Spreadsheets in Education (eJSiE)

Volume 4 | Issue 3 Article 1

4-6-2011

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Recommended Citation
Abramovich, Sergei (2011) Number Theory, Dialogue, and the Use of Spreadsheets in Teacher Education, *Spreadsheets in Education (eJSiE)*: Vol. 4: Iss. 3, Article 1.
Available at: http://epublications.bond.edu.au/ejsie/vol4/iss3/1

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Keywords

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Abstract

This paper demonstrates the use of a spreadsheet in teaching topics in elementary number theory. It emphasizes both the power and deficiency of inductive reasoning using a number of historically significant examples. The notion of computational experiment as a modern approach to the teaching of mathematics is discussed. The paper, grounded in a teacher-student dialogue as an instructional method, is a reflection on the author's work over the years with prospective teachers of secondary mathematics.

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1 Introduction

The use of computers to support teaching topics in number theory to prospective teachers of mathematics as well as to high school students has been known in the United States for almost five decades [14]. Until the mid 1990s, the pioneers of computer oriented mathematics instruction focused on the use of different programming languages like BASIC [22], [21], [19] and PASCAL [6]. This resulted in augmenting, if not, by some accounts [10], distorting, mathematics curriculum with the learning of syntax and semantic of computer programming. The advent of various computer applications enabled a qualitatively new didactic approach, which, in many cases, shifted the attention from teaching *about* the computer to teaching *with* a computer. Nowadays, the pedagogy of teaching with computer is considered as the most efficient way of using technology in mathematics education [15], [7], [27], [13]. Teaching mathematics with computer is often based on an appropriately designed computational experiment.

A powerful tool that enables easy access to ideas and concepts of number theory through a computational experiment is an electronic spreadsheet [3], [4], [24]. The tool represents one of the most popular general-purpose software used by educators for more than 30 years to promote the spirit of exploration and discovery by integrating experiment in the teaching of mathematics [5], [16], [23], [1]. Long before the computer age, the role of a mathematical experiment, especially in number theory, was emphasized by such mathematical giants as Euler and Gauss. Euler: "As we must refer the numbers to the pure intellect alone, we can hardly understand how observations and quasi-experiments can be of use in investigating the nature of the numbers. Yet, in fact, ... the properties of the numbers known today have been mostly discovered by observation, and discovered long before their truth has been confirmed by rigid demonstrations" (translated by Pólya [18, p3]. Gauss: "In arithmetic the most elegant theorems frequently arise experimentally as the result of a more or less unexpected stroke of good fortune, while their proofs lie so deeply embedded in darkness that they defeat the sharpest inquiries" (cited in [25]).

In the age of technology, an experimental approach to mathematics using a spreadsheet draws on the power of the tool to perform numerical computations and graphical constructions, thereby, enabling easy access to mathematical ideas and objects under study. The approach includes one's engagement in recognizing numerical patterns formed by modeling data and formulating properties of the studied objects through interpreting the meaning of the patterns and their graphic representations. This makes it possible to balance formal and informal approaches to mathematics allowing teachers to learn how the two approaches complement each other. In particular, a computational (i.e., empirical) approach to the development of concepts of number theory follows the historical evolution of the ideas of this classic area of mathematics. Indeed, many results about properties of numbers were first discovered by observations and only later have been confirmed using the language of formal proof. This approach also included observations supported by inductive reasoning alone that turned out to be erroneous generalizations. These examples can be introduced by using a spreadsheet.

Whereas mathematical activities that can be motivated by and presented within a spreadsheet can be quite significant, the software allows for hiding some of the complexity and formal structure of mathematics involved. This feature is especially important in the context of preparation of mathematics teachers for it enables their true engagement in rather advanced context without the need to have a rigorous understanding of the context expected from future professional mathematicians. In other words, using a spreadsheet, a prospective teacher can learn how mathematics can be approached initially through a computational experiment rather than through a less generally appreciated formal demonstration [7]. Many examples of such an approach to the preparation of secondary mathematics teachers can be found elsewhere [1].

This paper includes several historically significant examples from number theory demonstrating both the power and deficiency of inductive reasoning, which nowadays is greatly enhanced by computers. It demonstrates how a spreadsheet can be used in the classroom to support the empirical approach to the development of knowledge advocated by John Dewey—the most notable reformer of the modern era of American education—who argued that experience is educative only if it results in one's intellectual growth. To this end, Dewey [8] promoted the pedagogy of reflective inquiry—a problem-solving method that blurs the distinction between knowing and doing. Through a teacher-guided reflection new knowledge can be developed, as students are encouraged to inquire about the meaning of their experience. In the modern classroom, such experience can be provided by the appropriate use of a spreadsheet that allows for very inexpensive yet extremely powerful experiments with numbers. Whereas the appropriate use of the tool is determined by the teacher, "the ideas should be born in students minds and the teacher should act only as midwife" [19, p104]. Working as midwife, or "scaffolding" [26] one's thinking is grounded in an interaction between novice (student) and expert (teacher) aimed at carrying out a task beyond the novice's unaided performance requiring the grasp of solution prior to its production. In other words, the scaffolding pedagogy should allow a student to recognize a solution before he or she is able to produce steps leading to its formal demonstration.

The paper is structured as a series of (non-verbatim) vignettes of prospective teachers of secondary mathematics learning classic mathematical ideas using spreadsheets under the guidance of "the more knowledgeable other." Such a structure was chosen to emphasize the importance of the teachers' knowledge of mathematics for their students' learning. The paper concludes with a brief discussion of how mathematics of Euler and Fermat can be connected to that of Pythagoras and Euclid through the use of a spreadsheet. It is a reflection on the author's work over the years with the teachers in a number of computer-enhanced courses.

2 Developing basic summation formulas

One of the problems in number theory that goes back to the Babylonian mathematics (3000 B.C.—600 B.C.) is the summation of squares of counting numbers. According to Kline, "... the sum of the squares of the integers from 1 to 10 was given as though they [Babylonians] have applied the formula $1^2 + 2^2 + ... + n^2 = (1 \cdot \frac{1}{3} + n \cdot \frac{2}{3})(1 + 2 + ... + n)$. No derivation accompanied the special cases treated in their texts" [12, p10]. Nowadays, a similar result can be obtained through a numerical approach made possible by the use of a spreadsheet. For example, one can numerically model the partial sums of consecutive perfect squares with the goal to construct a closed formula for the general case of the sum of the first n squares of counting numbers. After constructing a five-column spreadsheet shown in Figure 1 (see Appendix for programming details), the teacher (T) begins a dialogue with the student (S).

	A	B	C	D	E
1	n	$1+2+...+n$	$1^2+2^2+...+n^2$	$(1+2+...+n)/n$	$(1^2+2^2+...+n^2)/(1+2+...+n)$
2	1	1	1	1	1
3	2	3	5	1 1/2	1 2/3
4	3	6	14	2	2 1/3
5	4	10	30	2 1/2	3
6	5	15	55	3	3 2/3
7	6	21	91	3 1/2	4 1/3
8	7	28	140	4	5
9	8	36	204	4 1/2	5 2/3
10	9	45	285	5	6 1/3
11	10	55	385	5 1/2	7
12	11	66	506	6	7 2/3
13	12	78	650	6 1/2	8 1/3
14	13	91	819	7	9
15	14	105	1015	7 1/2	9 2/3
16	15	120	1240	8	10 1/3

Figure 1: Numerical approach to finding the sum of consecutive squares.

T: What are the numbers in column A?

S: They are consecutive counting numbers.

T: What are the numbers in column B?

S: They are the sums of consecutive counting numbers.

T: Exactly! How can one generate numbers in column B from numbers in column A? For example, how to get 3 (cell B3) from 2 (cell A3)?

S: Number 3 is 2 times 3 divided by 2, that is, $3 = \frac{2 \cdot 3}{2}$.

T: Well! And, in general, what is the factor that transforms the counting number *n* into the sum of the first *n* counting numbers?

S: This factor is [writes on the board] $\frac{n+1}{2}$.

T: How does the formula for the sum of the first *n* counting numbers look like?

S: Here is the formula [writes on the board]

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2} \qquad (1)$$

T: How can this be explained? That is, what is the meaning of the map $n \to n \cdot \frac{n+1}{2}$ that turns any counting number *n* into the sum of the first *n* counting numbers?

S: I have to think about this question.

T: Good, we will discuss this later. But now, ... what are the numbers in column C?

S: They are the sums of squares of counting numbers starting from one.

T: How to generate numbers in column C from numbers in column B? In other words, how to generate the sums of squares from triangular numbers? For example, how to get 5 (cell C3) from 3 (cell B3), how to get 14 (cell C4) from 6 (cell B4), and so on?

S: Number 5 is 3 times 5 divided by 3, number 14 is 6 times 14 divided by 6, and so on. That is, $5 = 3 \cdot \frac{5}{3}, 14 = 6 \cdot \frac{14}{6}, 30 = 10 \cdot \frac{30}{10}, 55 = 15 \cdot \frac{55}{15}$.

T: Could you please reduce the fractions?

S: [writes on the board] 5/3, 7/3, 3, 11/3 .

T: Well, how to make 3 a number with the denominator 3?

S: I multiply 3 by 3 and divide by 3, that is, $3=\frac{9}{3}$. Thus, the above four fractions that transform the sums of counting numbers into the sums of their squares have the form 5/3, 7/3, 9/3, and 11/3.

T: What do the numerators 5, 7, 9, 11 have in common?

S: They are consecutive odd numbers.

T: Wonderful. How are they related to numbers in column A?

S: They are twice the corresponding numbers in column A increased by one: $5=2\cdot 2+1, 7=2\cdot 3+1, 9=2\cdot 4+1, 11=2\cdot 5+1$.

T: Exactly! What is the factor that transforms the sum of the first n counting numbers into the sum of their squares?

S: This factor is [writes on the board] $\frac{2n+1}{3}$.

T: How can this be explained? That is, what is the meaning of the map $(1+2+3+...+n)\rightarrow(1+2+3+...+n)\frac{2n+1}{3}$ that turns the sum of the first n counting numbers into the sum of their squares?

S: Once again, I have to think about this question.

T: Very well. But what is the sum of the first n squares of counting numbers?

S: This sum is [writes on the board] $\frac{n(n+1)(2n+1)}{6}$.

Comparing the obtained result to formula (1) one can represent the sum of the first n squares of counting numbers, s_n, as follows

$$s_n=\frac{2n+1}{3}\cdot\frac{n(n+1)}{2} \qquad (2)$$

or, in the language of Babylonians,

$$s_n=(1\cdot\frac{1}{3}+n\cdot\frac{2}{3})(1+2+3+...+n)$$

Note that in much the same way the formula for the sum of perfect cubes

$$c_n=(1+2+3+...+n)^2 \qquad (3)$$

can be conjectured by expanding the spreadsheet in Figure 1 to include a column filled with partial sums of consecutive cubes of counting numbers.

Relations (2) and (3) show an important role that a lower concept—the sum of the first n counting numbers—plays in the development of higher concepts of number theory. Formulas (1)-(3) can be proved by the method of mathematical induction supported by a spreadsheet as well. The use of a spreadsheet in aiding mathematical induction proof is described elsewhere [2].

3 Answering outstanding questions

The meaning of the map $n \to n \cdot \frac{n+1}{2}$ can be explained by drawing the diagram in Figure 2 (where n = 5) that reflects a geometric idea of making a rectangle out of two triangles each of which represents the sum of the first n counting numbers. The rectangle has the difference between its side lengths equal to one and the factor 1/2 is responsible for taking half of the rectangle; this half being a representation of the required sum.

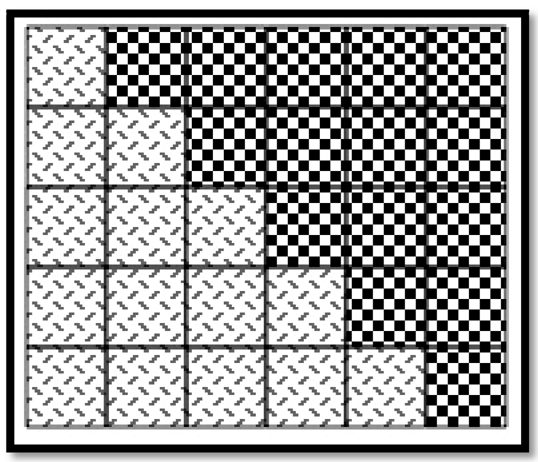

Figure 2: Doubling the sum 1 + 2 + 3 + 4 + 5.

Similarly, the meaning of the map $n \to (1+2+3+...+n) \cdot \frac{2n+1}{3}$ can be explained by drawing the diagram in Figure 3 (where n = 4) that reflects a geometric idea of making a rectangle out of three quasi-triangles each of which represents the sum of the first n squares of counting numbers. The rectangle has the side lengths equal to $1+2+3+...+n$ and $2n$ + 1; the factor 1/3 is responsible for reducing the rectangle to a single representation of the required sum.

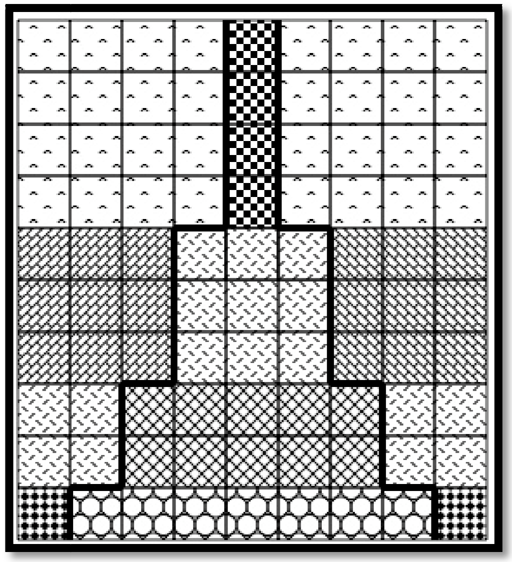

Figure 3: Tripling the sum $1^2 + 2^2 + 3^2 + 4^2$.

4 Constructing the Sieve of Eratosthenes

Another powerful application of a spreadsheet deals with the identification of prime numbers among counting numbers. Recall that counting numbers with exactly two different factors are called prime numbers. As Gauss noted, "The problem of distinguishing prime numbers from composite numbers and of resolving the latter into their prime factors is known to be one of the most important and useful in arithmetic" [9, p396]. The importance of prime numbers in not only fundamental in number theory (alternatively, arithmetic), but it is important to the whole of mathematics [11].

In the 3rd century B.C., a Greek scholar Eratosthenes designed a simple method of finding prime numbers among counting numbers. This method, known as the Sieve of Eratosthenes, allows one to obtain all the prime numbers less than any given integer *N*, by crossing out from the set of all counting numbers less than *N* the multiples of each of the primes up to the $INT(\sqrt{N})$. All numbers that remain undeleted are the primes sought. To start the process, one needs to know the smallest integer with exactly two different divisors. Obviously, the number 2 is such an integer; in other words, the smallest prime number. This introduction initiates the following dialogue between the teacher and the student.

T: How do we know if the number 3 is a prime or not?

S: It is not divisible by any number except itself and one.

T: Very well. But how can one test that formally?

S: Dividing 3 by 2 yields a non-integer greater than one; the next number used in the test is 3 and because in that case the quotient is equal to one, the test terminates, thereby, identifying 3 as a prime number.

T: Good. How do we know that the number 4 is not a prime?

S: Dividing 4 by 2 yields an integer. Therefore, 4 is not a prime number.

T: But dividing 3 by 3 resulted in an integer also. Yet the number 3 was identified as a prime number.

S: Oh, I see the difference. Dividing 4 by 2 yielded an integer greater than 1, which is another divisor of 4 besides 2 and 1.

T: What about 25? How does one test if the number 25 has more than two different divisors?

S: Well, first we divide 25 by 2; because the result is a non-integer greater than 1, the test continues. That is, it shows that 25 is not divisible by 2; finally, 25 turns out to be divisible by 5 with an integer quotient, 5, greater than 1. The test terminates identifying 25 as a composite number.

T: Very well. But should one test whether 4 divides 25 if 25 failed the test of divisibility by 2?

S: Oh, I understand. This is not necessary. We should use in the test prime numbers only!

T: Great! So, using only four primes—2, 3, 5, and 7—how many integers one can correctly test for primality?

S: All integers smaller than 121 because $121 = 11 \cdot 11$; therefore, 121 is the least composite number that survived divisibility by 2, 3, 5, and 7. In other words, all multiples of 2, 3, 5, and 7 which are smaller than 121 can be eliminated without using 11 (or any prime number greater than 11).

T: The test we have discussed can be implemented in the form of a chart. Similarly to the multiplication table one can create a division table (chart), which involves two integral variables—a tested integer n and prime number p that divides n. Let us create such chart.

S: [After working on the chart]. Here it is (Figure 4).

	2	3	5	7	11	13	
2	1	**PRIME**					
3	3/2	1	**PRIME**				
4	2	**COMPOSITE**					
5	5/2	5/3	1	**PRIME**			
6	3	**COMPOSITE**					
7	7/2	7/3	7/5	1	**PRIME**		
8	4	**COMPOSITE**					
9	9/2	3	**COMPOSITE**				
10	5	**COMPOSITE**					
11	11/2	11/3	11/5	11/7	1	**PRIME**	

Figure 4: A chart for identifying prime numbers.

T: Why do we identify the number 2 as a prime?

S: Because one gets an integer quotient in the cell with the coordinates (n, p) = (2, 2).

T: Good. But how can one instruct a computer to conclude that $2 \div 2$ is an integer and $3 \div 2$ is not?

S: In the case $2 \div 2$ the remainder is zero; in the case $3 \div 2$ the remainder is not zero.

T: Wonderful. So, one can carry out a logical test using either the greatest integer function, which, when applied to a number returns the largest integer smaller than the number.

S: If the value of the greatest integer function of the ratio $\frac{n}{p}$ is equal to $\frac{n}{p}$, then the test terminates otherwise it continues. Alternatively, if the remainder from the division $n \div p$ is zero, then the test terminates otherwise it continues.

T: Exactly! We can use a spreadsheet (Figure 5) to do this test using the function IF with the function INT embedded into it. Indeed, like in the chart, let us fill column A (beginning from cell A2) with consecutive positive integers n (numbers to be tested) and row *1* with consecutive prime numbers p (testing numbers); then, giving the number in the range B1:M1 name p, in cell B2 we define the formula =IF(INT(A2/p)=A2/p,0,A2). Furthermore, the spreadsheet can be set not to display zero values. To do that, one should enter the *Excel Preferences* menu, click at the *View* dialogue box, uncheck the *Zero Values* box, and click OK.

S: Is it possible to preserve prime numbers along all columns?

T: I have been awaiting this question. In other words, you want to distinguish between two cases of integral quotient—one case when this quotient is equal to one and another case when this quotient is not equal to one, don't you?

S: This is correct.

T: Well, what is the difference between tested and testing numbers when one gets into a prime and composite numbers, respectively?

S: In the case of a prime number this difference is zero, otherwise it is greater than zero.

T: There is the function *sign*(x), which, depending on whether $x > 0$, $x < 0$, or $x = 0$, assumes, respectively, three values only: 1 , -1, or 0. The use of this function can help to preserve primes. Indeed, if we change the content of cell B2 to

=IF(INT(A2/p)*sign(A2-p)=A2/p,0,A2),

then in the case of a prime number in cell A2, the condition is false and the action that must be taken is to display content of cell A2, i.e., this preserves a prime number.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1		2	3	5	7	11	13	17	19	23	29	31	37	
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
4	4													
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	6													
7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
8	8													
9	9	9												
10	10													
11	11	11	11	11	11	11	11	11	11	11	11	11	11	11
12	12													
13	13	13	13	13	13	13	13	13	13	13	13	13	13	13
14	14													
41	41	41	41	41	41	41	41	41	41	41	41	41	41	41
42	42													
43	43	43	43	43	43	43	43	43	43	43	43	43	43	43
44	44													
45	45	45												
46	46													
47	47	47	47	47	47	47	47	47	47	47	47	47	47	47
48	48													
49	49	49	49	49										
50	50													

Figure 5: A spreadsheet-based Sieve of Eratosthenes.

In order to find the smallest composite number which would be identified as a prime number by using all the primes from 2 to 37, one has to square the smallest prime number greater than 37. In that way, all the numbers smaller than $1681 = 41^2$ will be correctly tested for primality using just the first twelve primes.

5 Fermat primes and Euler's factorization method

One of the topics appropriate for a *History of Mathematics* course is a story about the so-called Fermat primes. Fermat—a French mathematician of the 17th century, one of the founders of modern number theory—conjectured that for all *n* = 0, 1, 2, ... the expression $2^{2^n}+1$ yields prime numbers only. This conjecture was based on inductive reasoning, as the cases *n* = 0, 1, 2, 3, and 4, indeed produce prime numbers 3, 5, 17, 257, and 65537, respectively. So Fermat believed that the number $2^{32} + 1$ also does not have divisors different from one and itself. However, a century later, Euler, using a factorization method that now bears his name, demonstrated that the case *n* = 5 produces a composite number. The method is based on the assertion that if an integer can be represented as a sum of two squares in two different ways, it is a composite number. The following dialogue introduces Euler's factorization method, which then will be applied to factoring $2^{32} + 1$ by using a spreadsheet.

T: Let *N* be an odd integer (otherwise, with the exception of two, *N* is a composite number) with two different representations as a sum of two squares, $N^2 = a^2 + b^2$ and $N^2 = c^2 + d^2$, where *a* and *c* are even numbers and, thus, *b* and *d* are odd numbers. How can the four numbers *a*, *b*, *c*, and *d* be connected?

S: This is easy, [writes on the board]

$$a^2 + b^2 = c^2 + d^2 \quad (4)$$

T: Is it possible to represent relation (4) as equality between the products of two factors?

S: A sum of two squares is not factorable.

T: What about a difference of two squares?

S: If I had $a^2 - b^2$, it could be factored as $(a-b)(a+b)$.

T: Could the terms in relation (4) be rearranged to allow for factoring its both sides?

S: [Writes on the board] $a^2 - c^2 = d^2 - b^2$ whence

$$(a-c)(a+c) = (d-b)(d+b) \qquad (5)$$

T: Very good! Now, I will help you a little bit to proceed from here. It is possible that $a - c$ and $d - b$ have common factors. Let $k = GCF(a-c, d-b)$. Then $a - c = kl$ and $d - b = km$. What can be said about l and m? Do they have common factors?

S: If they do, then k is not the greatest common factor.

T: Nice! How do we call numbers with no common factors different from one?

S: Relatively prime numbers.

T: Exactly. Thus, we can write

$$GCF(l,m) = 1 \qquad (6)$$

and it follows from (5) that $kl(a+c) = km(d+b)$ or, after cancelling out k,

$$l(a+c) = k(d+b) \qquad (7)$$

In turn, relation (7) implies that $a + c = nm$ and $d + b = nl$; therefore

$$N = \frac{1}{4}(2a^2 + 2b^2 + 2c^2 + 2d^2) = \frac{1}{4}[(a+c)^2 + (a-c)^2 + (b+d)^2 + (d-b)^2]$$
$$= \frac{1}{4}(n^2m^2 + k^2l^2 + n^2l^2 + k^2m^2) = \frac{1}{4}(k^2 + n^2)(l^2 + m^2).$$

S: How can this be explained numerically?

T: Well, consider the number 65. We have $65 = 8^2 + 1^2 = 4^2 + 7^2$. So,

$$65 = \frac{1}{4}(2\cdot 8^2 + 2\cdot 1^2 + 2\cdot 4^2 + 2\cdot 7^2) = \frac{1}{4}[(8+4)^2 + (8-4)^2 + (1+7)^2 + (1-7)^2]$$
$$= \frac{1}{4}(12^2 + 4^2 + 8^2 + 6^2) = \frac{1}{4}(4^2\cdot 3^2 + 2^2\cdot 2^2 + 4^2\cdot 2^2 + 2^2\cdot 3^2)$$
$$= \frac{1}{4}(4^2 + 2^2)(3^2 + 2^2) = \frac{1}{4}\cdot 4\cdot (2^2 + 1^2)(3^2 + 2^2) = 5\cdot 13.$$

Note that both factors of 65 are prime numbers and, along with their product, are congruent to one with modulus four; that is, when divided by four give the remainder one. [This observation is of a special importance for a discussion that follows].

In order to apply this method to $2^{32} + 1$, a spreadsheet can be used. Figures 6-9 show how using Euler's factorization method the following remarkable factorization can be found:

$$2^{32} + 1 = 4294967297 = 641\cdot 6700417.$$

It should be noted that one can find the two factors without much difficulty by using an on-line computational engine *WolframAlpha* (www.wolframalpha.com). However, the purpose of using a spreadsheet in this context is not the factoring per se, but rather the demonstration of a classic gem from the history of mathematics—Euler's factorization method. Likewise, one can use *WolframAlpha* to find out that both numbers, 641 and 6700417, are prime numbers.

	A	B	C	D	E	F	G	H	I
1	4294967297								
2	1	1	4294967296	65536	1	1	65536	1	YES
3	2	4	4294967293						
4	3	9	4294967288						

Figure 6: Locating the first sum of squares, $a^2 + b^2$.

	A	B	C	D	E	F
1004	a	c	a-c	a+c	k	
1005	65536				l	
1006	b	d	d-b	d+b	m	
1007	1				n	
1008	N	x	y			
1009	4294967297					

Figure 7: Entering *a* and *b* into the spreadsheet.

	A	B	C	D	E	F	G	H	I
1	4294967297								
2	20001	400040001	3894927296			1	62264	20449	YES
3	20002	400080004	3894887293						

Figure 8: Locating the second sum of squares, $c^2 + d^2$.

◇	A	B	C	D	E	F
1004	a	c	a-c	a+c	k	8
1005	65536	62264	3272	127800	l	409
1006	b	d	d-b	d+b	m	2556
1007	1	20449	20448	20450	n	50
1008	N	x	y			
1009	4294967297	641	6700417			

Figure 9: Completing prime factorization of $2^{32}+1$ after locating *c* and *d*.

6 Connecting number theory to geometry

Using the spreadsheet in Figure 6 one can find two unique representations of the two prime factors, $641 = 25^2 + 4^2$and $6700417 = 2556^2 + 409^2$. Due to the identities

$$(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2 \text{ and } (a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2,$$

this implies that their product would have two such representations, something that we have already discovered. However, the very factors, while could have been found without using a spreadsheet, have resulted from the use of a spreadsheet in demonstrating Euler's factorization method.

Note, that both the fifth Fermat "prime" and its factors are of the form 4*n* + 1. With this in mind, one can construct a spreadsheet to relate consecutive numbers of the form 4*n* + 1, *n* = 1, 2, 3, ... , to the number of their representations as a sum of two squares. Such a spreadsheet is shown in Figure 10. The number 65, used to illustrate Euler's factorization method, is the smallest number of the form 4*n* + 1 that has two representations as a sum of two squares. The spreadsheet in Figure 10 shows that, for example, the number 593 (cell A1) is the sum of two squares, $8^2 + 23^2$ (cells A10 and B10), and that such a representation is unique (cell E1). The uniqueness of the representation of a number of the form 4*n* + 1 as a sum of two squares does guarantee that the number is a prime number. For example, $45 = 6^2 + 3^2$, no other two squares can add up to 45, yet 45 is a composite number. By the same token, the form 4*n* + 1 does guarantee a representation of the corresponding number as a sum of two squares. Consider the range O3:P18—a part of the table relating an integer of the form 4*n* + 1 to its number of representations as a sum of two squares. The numbers 9, 21, 33, 49, 57 (being in an arithmetic progression) cannot be represented as the sum of two squares. An interesting question to explore is: Would this property continue with the growth of the terms of this arithmetic series?

	A	B	C	D	E	F	G
1	593				1		
2							
3	1					5	1
4	2					9	0
5	3					13	1
6	4					17	1
7	5					21	0
8	6					25	1
9	7					29	1
10	8	23	465	368	PT	33	0
11	9					37	1
12	10					41	1
13	11					45	1
14	12					49	0
15	13					53	1
16	14					57	0
17	15					61	1
18	16					65	2

Figure 10: Counting the number of representations as a sum of two squares.

One can also find out that the number 1105, which is the product $5 \cdot 13 \cdot 17$, is the smallest number of the form 4*n* + 1 that is expressible as a sum of two squares in four ways. Note, that just like 5 and 13, the number 17 is a prime number of the form 4*n* + 1. One can observe that multiplying prime numbers of the form 4*n* + 1 yields a composite number (also of that form) the number of representations of which as a sum of two squares doubles with each new factor. These computational experiments can facilitate the introduction of the following remarkable proposition inductively discovered by Fermat and only 100 years later formally proved by Euler.

THEOREM: *A prime number of the form 4n+1 is expressible as a sum of two squares in only one way, the product of two different primes of the form 4n+1 is expressible as a sum of two squares in two ways, the product of three different primes of the form 4n+1 is expressible as a sum of two squares in four ways, the product of four different primes of the form 4n+1 is expressible as a sum of two squares in eight ways. In general, the product of k different primes of the form 4n + 1 is expressible as a sum of two squares in* 2^{k-1} *ways.*

Note that the fact that every prime number of the form 4*n* + 1 can be represented as the sum of two squares in one and only one way, Fermat called the fundamental theorem of right triangles. The reason Fermat called his discovery the fundamental theorem of right triangles is that, as was known from the time of Euclid, the length side of the hypotenuse of a right triangle is a sum of two squared integers. Using the formulas $c = n^2 + m^2, b = 2nm, a = m^2 - n^2$, known already to Euclid—the most prominent Greek mathematician of the 3rd century B.C., for the elements of a Pythagorean triple (*a*, *b*, *c*), with *c* being the largest one, each such representation uniquely determines the side lengths of the legs. For example, when the side length of the hypotenuse of a right triangle is equal to 593, we have $593 = 23^2 + 8^2$ and, thereby, $23^2 - 8^2 = 465$, and $2 \cdot 23 \cdot 8 = 368$. From here, the triple of side lengths (465, 368, 593) of a Pythagorean triangle results.

At the same time, the number 1105 has four different representations as a sum of two squares and, therefore, there exist four different Pythagorean triangles with the hypotenuse equal to 1105. The spreadsheet pictured in Figure 11 demonstrates how those four triangles can be found computationally. In that way, several classic ideas of number theory and geometry that cover the time span of more than two thousand years have come together in a modern spreadsheet environment.

	A	B	C	D	E
1	1105				4
2					
3	1				
4	2				
5	3				
6	4	33	1073	264	PT
7	5				
8	6				
9	7				
10	8				
11	9	32	943	576	PT
12	10				
13	11				
14	12	31	817	744	PT
15	13				
16	14				
17	15				
18	16				
19	17				
20	18				
21	19				
22	20				
23	21				
24	22				
25	23	24	47	1104	PT

Figure 11: There exist four Pythagorean triangles with hypotenuse 1105.

7 Conclusion

The paper described the use of a spreadsheet in teaching topics in elementary number theory through a teacher-student dialogue. Both the power and deficiency of inductive reasoning has been demonstrated. The focus was on the experimental approach to mathematical ideas made possible by the use of a spreadsheet. In turn, an experiment motivates introduction and facilitates understanding of formal theory. Such an approach appears to be especially instructive in the context of preparation of mathematics teachers.

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Appendix

All computational environments presented in this paper are based on Excel 2008 (Mac)/2007 (Windows) versions. Note that syntactic versatility of the software enables the construction of both visually and computationally identical environments using syntactically different formulas. Below the notation (A1)→ will be used to present a formula defined in cell A1.

Figure 1.

(A2)→ = 1; (A3)→ =A2+1—replicated down column A; (B2)→ = 1; (B3)→ =B2+A3—replicated down column B; (C2)→ =A2^2; (C3)→ =A3^2+C2—replicated down column C; (D2)→ =B2/A2—replicated down column D; (E2)→ = C2/B2—replicated down column E.

Figure 6.

(A1)→ = 2^32+1

Cell A2 is slider-controlled and set to display numbers congruent to one modulo 1000;

(A3)→ =A2+1—replicated to cell A1001; (B2)→ = A2^2—replicated to cell B1001;

(C2)→ =IF(A$1-B2>0,A$1-B2," ")—replicated to cell C1001;

(D2)→ =IF(OR(C4=" ",C4<B4)," ",IF(SQRT(C4)=INT(SQRT(C4)),SQRT(C4)," "))—replicated to cell D1001;

(E2)→ =IF(COUNT(D2)>0,1," ");

(F2)→ =COUNTIF(D2:D30000,">0");

(G2)→ =IF(SUM(E2:E1001)=0," ",LOOKUP(1,E2:E1001,D2:D1001));

(H2) → =IF(SUM(E2:E1001)=0," ",LOOKUP(1,E2:E1001,A2:A1001));

(I2)→ =IF(G2=" "," ",IF(G2^2+H2^2=A1,"YES"," ")).

Figure 7.

(A1005)→ =IF(COUNT(D2)>0,G2,A1005);

(A1007)→ =IF(COUNT(D2)>0,SQRT(A1-A1005^2),A1007);

(A1009)→ =A1; (B1005)→ =IF(D2=G2," ",IF(COUNT(F2)>0,G2,C1005));

(B1007)→ =IF(B1005=" "," ",SQRT(A1-B1005^2));

(B1009)→ =IF(B1005=" ", " ", (F1004/2)^2+(F1007/2)^2);

(C1005)→ =IF(B1005=" "," ",A1005-B1005); (C1007)→ =IF(B1005=" "," ",B1007-A1007);

(C1009)→ =IF(B1005=" "," ", F1006^2+F1005^2);

(D1005)→ =IF(B1005=" "," ",A1005+B1005); (D1007)→ =IF(B1005=" "," ", A1007+B1007);

(F1004)→ =IF(B1005=" "," ",GCD(C1005,C1007));

(F1005)→ =IF(B1005=" "," ", C1005/F1004); (F1006)→ =IF(B1005=" "," ", C1007/F1004);

F(1007)→ =IF(B1005=" "," ",D1007/F1005).

Figures 10 and 11.

(A1)→ =4*C1+1; cell A1 is given name n; (A3)→ =1; (A4)→ =A3+1—replicated down column A; cell C1 is slider-controlled;

(B3)→ =IF(AND(n-a^2>0),IF(AND(a<=SQRT(n-a^2),SQRT(n-a^2) =INT(SQRT(n-a^2))),SQRT(n-a^2)," ")," ")—replicated down column B;

(C3)→ =IF(B3=" "," ",B3^2-A3^2)—replicated down column C;

(D3)→ =IF(B3=" "," ",(n^2-C3^2)^0.5);

(E1)→ =COUNT(B3:B1000); (E3)→ =IF(B3=" "," ",IF(n^2=C3^2+D3^2,"PT"," "));

(F3)→ =4*A3+1—replicated down column F;

(E3)→ =IF(n<5," ",IF(F3=n,E$1,G3))—replicated down column E.