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Teaching the Logistic Growth Difference Equation Using Spreadsheets

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Keywords

Scroll Bar, Logistic Growth Model, Difference Equation

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Teaching the Logistic Growth Difference Equation Using Spreadsheets

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Abstract

The logistic growth difference equation is often used in biology to model population growth. The terms that satisfy the difference equation have many remarkable mathematical properties such as exhibiting chaotic behavior. Using spreadsheet modeling tools, the properties of logistic growth can be investigated by students in a user friendly environment. Students will learn about useful computational and modeling tools, while also learning about a new area of mathematics that has fascinated many (e.g., James Gleick's *Chaos: Making a New Science* is a national best seller). Moreover, the model has many real world applications in biology. Unfortunately, many mathematics and computer science students do not see the logistic growth model because it does not appear in the standard set of required courses. In this paper we describe a how to implement the logistic growth model, and describe related applications and student exercises.

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Key words: Scroll Bar, Logistic Growth Model, Difference Equation

1 Introduction

Predicting the size of a population is a fundamental concern in the study of population ecology. One model often used to make predictions is a geometric growth model which assumes that a population grows by the same percentage every year. Such models have no limit on the size of a population. This is unrealistic in the long run because geometric growth models ignore issues such as competition for food and living space that act to limit the size of a population. These aspects may be safely ignored with negligible error when the population is small. But as the population grows larger, as competition for resources becomes more pronounced, it becomes unlikely that the population will continue to grow by the same percentage that occurred when the population was small. Hence, ecologists attempt to identify the maximum population that a given habitat can

sustain. This is known as the *carrying capacity* which we shall denote as K . Using K , ecologists seek to construct a model that has a population whose growth rate slows down when the population is close to K , and the growth rate becomes negative when the population exceeds K .

To model population dynamics consider the sequence N_t , where N_t is the population size at time period t , and N_0 is the starting population satisfying $0 < N_0 < K$. The intrinsic rate of growth r is defined to be the difference between the birth rate and the death rate. In the logistic growth model, we will assume that r remains constant, but the actual rate of change in N_t will depend on how close N_t is to K . In particular, the rate of change in N_t from one period to another is given by the product $r(K - N_t)$. The complete logistic growth difference equation model is as follows.

$$N_{t+1} = r(K - N_t)N_t$$

For example, consider, the difference equation $N_{t+1} = 0.000025(90,000 - N_t)N_t$ discussed in [3] to model the population of fish in a lake. Here, the carrying capacity is $K = 90,000$ and the intrinsic rate of growth is $r = 0.000025$. If $N_t = 50,000$, then $0.000025(90,000 - N_t) = 1$. In this case the population does not grow at all. When $N_t < 50,000$, the population will grow larger, and when $N_t > 50,000$ the population will decrease. The population of 50,000 is sometimes called a steady state value or an equilibrium.

Whenever an ecologist studies population dynamics it is common to determine if a population reaches an equilibrium, and if so, what is the size of the equilibrium population? In other words, we want to know if there a positive integer T such that $N_{t+1} = N_t$, for every $t \geq T$. When an equilibrium value exists, we shall denote it by \bar{N} . If there is no equilibrium, then one might seek a small set of values that N_t cycles through. It turns out that answers to these questions depend on r and K . For example, consider the equation $N_{t+1} = r(1 - N_t)N_t$, where $N_0 = .90$. Here, we use $K = 1$ for the carrying capacity and think of N_t as the density of a population with respect to its carrying capacity. Using a starting population of $N_0 = .90$ implies that at time designated by $t = 0$, the population is at 90% of its carrying capacity. Graphs of the first 50 values are given for the cases $r = 2.90$ and $r = 3.55$ in Figures 1 and 2 respectively. Observe that the terms $\{N_t\}$ for $r = 2.90$ displayed in Figure 1 converge to a single value $\bar{N} = 0.655$, whereas the terms for $r = 3.55$ in Figure 2 cycle through the eight values: .882, .370, .828, .506, .887, .335, .813, .540, for $t > 10$.

After working with the logistic growth model students will see how critical it is to find the correct value of r when modeling growth. As one can see from the figures, for some values of r , $\{N_t\}$ will converge to an equilibrium. For other values of r , $\{N_t\}$ may cycle among two or more values. Moreover, there exist threshold values for r for which the behavior of the terms $\{N_t\}$ changes radically. Hence, it is natural to ask, for what values of r does $N_{t+1} = r(1 - N_t)N_t$, with $N_0 = .90$, converge to one value, two values, three values, and so on? It turns out that a spreadsheet is an ideal computing environment to numerically and graphically introduce students to the behavior of the logistic growth equation. This is because it is easy to compute values of a difference

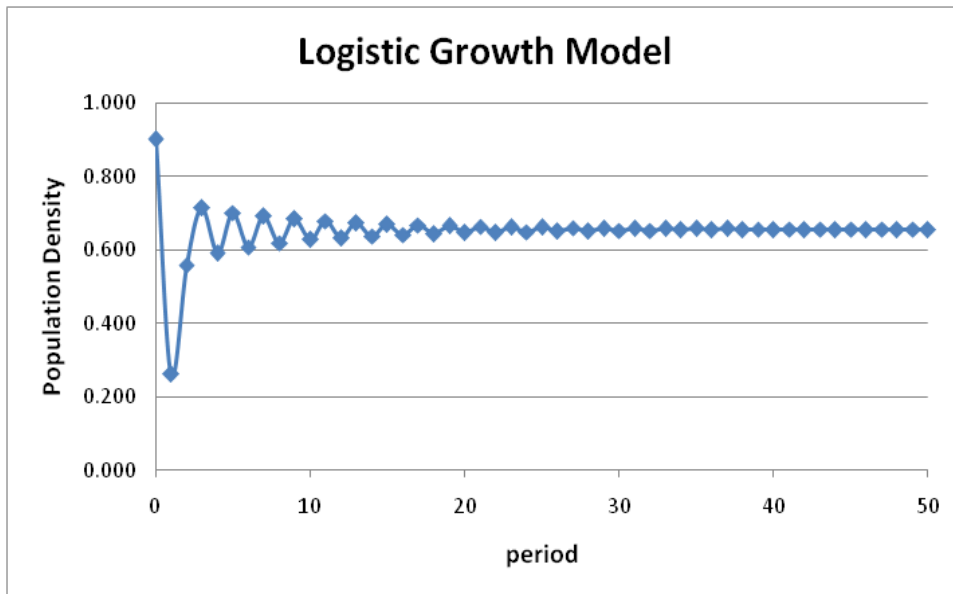


Figure 1: The first 50 periods for the Logistic Growth Model with $r = 2.90$

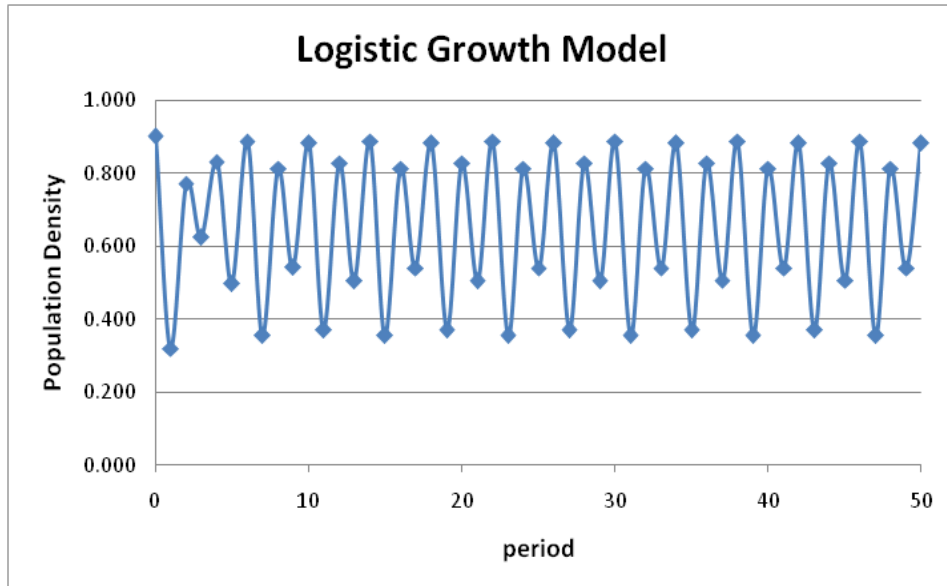


Figure 2: The first 50 periods for the Logistic Growth Model with $r = 3.55$

equation, create graphs and vary key parameters with a spreadsheet. In addition, we can insert a scroll bar, which is an excellent way to vary a parameter such as r , and watch what happens to the graphs as the numbers are varied. In the next section we discuss the construction of the spreadsheet model. In Section 3 we will discuss some mathematical investigations for student exploration. Section 4 discusses specific applications and extensions of the model recently studied by ecologists.

2 Developing the spreadsheet model

The models described in this paper were developed in Excel 2007, and this version is assumed in the description to follow. An illustration of an Excel spreadsheet model implementing the logistic growth equation $N_{t+1} = r(1 - N_t)N_t$ is given in Figure 3. The initial term is $N_0 = .90$ and we input this by entering 0.900 in cell C16. For convenience, we assume that the values are scaled and K is kept constant at 1. In cell C17 the Excel formula " $=\$C\$12*(1-C16)*C16$ " is entered and then copied down in column C from C18 to C66. To study how sensitive the model is to changes in r , we insert a scroll bar which will be used to control the values of r .

Before a scroll bar is inserted we must first obtain the "Developer" tab in the Excel Ribbon. To do this, begin by clicking on the Office Button located in the top left corner. Then click on Excel Options. Under "Top options for working with Excel" click on the box next to "Show Developer tab in the Ribbon". Then click on OK. Now the Developer tab should be visible. Next click on the Developer tab, and then click on the triangle below Insert. This should display a table labeled "Form Controls." Now click on the Scroll Bar control option, and then use the mouse to trace out a scroll bar making it any size you wish. The last step is to "Format Control" of the scroll bar which links the scroll bar to a spreadsheet cell. To format control we place the mouse pointer on the scroll bar and make a right click, followed by a left click on the "Format Control" option. In the dialog box we must enter minimum and maximum values, which we choose to be 0 and 500. The incremental change must be an integer, so we set it equal to 1. For the cell link, we enter G12, then click OK. Notice that G12 acts as an auxiliary cell. This is because we want the scroll bar to control cell C12 which contains our value of r , but an incremental change value less than one is not allowed. So we use the scroll bar to vary cell G12 from 0 to 500, and in cell C12 we use the formula " $=G12/100$ ". This allows us to vary r from 0 to 5 using two decimal place resolution.

The graph is obtained by highlighting the values of t (cells B15 to B66) and the values of N_t (cells C15 to C66). Then click on Insert, and choose the "Scatter" option. Next choose a subtype that plots points connected by line segments (top right). Complete the chart by selecting a chart layout and entering appropriate titles. Once the chart is completed, one can rescale the axes by pointing the mouse to an axis and then double click. Next, go to the Scale option and make any desired changes.

At this point it is always a good idea to begin experimenting with some extreme values. Try setting $r = 0$. The result should be a horizontal line at 0. Next try setting $r = 5$. Now, one must look carefully at what is going on. The graph will show a spike

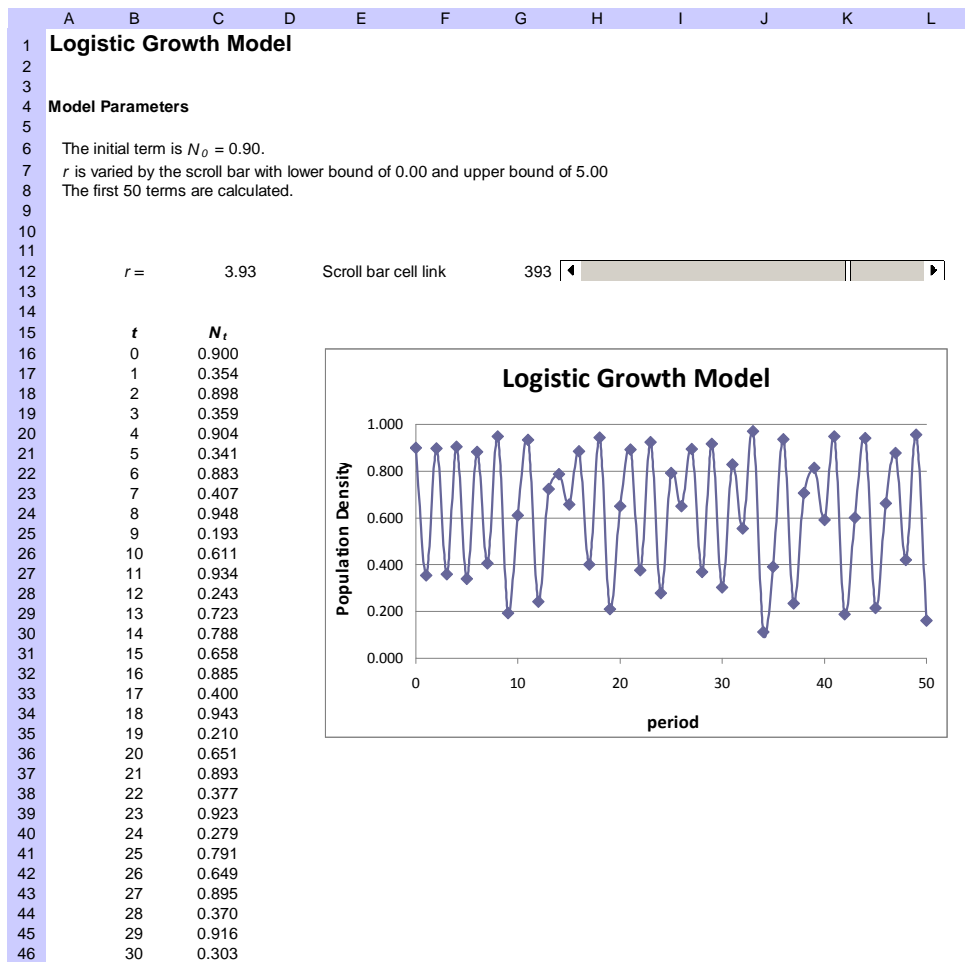


Figure 3: Spreadsheet implementation of the Logistic Growth Model

and very large numbers on the vertical scale of the graph. What happens is that the numbers N_t "blow up"; that is, they become unbounded. When numbers get very large, Excel displays #NUM! in the cell and actually stores the number as 0. So instead of the graph displaying an unbounded behavior, we see the graph return to 0. This in itself is a valuable lesson. One should always be careful because assumptions are made by the software designers that a user must be aware of.

3 Mathematical investigations using the model

There are many interesting aspects of the logistic growth equation. So far, we have seen that values from the equation $N_{t+1} = r(1 - N_t)N_t$, with $N_0 = .900$, can either be constant ($r = 0$), converge to a unique value ($r = 2.9$), cycle through eight values ($r = 3.55$), or become unbounded ($r = 5$). We begin with a question that is useful in

population growth studies.

Question 1. For which values of r does $\{N_t\}$ reach an equilibrium? Can you determine the equilibrium value as a function of r and K ?

To help guide students it may be useful to have them fill out a table containing values of r in increments of say 0.2, together with equilibrium values when they exist. Students should be able to quickly see that as r gets close to 3.0, there are significant changes in the behavior of the terms. It may also help to extend the model described above to look at the first 200 terms instead of just the first 50 terms. There is a known result in [3] that states:

Theorem Suppose that a logistic growth model has the difference equation

$N_{t+1} = r(K - N_t)N_t$ and N_0 is between 0 and K . Then if $1 < rK < 3$, the population will eventually level off at an equilibrium value of $K - \frac{1}{r}$.

For example, when $r = 2$ and $K = 1$, the equilibrium value is 0.50. Chaos is often described as a dynamic system that has a sensitive dependence on its parameters and initial conditions. In particular, small errors or changes in the parameters or starting values lead to vastly different long term behaviors. In the next set of questions students can observe this in a "hands on" manner.

Question 2. What happens to the cycle lengths of $\{N_t\}$ as r varies between 3 and 4? Does $\{N_t\}$ ever cycle between 8 values? Does the length of the cycles necessarily increase as r varies from 3 to 4? What happens to the terms when $r > 4$?

With the proper scaling of the graph and enough terms of $\{N_t\}$ students should notice the period (i.e., cycle length) repeatedly double from 1 to 2, to 4, and then to 8, as r increases past 3. When $r = 4$, the terms $\{N_t\}$ are all between 0 and 1. But at $r = 4.01$, the terms become unbounded! Every time the qualitative behavior of the solution of a difference equation changes as the result of a change in a parameter such as r , we say a *bifurcation* has occurred. The period doubling that we see as r increases for the logistic difference equation is called the *period doubling route to chaos* and is a common effect when chaos occurs for a difference equation.

With basic algebra, students can determine for themselves why the theorem states the equilibrium value is $K - \frac{1}{r}$. The algebraic method of solving a difference equation is to substitute \bar{N} for every term N_t in the difference equation, and then solve the equation for \bar{N} . So for the logistic difference equation $N_{t+1} = r(K - N_t)N_t$, we get $\bar{N} = r(K - \bar{N})\bar{N}$. Solving for \bar{N} , gives either $\bar{N} = 0$ or $\bar{N} = K - \frac{1}{r}$. Thus, there are two equilibria for the equation. One of these ($\bar{N} = 0$) is an equilibrium that is called a *source*. Only a set of terms starting with this value will converge to it. The other equilibrium is an attracting equilibrium called a *sink*. Since the nontrivial equilibrium is attractive, the terms will eventually converge to it rather than the source. Regarding equilibria, one other possibility exists (although it does not occur for the logistic equation) and that is called a "node". A *node* equilibrium is attractive from one direction and repelling from the other direction.

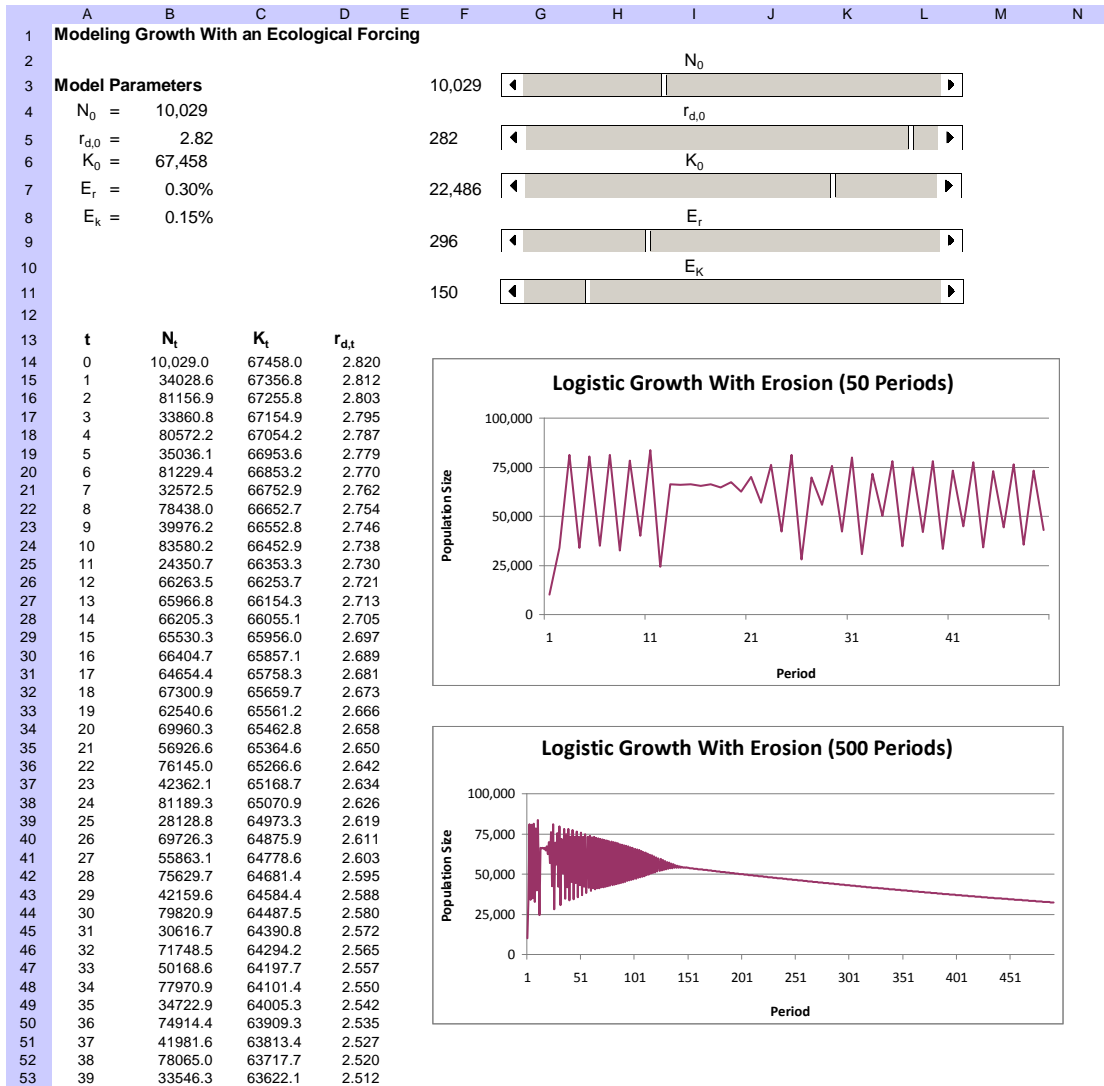


Figure 4: A graph of the logistic growth model with erosion parameters

To illustrate the different types of equilibria, consider the difference equation $N_{t+1} = (2 - N_t)(N_t)^2$. Solving for equilibria, we get $\bar{N} = (2 - \bar{N})(\bar{N})^2$. From which we can see that either $\bar{N} = 0$, or $1 = (2 - \bar{N})(\bar{N})$. Hence, the nontrivial equilibria must satisfy $\bar{N}^2 - 2\bar{N} + 1 = 0$, which has the solution $\bar{N} = 1$. In order to determine whether an equilibrium is a sink, source, or node, an approach analogous to the first derivative test from calculus is used. We rewrite the difference equation in the form $N_{t+1} = N_t f(N_t)$. When $f(N_t) > 1$, the solution increases (i.e. $N_{t+1} > N_t$) and when $f(N_t) < 1$, the solution decreases. For this difference equation,

$$f(N_t) = N_t(2 - N_t).$$

When $N_t > 1$ and $f(N_t) < 1$, the terms decrease. If the initial term is greater than 1 and sufficiently close 1, then the terms converge to 1. The set of points for which the terms of a solution will converge to an equilibrium is called the *basin of attraction*. When $0 < N_t < 1$, we have $f(N_t) < 1$ and the solution decreases away from $\bar{N} = 1$ and towards $\bar{N} = 0$. This makes $\bar{N} = 1$ a node and $\bar{N} = 0$ a sink (on the given domain). If this were modeling the population of a species, it might indicate a minimum population level is needed for the species to survive (and also that there are only enough resources to sustain this minimum level, quite a dangerous situation for this species!).

Question 3. Construct a spreadsheet model for the difference equation $N_{t+1} = (2 - N_t)(N_t)^2$, where $0 \leq N_0 \leq 2$. Insert a scroll bar that controls N_0 using two decimal place resolution, with a minimum of 0 and a maximum of 2. Use the model to find the basin of attraction of the equilibria $\bar{N} = 0$ and $\bar{N} = 1$. Can you formulate an algebraic argument that proves your conjecture concerning the basin of attraction?

After some experimenting with the model, students should find the basin of attraction for $\bar{N} = 1$ is $[1, \phi]$, where ϕ is the famous golden ratio $\phi \approx 1.62$. The basin of attraction for $\bar{N} = 0$ is $[0, 1) \cup (\phi, 2]$. A proof of this can be obtained by setting $(2 - \bar{N})(\bar{N})^2$ equal to 1, and then solving for \bar{N} .

4 Ecological extensions and applications

Many natural populations exhibit population cycles, and ecologists have long been interested in understanding what causes cyclic population behavior (e.g., see [5] and [6], and references therein). Over time, the number of field-collected population time series have increased, with as many as 30% exhibiting cyclicity of some sort [5]. Even with field-collected time series in hand, there are still several questions that population ecologists must tackle. First, is there really a cycle? Natural populations are notoriously noisy; to detect cycles ecologists employ a number of sophisticated statistical techniques to see if recognizable patterns emerge through the noise. Though mathematically challenging and fascinating, we do not address this aspect of ecology at this time. Second, what are the underlying biological mechanisms that drive these dynamics? Ecologists may offer predator-prey dynamics, parasite-host dynamics and lag effects as some of the

many possible drivers of cycles [5]. This work is often incredibly time consuming because making connections between population growth, the natural histories of predators and prey requires a great deal of biological expertise plus extensive field studies. This kind of information, while valuable and informative in explaining aspects of population dynamics, is often simply not available. Nonetheless, even in the presence of stochastic noise and mechanistic ignorance, we know that organisms mate, reproduce and die. Thus, ecologists tackle a third question. If there is a cycle, is it consistent with a reasonable mathematical model of population growth? “Only mathematical models can show which factors are even capable of generating cyclic dynamics; without such models the experimental approach will not be able to solve the question of population cycles” [5]. It is this third question that we will address in this paper. In particular, we will develop an ecological scenario in which students may use the spreadsheet model to explore the implications of varying a given set of parameters.

With the model developed in Section 2, one can easily generate cycles. Although the behavior of the population growth displayed in Figure 3 exhibits chaotic behavior, it is still bounded and cyclic. A population with this underlying structure could persist indefinitely. However, natural populations may exhibit extreme changes and emerge from the bounded situation to yield either a boom or a bust year, perhaps even driving to extinction. Sometimes the reasons for an extinction event are obvious; for instance, a deforestation event will clearly lead to local extinction of a forest (e.g., see [7]). In other cases, dramatic declines are not so easy to explain. In such instances, ecologists have historically made the assumption that large population declines must be the result of some dramatic ecological or environmental change. Indeed, Vandermeer and Yodzis [7] argue that this assumption leads to the obverse assumption: “. . . that gradual secular changes in environmental parameters lead to gradual secular changes in populations and communities,” they go on to say, “Frequently, gradual secular changes in environmental parameters can also lead to sudden and dramatic changes in population and community variables.” Given the sensitive nature of the model we developed in section 2, it is not unreasonable to expect that small perturbations in the model parameters could have dramatic mathematical effects. If that is so, then gradual secular changes, sometimes referred to as “forcings,” may radically alter the behavior of the population dynamics.

Recently, population ecologists have been examining the issue of climate forcing. “It has been shown that just a few more days with above zero temperatures during the winter may dramatically lower the survival rate of voles (small arctic rodents) and disrupt the dynamics of populations” [6]. This has been noted in other studies as well, and many suggest that increased CO₂ levels and global climate change, which has been happening gradually since the 1800’s, may have driven these dynamic changes. To examine population growth, we use a variation of the models presented in Section 2; however, now we examine how the numbers of individuals change over time rather than population density. Though the behavior of the model is the same with regard to cyclicity, populations can now cycle above and below K as they often do in nature. For our ecological scenario, we start with the following model as presented in [2]: $N_{t+1} = N_t + r_{d,t}N_t(1 - \frac{N_t}{K_t})$, where $r_{d,t}$ is the discrete rate of growth at time t , and K_t is the carrying capacity at

time t . To this model we introduce two new parameters related to population "erosion": E_r and E_k . These erosion parameters represent a gradual decline in the discrete rate of growth $r_{d,t}$ and the carrying capacity K_t respectively. These parameters are used to model a slow gradual decline in $r_{d,t}$ and K_t , perhaps due to some environmental effect such as increased temperature arising from climatic forcing. Incorporating the erosion parameters gives the following difference equation:

$$N_{t+1} = N_t + (r_{d,t} - E_r)N_t\left(1 - \frac{N_t}{(K_t - E_k)}\right).$$

We acknowledge that this is somewhat biologically naïve [5] as we have provided no biological mechanism to explain the decline. However, we submit that this model is not inconsistent with a natural world in which habitat destruction, pollution and global climate change are realities. Moreover, we argue that students will readily understand the underlying biological principle as it applies to the model.

In Figure 4 a spreadsheet model similar to the one produced in Figure 3 is presented, but with several modifications. First, we have added additional scroll bars to give greater control for the basic parameters of initial population size N_0 , initial carrying capacity K_0 , as well as the discrete rate of growth $r_{d,t}$. Because the model is quite sensitive to starting conditions, these extra scroll bars allow students to undertake a greater exploration of the parameter space. We have also added two additional scroll bars to control the erosion parameters E_r and E_k . The scroll bars should be set with the following ranges: for N_0 , we use 0 to 30,000; for K_0 , we use 0 to 30,000; for $r_{d,t}$, we use 0 to 3.00; for E_r , 0 to 1,000; and finally for E_k , we use 0 to 1,000. The starting conditions of the model and the erosion values are listed in the upper left corner under "Model Parameters" and linked to an appropriate auxiliary cell, which is always to the immediate left of the scroll bars. The initial population size is set to exactly match the N_0 scroll bar value, located in cell F3.

The initial discrete rate of growth $r_{d,0}$ is set to equal the scroll bar value, located in cell F5, divided by 100 so that the range of $r_{d,t}$ falls between 0.0 and 3.0. Since the scroll bars have a maximum value of 30,000, and we would like a greater number of individuals for this example, we set K_0 under "Model Parameters" to equal three times the value of the K_0 scroll bar located in cell F7. For both erosion values, under "Model Parameters" we set them equal to the scroll bar values located in the auxiliary cells for E_r and E_k (cells F9 and F11 respectively) divided by 100,000. The high resolution is used to model a very small incremental decline. Such small declines are consistent with a nearly imperceptible, gradual secular decline as described by Vandermeer and Yodzis [7]. In Figure 4, we show one interesting result of the model. With both E_r and E_k set at a low rate of about a third of one percent, the model shows a markedly different short term and long term behaviors. Running the model out 500 periods shows that the population size converges to a glide path that will ultimately lead to extinction. What is interesting about this model output is that if you view the behavior over the first 50 years, there would likely be no way for an ecologist measuring this in the field, to predict the ultimate demise of the population in the future. This result is superficially

similar to the pattern of decline in arctic foxes, in which a series of seemingly stable cycles suddenly takes a turn for a tight downward path (as noted in [6]).

Question 4 Consider the scenario displayed in Figure 4. Suppose that after 75 years a conservation effort is implemented. You are faced with a choice. Is it better to rear a large batch of individuals and supplement the population with an infusion of new individuals, or is it better to focus on ecoremediation to stabilize $r_{d,t}$ and K_t ?

If a student were to insert, say 20,000 individual in year 79 simply by adding to the formula in cell B79, the model output shows us that it will have no effect on the final outcome. The population is still doomed. Reducing the erosion of $r_{d,t}$ from 0.30% to 0.15% also does not change the fate of the population; however, the cyclic nature of the population becomes more apparent as E_r declines. It is only with the stabilization of K that the population takes on a more favorable population trajectory. Reducing E_k by about half to 0.15% dramatically slows the population decline. Based on these results an ecologist might argue that preserving and maintaining an ecosystem, thereby stabilizing the ability of a habitat to support a population, is by far the most useful application of money and effort.

5 Conclusion

As demonstrated above, there are many interesting aspects of the logistic growth model and here we have just given a brief introduction. The interested reader can learn more from the references listed below. It should now be clear from the above models that spreadsheet software is an ideal computational tool to introduce students to this model. Our spreadsheet model is available at the *Spreadsheets in Education* website. If the reader has any questions, do not hesitate to send the first author an email.

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