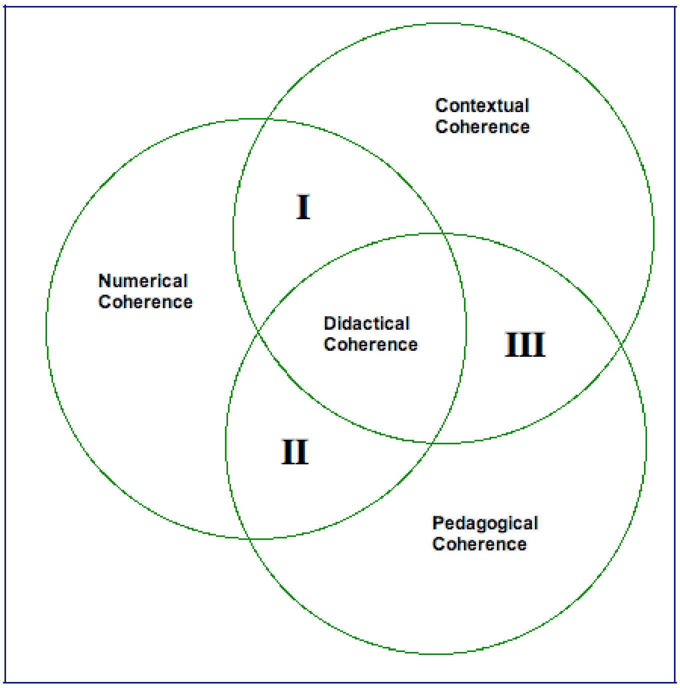

Figure 1: Didactical coherence defined through a Venn diagram

of a problem includes, but is not limited to, attention to students' on-task behavior, the absence of (or minimizing) extraneous data, the level of syntactic complexity [39], grade appropriateness, and a method of solution expected. In other words, a problem should be designed "with the learner in mind" [41, p416]. At the junction of the three components, didactical coherence in problem posing is achieved. In particular, the notion of didactical coherence can be applied to posing problems using spreadsheets.

The main thesis of this article is that attending to the notion of didactical coherence of a problem allows for a greater effect of computational support on problem posing provided by a spreadsheet. The Venn diagram of Figure 1, being a cultural tool itself, didactically supports spreadsheet-based problem posing. By learning to use the Venn diagram as a tool that informs problem posing, teachers develop higher order thinking and reasoning skills and gain valuable research-like experience in preparing their own curriculum materials.

3 Numerical coherence

In order to decide whether a posed problem is numerically coherent or not; that is, whether, numbers involved have been chosen correctly, one has to solve the problem. As mentioned by Grossman, Schoenfeld, and Lee [22], the effectiveness of learning mathematics through problem solving depends on teachers' knowledge of "how to solve problems they pose to students" (p. 205). This problem-solving phase of a problem-posing activity has been a reasonable expectation both for traditional [44] and technology-enhanced [46] problem-posing activities at the elementary and middle school levels, respectively. The concept of numerical coherence forms a robust link between problem posing and problem solving.

Nevertheless, as the authors have observed, teachers often fail to recognize that problem posing "is a platform from which further development proceeds" [14, p23]. Whereas research typically supports this unifying position [39], [40], there are studies that suggest the existence of a dichotomy between problem posing and problem solving. For example, Crespo [13] argued that because "knowing mathematics for oneself may not be a reliable predictor of good problem posing practice" (p. 266), no clear connection between problem posing and problem solving can be established. This view may be due to the fact that just like many children traditionally see their role at school as an engagement in answering, not asking questions [42], many teachers believe (or want to believe) that problem posing is a practice in asking, not answering questions. However, in the case of the spreadsheet-based support of problem posing, the answer is there as a computational setting is designed to generate the answer. In order to recognize the answer within a computational environment, one has to interpret correctly the results of spreadsheet modeling. As an illustration, consider a problem posed by one of the teachers.

Problem 1 *You want to buy a 40-cent [Figure 2, cell* `A2`*] piece of candy at Kinney's. You have a piggy bank full of quarters, dimes and nickels [cells* `E2, F2, G2`*, respectively]. What are the different combinations of these coins you could take with you to the store to pay for the piece of candy?*

The spreadsheet of Figure 2 (see Appendix for programming details) shows that there are seven ways to pay for the candy: the software counts the number of non-empty cells in the range `D5:J9` and displays this number in cell `A8`. One can say that Problem 1, which mathematical model is the equation $25x + 10y + 5z = 40$, is numerically coherent. A simple alteration of data (that can be done in multiple ways) can bring about many more numerically coherent problems of this type. For example, all Diophantine equations of the form $5kx + 2ky + kz = 8k$ when modeled within the spreadsheet yield the same output in the range `D5:J9` (Figure 2). Without changing input in the spreadsheet (i.e., the content of cells `A2, E2, F2, G2`), the insight into numerical coherence of Problem 1 may become a source of new problems.

4 Contextual coherence

The universal character and descriptive power of mathematical concepts allows for a single mathematical model to describe multiple phenomena arising in diverse situations. Being embedded by a designer in a problem-posing tool, a mathematical model can be associated with multiple contexts. Rather than being represented by a physical environment, a context is epitomized "by what people are doing and where and when they

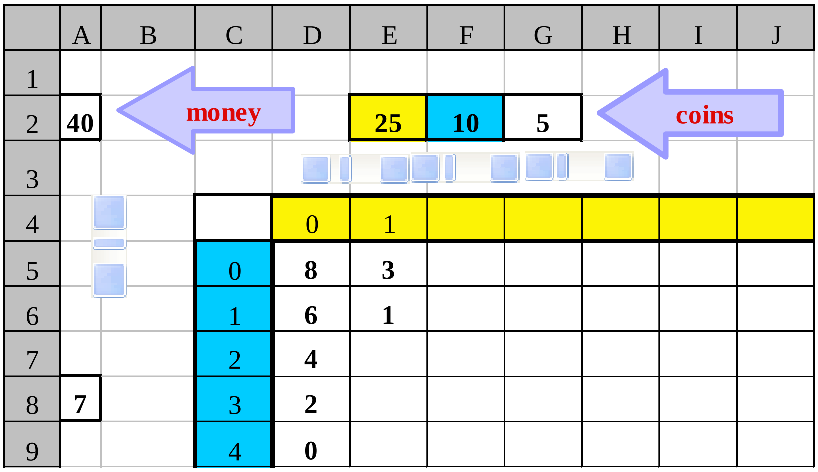

Figure 2: Spreadsheet solution to Problem 1

are doing it" [17, p22]. To understand a context, one needs to possess a certain amount of social competence, which may vary across cultures, age levels, and social groups. Moreover, as Silver [38] noted, "issues of morality, justice and human relationships may have been as important to some students as issues of formal mathematics" (p. 26) when posing problems. All this requires a problem statement to be consistent with cultural background and social competence of a mathematics classroom involving culturally heterogeneous groups of pupils.

It should be noted, whereas Problem 1 may typically be considered contextually coherent for an U.S. classroom, pupils from immigrant communities may have difficulty in ascribing numerical meaning to the names of the coins. (For example, many countries, including Australia, Korea, Mexico, Russia, to name just a few, do not have a numerical equivalent to an U.S. quarter dollar coin.) Therefore, one cannot say that Problem 1 is contextually coherent for this group of pupils. Generally speaking, contextual coherence of a problem is a variable attribute. Just as without the mastery of base ten system – a cultural tool designed to support one's counting abilities – one cannot understand the numerical meaning of a multi-digit number, without the mastery of another cultural tool – a currency system of a particular country – one cannot solve a problem which context does not relate well to one's cultural background. This is consistent with Cobb's [11] position that the learning of arithmetic involves the mastery of the numeration system as a cultural tool. In formulating problems in context [36], one has to be sensitive to this issue, as many pupils tend to tolerate mathematical errors when "they feel no personal ownership of mathematics" [38, p26].

5 Pedagogical coherence

The effectiveness of using an open-ended approach in the teaching of mathematics has been repeatedly emphasized in mathematics education research [8], [25], [30], [32], [47]. This approach includes the use of problems with more than one correct answer. By analyzing problems of that type posed by elementary pre-teachers using a spreadsheet, it has been found that the straightforward technological support of problem posing often leads to problems that lack pedagogical coherence.

To illustrate this concept, consider another problem posed by one of the teachers.

Problem 2 *Andy and Sarah have, respectively, $2 and 50 cents in quarters, dimes, and nickels. In how many [different] ways can they share money so that Andy would have as much money as Sarah?*

One can first conclude that if Andy gives to Sarah 75 cents, they would have equal amounts of money. Discounting the cases of money exchange and disregarding specific coins involved [1] – otherwise Problem 2 would have too many levels of complexity, Figure 3 shows there might be eighteen different ways of making 75 cents out of quarters, dimes, and nickels. However, if one is really concerned with solving self-posed problems, one may wonder if a problem with so many answers, let alone a multitude of its interpretations, is appropriate in terms of expected on-task behavior and the development of insight from which a method – a way of thinking that differs "from mere observation and speculation" [37, p7] can emerge. It appears that one would hardly become motivated to stay on task that requires adding numbers over and over without 'seeing light at the end of the tunnel.' By the same token, the development of a method would be shadowed by an endless addition exercise. This raises the question: Is the support of problem posing provided at the computational level effective in this case?

Some teachers the authors worked with have been observed formulating problems similar to Problem 2, thus paying little attention to the notion of pedagogical coherence. While being users of technology themselves, the teachers seemed to overlook the fact that pupils – the primary consumers of their pedagogical knowledge – would typically not be using a spreadsheet to solve a problem. Rather than reflecting on pedagogy associated with their problems, the teachers focused on answers as numerical entities. This observation is consistent with the finding of Thompson, Carlson and Silverman [41] regarding pre-service teachers' belief that the purpose of mathematical tasks they were given was for them to answer specific questions rather than to build coherent meaning through exploring the tasks and their non-traditional extensions.

In revising course materials for the teachers related to problem posing, the notion of pedagogical coherence of a problem was introduced. As the authors' analysis of teachers' portfolios (developed for the purpose of assessment) indicates, this resulted in a more thoughtful use of technology in problem posing by teachers. That is, the notion of pedagogical coherence, by providing support of problem posing at the coherence level, can make a positive effect on one's problem posing abilities.

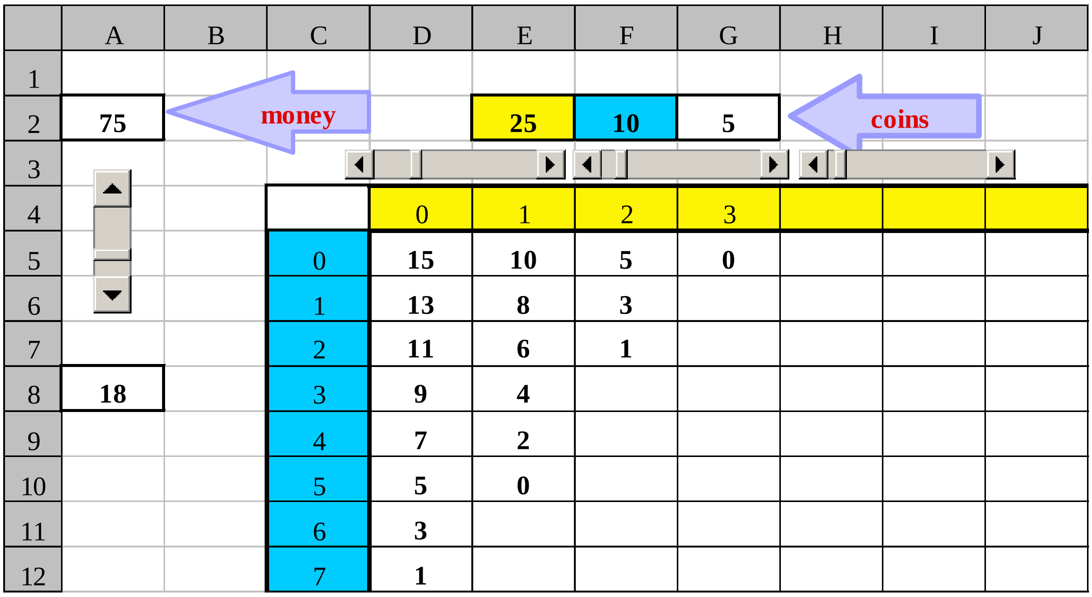

Figure 3: A problem with 18 answers

6 Analysis of problem posing activities by teachers

Typically, problems posed by teachers using a spreadsheet are numerically coherent. Yet, because of the importance of understanding of how the environment works, the teachers were asked to reflect on their problems by answering a number of questions about problems posed, among them: "Why do you think the spreadsheet generated all solutions to your problem?" Following is the answer for Problem 1:

> *"I do think the spreadsheet [Figure 2] generated all solutions to my problem. The smallest coin one could use to make 40 cents was a nickel. The spreadsheet listed that 8 nickels [cell D5] would equal 40 cents. The largest coin one could use to make 40 cents was a quarter. The spreadsheet also listed that 1 quarter +1 dime + 1 nickel [cells E4, C6, E6, respectively] would equal 40 cents and 1 quarter +3 nickels [cells E4 and E5, respectively] would equal 40 cents."*

In this response, the teacher attempted to develop a systematic way of solving the problem in the absence of technology by analyzing the results of spreadsheet modeling. Admitting an important role of technology in problem posing, she went on to suggest another (simpler) question that may yield from this analysis: *"If you use no quarters and no dimes, how many nickels will you use to make 40 cents?"* Again, it is the understanding of how the spreadsheet supports problem posing that allows one to develop skills in organized mathematical exploration thereby being able to operate at a higher cognitive level without pressing buttons on a computer.

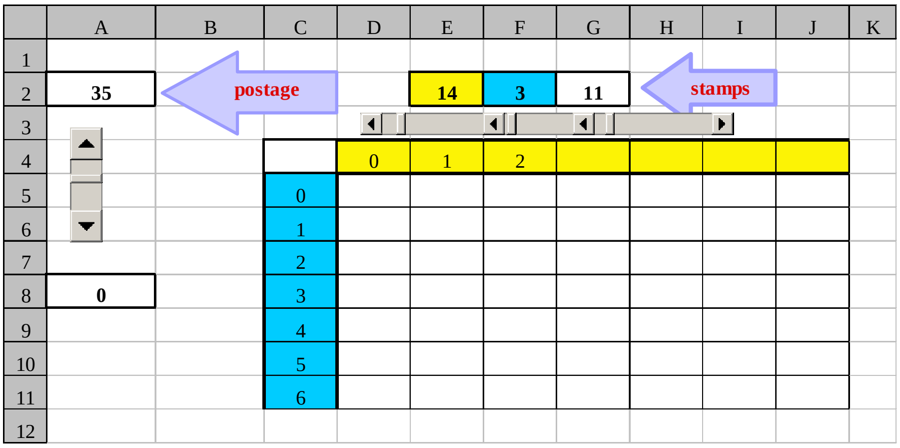

Figure 4: Pseudo-numerical incoherence

An appreciation of what the spreadsheet actually adds to problem posing can be found in the following remark by a teacher: *"I also think that the use of the spreadsheet will build confidence in educators as it is a tool to use to confirm one's problems are correct and appropriate for the students who will solve them."* In terms of the appropriateness of problems posed for students, another teacher noted *"Things like the partitions spreadsheet allow teachers to challenge their academically gifted students while still allowing other more challenged students the opportunity to succeed. Technology allows teachers the chance to quickly, yet effectively gear the curriculum and problem sets toward individual learners. The efficiency in which problems can be generated allows less stress to be put on teachers which should allow them more time to concentrate on students rather than creating worksheets."* This note indicates the usefulness of the tool to support teachers' design of differentiated curriculum for students of different abilities. It shows how technology, in general, can provide a paradigm shift from product-oriented, worksheet-based assessment to process-oriented teaching through problem solving [28].

It should be noted that the effect of support at the coherence level could be expected only if one understands as to how support at the computational level operates. For example, the spreadsheet of Figure 4 appears displaying a numerically incoherent problem (showing zero in cell `A8`); nonetheless, there are two ways to make the 35-cent postage (cell `A2`) out of the 14-cent, 11-cent and 3-cent stamp denominations (cells `E2, G2, F2`, respectively). By placing the smallest denomination in cell `F2`, one does not take into account the relationship between data and the corresponding ranges (see Appendix for details). In doing so, one deals with an incomplete range for the 3-cent stamp (range `C5:C11`) reducing it from eleven stamps to six stamps only. As a result, one misses combinations with seven and eight stamps of that denomination. In order to avoid such a pseudo-numerical incoherence, the smallest denomination should be placed in cell `G2`. These issues are discussed with the teachers as the problem-posing environment is introduced.

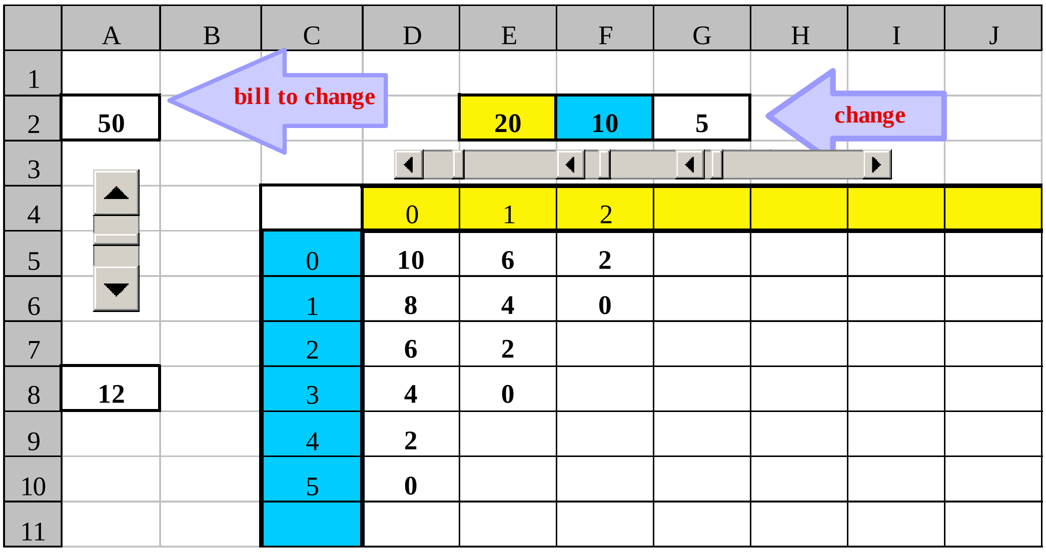

Figure 5: A problem proposed by a student

Most of the teachers, when asked to evaluate their own problems in terms of pedagogical coherence, focused on the appropriate number of solutions: (e.g., *"Yes, there are not too many solutions"* or *"Yes, ... there are not so many solutions ... it would be possible for the students to find them all"*). Yet, some responses focused on grade level, pupil's on-task behavior, group work and the development of systematic reasoning (method) by using a problem with multiple answers as a tool. As one teacher (who posed a problem of finding all the ways to change a $50 bill using $20, $10, and $5 bills – see Figure 5) remarked:

"My problem is definitely pedagogically coherent for third and fourth graders. Pedagogically, it makes a great pair-share opportunity for third graders and gives an appropriate challenge to fourth graders. They are capable of finding all [twelve] solutions ... because they have begun to develop an organization about their thinking."

Finally, some pupils may have difficulty ascribing numerical meaning to the names of the coins mentioned in Problem 2. For that group, the problem may be contextually incoherent. In that case, it does not belong in region I in the Venn diagram of Figure 1. A useful practice for teachers could be to alter the context while preserving the numerical structure of the problem as well as to reduce its linguistic complexity through the recourse to meta-context – a picture in which coins, words, and numbers are put into one-to-one correspondence [1].

7 Overcoming limitations of a three-dimensional spreadsheet

It should be noted that pedagogical coherence of a problem depends on the expected method of solution. Often, as pupils learn to use more and more sophisticated mathematical tools, a pedagogically incoherent problem for a lower-grade level becomes pedagogically coherent for a higher-grade level. A classic example of that kind is a legend about Carl Friedrich Gauss who, at an early age, was able to avoid the straightforward summation of 100 consecutive natural numbers (a pedagogically incoherent problem) by recognizing a pattern that the numbers follow [16, p237]. However, as mentioned elsewhere [5], the opposite relationship can be observed: a pedagogically coherent problem for a lower-grade level may become pedagogically incoherent for a higher-grade level. For example, whereas for a six years old pupil (who uses concrete materials – a non-computational technology – as means of problem solving) the tasks of arranging 24 students and 25 students into four groups to do a team work are at the same level of complexity, for a ten years old pupil the latter case is conceptually more difficult as it requires the interpretation of the meaning of remainder [31].

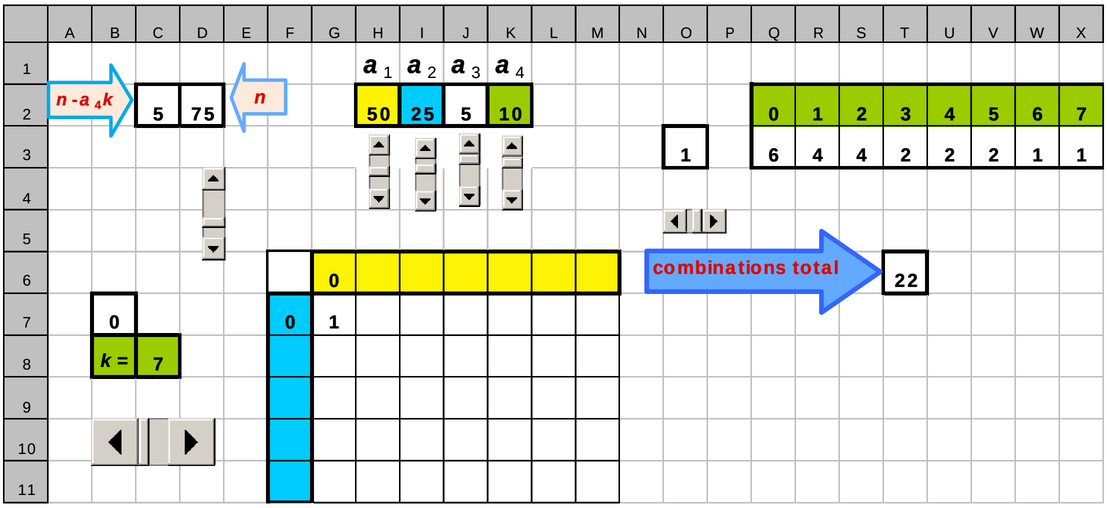

Figure 6: Spreadsheet as a four-dimensional partitioner

Just like a pedagogically incoherent problem for a lower grade level can motivate the development of a new method that would make it coherent for a higher-grade level, in the age of technology such a problem can motivate the development of a new computational environment within which similar yet pedagogically (and numerically) coherent problems can be posed. For example, one of the teachers inquired whether a spreadsheet could be used to extend the problem of making 75 cents to include half a dollar also. To address this inquiry, a new computational environment was developed (Figure 6, see Appendix for description and programming details). This new environment, being qualitatively different from the one used for a three-dimensional modeling, enables teachers to formulate didactically coherent problems of the higher degree of complexity. By using the environment, one can also specify coins that Andy has. For example, the spreadsheet pictured in Figure 7 shows that one way (out of 22 total – see Figure 6, cell `T6`) of making 75 cents (cell `D2`) out of nickels, dimes, quarters, and half-dollars (cells `J2, K2, I2, H2`, respectively) is through the combination of four dimes (cell `C8`), two nickels (cell `G8`), one quarter (cell `F8`), and zero half-dollars (cell `G6`). In that way, when no tool is available to decide if a problem, posed as an extension of an existing one, is didactically coherent or not, by constructing such a tool one provides problem posing with cultural support of a higher order through which two qualitatively different levels of problem posing – computational and didactical – can be supported.

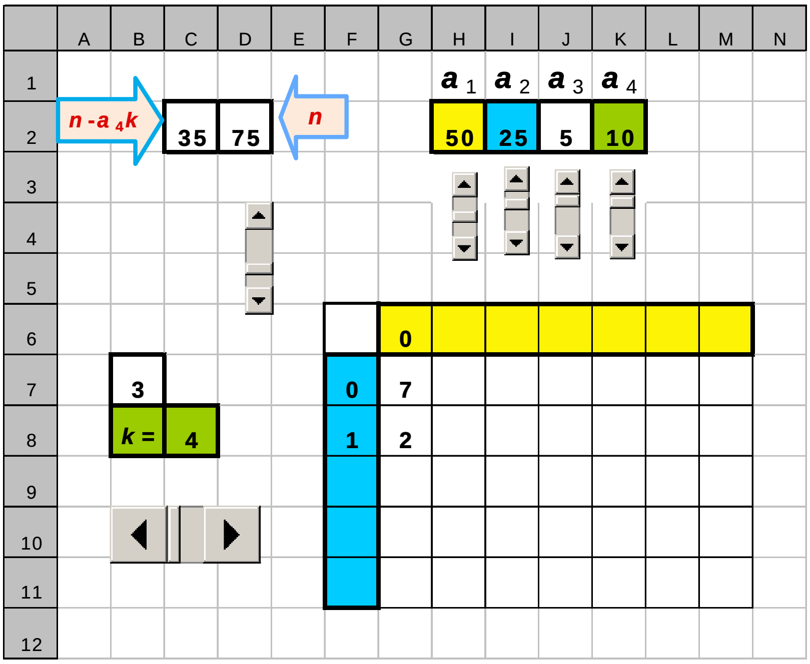

Figure 7: Showing $75 = 2 \times 5 + 4 \times 10 + 1 \times 25 + 0 \times 50$

8 Concluding remarks

This article discussed the use of an electronic spreadsheet by elementary pre-teachers in the context of mathematical problem posing. The authors argued that whereas computational support provided by a spreadsheet (designed by 'a knowledgeable other') enables formulated problems to be free from conceptual flaws, an important didactical task for the teachers is to decide the appropriateness of the problems in terms of their contextual, numerical, and pedagogical coherences. By being able to make this kind of decision, one demonstrates higher order thinking skills that support problem posing beyond operating a spreadsheet at the computational level. It is at the coherence level that the integration of problem posing and problem solving occurs.

By gaining a research-like experience in curriculum development through the use of technology, teachers can use their technological expertise in problem posing as an agency for recognizing both the profound effectiveness of systematic reasoning as well as the complexity of mathematical ideas that may emerge from familiar situations. In particular, the pedagogical and mathematical power of a spreadsheet allows one to delve into the way the tool supports problem posing and inductively discover interesting results about mathematical structures involved; results that traditionally are not available to teachers because of the complexity of mathematics associated with formal methods of exploring those structures. For example, the spreadsheet of Figure 5 shows 12 ways of changing a $50 bill using $20, $10, and $5 bills. One can also note that solutions generated by the spreadsheet in the range `D5:F10` are arranged in blocks forming an arithmetic progression. Such geometry of numbers and their connection to the number of solutions of the corresponding Diophantine equation, are due to the coefficients of the equation. Further elaboration of this remark would require considerations that are beyond the scope of this article. The authors plan to discuss this unexpected by-product of mathematical problem posing with a spreadsheet in a separate article.

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A Spreadsheet details

The purpose of this section is to provide details of spreadsheet programming of problem-posing environments discussed in this article. Note that mathematical ideas underlying the programming of similar environments, especially those based on the use of inequalities, are discussed in [6]. A connection between spreadsheet modeling of three-dimensional problems and systematic reasoning with three variables is discussed in [1]. All environments introduced in this article are based on Excel 2008 (Mac) or 2007 (Windows) spreadsheets but they are equally effective with Excel 2004 (Mac) or 2003 (Windows) versions. Below, following [6], the notation `(A1)=` will be used to present a formula defined in cell `A1`.

A.1 Spreadsheet programming for Figures 2-5

Consider the Diophantine equation

$$a_1x + a_2y + a_3z = n \tag{1}$$

which describes a partition of n into summands a_1, a_2, a_3. The four parameters are defined, respectively, in the slider-controlled cells `A2, E2, F2`, and `G2`, which, in turn, are given the names `n, a_1, a_2,` and `a_3`. The spreadsheet of Figure 2 (as well as Figures 3-5) is designed to generate values of variables x, y, and z that provide solutions to equation 1.

One can establish the inequalities $x \leq INT(n/a_1)$ and $y \leq INT(n/a_2)$, which enable ranges for variables x and y in equation 1 to be generated through the following spreadsheet formulas

`(D4)= =0; (C5)= =0; (E4)= =IF(D4<INT(n/a_1),1+D4," ")` – replicated across row 4; `(C6)= =IF(C5<INT(n/a_2),1+C5," ")` – replicated down column `C`.

The values of z satisfying equation 1 can be found as follows:

```
(D5)= =IF(OR(D$4=" ",$C5=" ")," ",
IF(AND(n-D$4*a_1-$C5*a_2>=0,
MOD(n-D$4*a_1-$C5*a_2,a_3)=0),(n-D$4*a_1-$C5*a_2)/a_3, " ")).
```

The formula in cell `D5` is replicated to cell `J9`.

Finally, the total number of solutions of equation 1 can be counted as follows:

`(A8)= = COUNT(D4:J9).`

A.2 Spreadsheet programming for Figure 6

Consider the Diophantine equation

$$a_1x + a_2y + a_3z + a_4k = n \tag{2}$$

which describes a partition of n into summands a_1, a_2, a_3, a_4. One can establish the inequality $k \leq INT(n/a_4)$, which enables the reduction of equation 2 to the family of equations

$$a_1x + a_2y + a_3z = n - a_4k \tag{3}$$

where $k \in \{0, 1, 2, \ldots, INT(n/a_4)\}$. For each such value of k, equation 3 represents a three-variable equation that can be modeled using a three-dimensional spreadsheet like the one shown in Figure 2. In finding ranges for variables x and y, in equation

3, one should take into account that these ranges depend on the value $n_k = n - a_4k$, $k = 0, 1, 2, \ldots, INT(n/a4)$. In other words, for each such value of k the inequalities

$$x \leq INT(n_k/a_1) \quad (4)$$

$$y \leq INT(n_k/a_2) \quad (5)$$

determine the ranges for variables x and y.

For example, the equation $10x+9y+8z+7k = 25$, due to the equality $INT(25/7) = 3$, can be reduced to the family of four equations $10x+9y+8z = 25$ ($k = 0$), $10x+9y+8z = 18$ ($k = 1$), $10x + 9y + 8z = 11$ ($k = 2$), $10x + 9y + 8z = 4$ ($k = 3$), each of which can be modeled within a three-dimensional spreadsheet. However, a modification of the three-dimensional spreadsheet allows for the full automation of the process of calculating the total number of solutions for any four-variable Diophantine equation of the form (2). Such a four-dimensional modeling tool (Figure 6) can be programmed as follows.

1. The values a_1, a_2, a_3, a_4, and n are defined, respectively, in the slider-controlled cells `H2, I2, J2, K2,` and `D2`, which, in turn, are given the names `a_1, a_2, a_3, a_4,` and `n`.

2. Cell `C8` is slider-controlled and given the name `k`.

3. `(C2)= =n-a_4*k`; this cell is given the name `n_k`.

4. `(B7)= =INT(n_k/a_4)`.

5. `(G6)= =0; (F7)= =0; (H6)= =IF(G6<INT(n_k/a_1),1+G6, " ")` – replicated across row 6; `(F8)= =IF(F7<INT(n_k/a_2),1+F7," ")` – replicated down column F. The last two conditional formulas are based on inequalities (4) and (5) allowing for the exact ranges for variables x and y to be generated by the spreadsheet.

6. `(G7)= =IF(OR(G$6=" ",$F7=" "), " ", IF(AND(n_k-G$6*a_1-$F7*a_2>=0, MOD(n_k-G$6*a_1-$F7*a_2,a_3)=0), (n_k-G$6*a_1-$F7*a_2)/a_3," "))` – replicated to cell `M11`. Note that the last formula is similar to the one used in the programming of the three-dimensional spreadsheet of Figure 2.

7. The chart pictured in the range `Q2:X3` is designed to record the number of solutions of the family of equations (3) for each value of k generated in the top row of the chart beginning from cell `Q2`. These values are generated through the formula defined in cell `R2` and replicated to the right; namely `(Q2)= =0; (R2)= =IF(Q2<INT(n/a_4),Q2+1," ")`. Slider-controlled cell `O3` is assigned two values, 0 and 1, and given the name *iterate*. When the content of `O3` is zero, all cells in the bottom part of the chart are blank; otherwise the of formula `(Q3)= =IF(iterate=0," ",IF(Q2=k,COUNT($G$7:$M$11),Q3))` records the number of integers displayed in the range `G7:M11`.The basic computational idea that enables such a chart representation of the function relating the value of k to the number

of solutions of the corresponding equation 3 is the use of a circular reference in a spreadsheet formula (that is, a reference to a cell in which the formula is defined). This circular reference makes it possible to keep the values of already computed the number of solutions function unchanged as the value of k varies in the slider-controlled cell C8. Alternatively, one can use the Data Table method described in [29]. Although this method enables one to avoid carrying out manual iterations by k in cell C8, the authors prefer using the circular references approach. The latter approach allows one to see solutions of equation 2 for each value of k displayed in the main table (range F6:M11); this information can be used as yet another source for posing problems. Technically, in Excel 2008 (Mac) Preferences one has to open the Calculation dialog box and check the bullets Automatically and Limit iteration, limiting the number of iterations to, say, 100. Similarly, in Excel 2007 (Windows) Options, within the Formulas category one has to open the Calculation, check Enable iterative calculation and set the limit.

8. Finally, by defining (T6)= =SUM(Q3:X3) one can find the total number of solutions of equation (2). Note that the inequalities $a_1 > a_2 > a_3$ ($50 > 25 > 5$ in Figure 6) follow the order used in programming the three-dimensional spreadsheet of Part 1; yet the inequality $a_3 < a_4$ ($5 < 10$ in Figure 6), by defying the order, allows one to minimize the number of iterations.

9. To begin iterations, one has to set zero in cells C8 and O3 by using sliders that control the cells. The next step is to set one in cell O3 by using the corresponding slider – this action generates a number in otherwise empty cell Q3. Finally, by using the slider that controls cell C8, its content has to be varied until zero is displayed in cell B7.