Spreadsheets in Education (eJSiE)

Volume 5 | Issue 2 Article 2

May 2012

Fourier Analysis: Graphical Animation and Analysis of Experimental Data with Excel

Margarida Oliveira Prof.
Escola E.B. Piscinas-Lisboa, mcoliveira@fc.ul.pt

Suzana Nápoles Prof.
Departamento de Matemática da Universidade de Lisboa, napoles@ptmat.fc.ul.pt

Sérgio Oliveira Eng.
Laboratório Nacional de Engenharia Civil, sbmoliveira@gmail.com

Follow this and additional works at: http://epublications.bond.edu.au/ejsie

This work is licensed under a Creative Commons Attribution-Noncommercial-No Derivative Works 4.0 License.

Recommended Citation

Oliveira, Margarida Prof.; Nápoles, Suzana Prof.; and Oliveira, Sérgio Eng. (2012) Fourier Analysis: Graphical Animation and Analysis of Experimental Data with Excel, *Spreadsheets in Education (eJSiE)*: Vol. 5: Iss. 2, Article 2.
Available at: http://epublications.bond.edu.au/ejsie/vol5/iss2/2

This Regular Article is brought to you by the Bond Business School at ePublications@bond. It has been accepted for inclusion in Spreadsheets in Education (eJSiE) by an authorized administrator of ePublications@bond. For more information, please contact Bond University's Repository Coordinator.

Fourier Analysis: Graphical Animation and Analysis of Experimental Data with Excel

Abstract

According to Fourier formulation, any function that can be represented in a graph may be approximated by the "sum" of infinite sinusoidal functions (Fourier series), termed as "waves".The adopted approach is accessible to students of the first years of university studies, in which the emphasis is put on the understanding of mathematical concepts through illustrative graphic representations, the students being encouraged to prepare animated Excel-based computational modules (VBA-Visual Basic for Applications).Reference is made to the part played by both trigonometric and complex representations of Fourier series in the concept of discrete Fourier transform. Its connection with the continuous Fourier transform is demonstrated and a brief mention is made of the generalization leading to Laplace transform.As application, the example presented refers to the analysis of vibrations measured on engineering structures: horizontal accelerations of a one-storey building deriving from environment noise. This example is integrated in the curriculum of the discipline "Matemática Aplicada à Engenharia Civil" (Mathematics Applied to Civil Engineering), lectured at ISEL (Instituto Superior de Engenharia de Lisboa. In this discipline, the students have the possibility of performing measurements using an accelerometer and a data acquisition system, which, when connected to a PC, make it possible to record the accelerations measured in a file format recognizable by Excel.

Keywords

Fourier Analysis; movie clips in spreadsheets; spectral analysis; experimental data.

Distribution License

This work is licensed under a Creative Commons Attribution-Noncommercial-No Derivative Works 4.0 License.

1 Introduction

If we had to choose a mathematical topic extensively used in science and technology, the Fourier Analysis would be one of the first choices. Due to its various applications, this subject, which is included in most curricula of the first years of mathematics and engineering courses, is a privileged theme to illustrate a mathematical teaching methodology based on the development of interactive applications that interconnect the different types of mathematical knowledge. Using visual representations of the main mathematical notions involved, we promote a critical consideration on the associated concepts, which is essential to a deep understanding of the subject.

The use of such dynamic applications [8], [9] becomes more meaningful when compared with the mere use of text books. For a long time, we have used text books which contained only images of an inevitably static nature. The development of technologies led to the creation of appealing software containing animated graphs, usually developed in a sophisticated programming language, and thus becoming a possible teaching tool, but only from the user's point of view.

The approach addressed here aims to demonstrate that this subject can be studied in a structured and cohesive manner by interconnecting students' previous mathematical knowledge.

For the purpose, we use visual representations of the main mathematical notions involved to encourage students' critical thinking.

The representation of mathematical ideas has been one of the mathematicians' concerns over the years. Regardless of its type of representation, whether geometric, algebraic or graphic, it has been of the general consensus that mathematical learning becomes easier when different types of representations are used [1].

Based on the belief that the construction of different representations by the students themselves is an advantage, this article suggests the development and/or use of Excel-based computational applications [7], [6] with animations, in order to visualize some of the mathematical concepts involved.

For instance, using the application ***waves.xls*** it is possible to study trigonometric functions of the type $f(t)=a\cos(\omega t)+b\sin(\omega t)$ and to observe the changes in the graphs of these functions when the values of a, b and ω are modified. Using the application ***Fourier_movie.xls***, it is possible to visualize a movie showing the approximation of a function by a Fourier series in a given interval.

The use of a spreadsheet enables students to develop animated computational modules on their own, being an advantage when compared with the use of previously built modules. This tool makes it possible to obtain remarkable results in terms of graphic animations (of a high pedagogical value) with just some basic programming notions of Visual Basic (VBA – Visual Basic for Applications).

Lastly, we present an example, which, despite its simplicity, enables the students to understand the relevance of studying Fourier Analysis.

2 Decomposition of functions into sinusoidal waves. Fourier series

In many science and engineering areas, some of the situations studied involve the analysis of time-changing magnitudes. These variations are described by time functions f = f(t), which usually have a random variation and, hence, cannot be represented by mathematical expressions. Examples of this are the velocity of wind on a bridge v = v(t), or the displacement on top of a building during an earthquake u = u(t) [2].

Below, we will show that in these phenomena, which can be described by functions varying randomly over time, we can use a property of the functions, discovered by Fourier (1768-1830): "any function that can be represented in a graph may be decomposed into the sum of infinite sinusoidal waves".

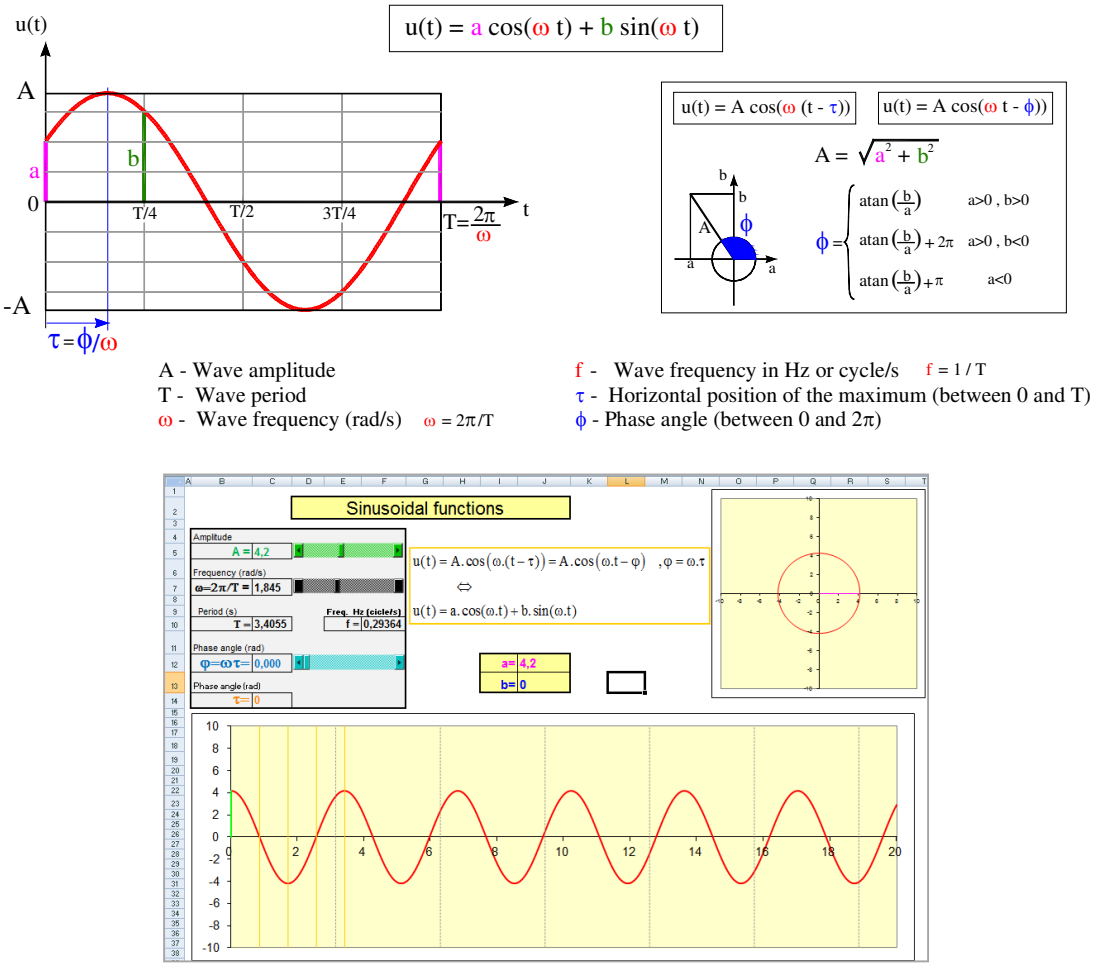


Figure 1: Sinusoidal wave functions. Geometric interpretation of parameters involved and visualization with the computational module ***waves.xls***.

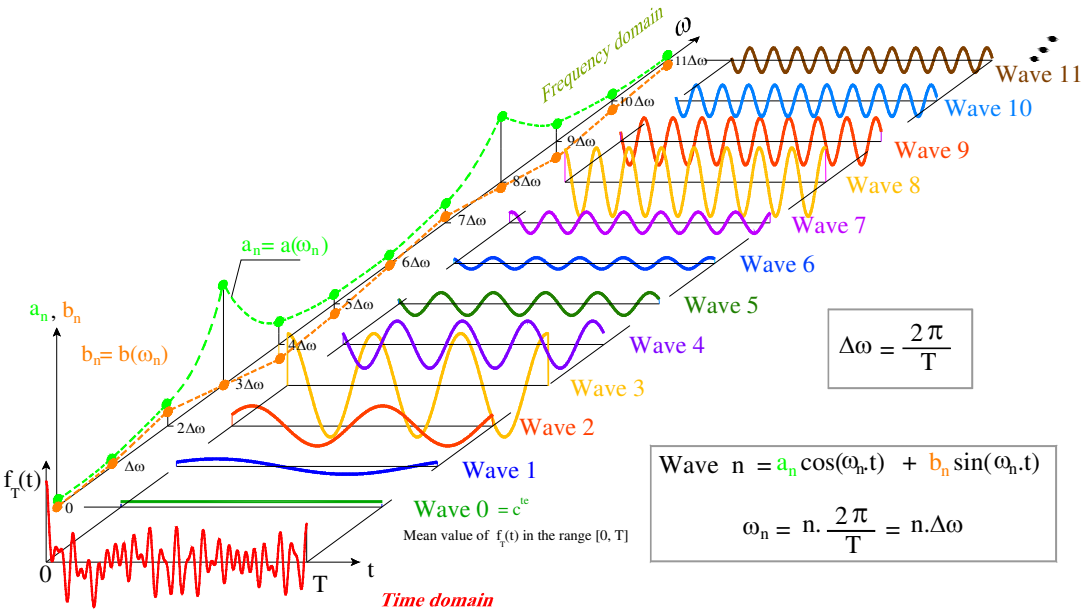
The application ***waves.xls*** makes it possible to experiment at the level of the alteration in the values of the wave amplitude and frequency and to observe the associated effects on the graph.

To make writing easier, we will designate as "wave" of frequency ω a sinusoidal function of the type

$$u(t)=a\cos(\omega t)+b\sin(\omega t) \tag{1}$$

of which the graphic representation is presented in figure 1, by emphasizing the geometric meaning of parameters a, b and ω.

Figure 2 shows the graphic representation of a function f(t) defined in the interval $[0,T]$, and its decomposition into "waves" (of increasing frequency), which corresponds to the mathematical concept on which Fourier Analysis is based.



Fourier serie approximation of the function: $f_T(t)$, $t \in [0, T]$

$$f_T(t) = \underset{\gamma_m}{c^{te}} + \text{Wave } 1 + \text{Wave } 2 + \text{Wave } 3 + \text{Wave } 4 + \ldots$$

$a_1 \cos(\omega_1.t) + b_1 \sin(\omega_1.t)$, $\omega_1 = 1.\Delta\omega$

$a_2 \cos(\omega_2.t) + b_2 \sin(\omega_2.t)$, $\omega_2 = 2.\Delta\omega$

$a_3 \cos(\omega_3.t) + b_3 \sin(\omega_3.t)$, $\omega_3 = 3.\Delta\omega$

...

The Discrete Fourier Transform of $f_T(t)$ is defined as the complex function:

$$F_T(\omega_n) = \frac{a_n - i\, b_n}{2} T \quad , -\infty < \omega_n < +\infty$$

where $a_n = a(\omega_n)$ and $b_n = b(\omega_n)$ are real functions representing the waves coefficients.

Figure 2: Decomposition into sinusoidal waves of a function f(t), defined in an interval [0, T] [4].

Afterwards, we will show that the coefficients a_n and b_n of each "wave" n can be determined through elementary mathematical concepts (particularly using the concept of mean value of a function in a given interval) and that the concept of Fourier Transform can be understood, in easier terms, as the result of the "junction" of values a_n and b_n into a single (complex!) figure of the form $(a_n - i b_n)T/2$. Actually, $a_n = a(\omega_n)$ and $b_n = b(\omega_n)$ are real functions of a discrete variable and

Fourier Transform $F(\omega_n) = (a_n - i\,b_n)T/2$ is a complex function of a discrete variable (Discrete Fourier Transform - DFT).

Usually, this mathematical subject is addressed in the first years of some science and engineering courses, and it is presented to the students as a theorem, from which a corresponding demonstration is shown and an example is presented. This approach is insufficient because it does not offer a deep insight into such an important subject as this [3].

It is essential for the students to know the process that led to Fourier's discovery and to be able to develop an application in a spreadsheet, in which it will be possible to visualize a function and its approximation by waves in a dynamic way, and to eventually be able to apply their knowledge to a specific case. This approach is addressed in the present work.

Fourier, in his studies about heat propagation in solids, has discovered that he could approximate "any" function f in a finite length interval T by a series – Fourier series – corresponding to the sum of a constant and of a set of infinite sinusoidal "waves" with periods equal to T and to its submultiples T, T/2, T/3, T/4, ...

Hence, these are waves with increasing frequencies given by

$$\frac{1}{T},\quad \frac{2}{T},\quad \frac{3}{T},\quad \ldots\quad \frac{10}{T},\quad \ldots \qquad \text{(in Hz)} \tag{2}$$

or

$$\frac{2\pi}{T},\quad 2.\frac{2\pi}{T},\quad 3.\frac{2\pi}{T},\quad \ldots\quad 10.\frac{2\pi}{T},\quad \ldots \qquad \text{(in rad/s)} \tag{3}$$

Considering $\Delta\omega = \frac{2\pi}{T}$, the frequencies of the mentioned "waves" are written as follows

$$\omega_1 = \Delta\omega,\quad \omega_2 = 2\Delta\omega,\quad \omega_3 = 3\Delta\omega,\quad \omega_4 = 4\Delta\omega,\quad \ldots\quad \omega_n = n\Delta\omega \tag{4}$$

Therefore, the expression corresponding to the approximation, in a Fourier series, of a given function f, in a certain interval of length T, can be written in simpler terms as follows

$$f_T(t) = c + \underbrace{\text{wave } 1}_{\omega_1 = \Delta\omega} + \underbrace{\text{wave } 2}_{\omega_2 = 2\Delta\omega} + \underbrace{\text{wave } 3}_{\omega_3 = 3\Delta\omega} + \quad \ldots \quad + \underbrace{\text{wave } n}_{\omega_n = n\Delta\omega} + \quad \ldots \tag{5}$$

In this expression, each sinusoidal wave ("wave n") can be written as the linear combination of trigonometric functions (cosine and sine), i.e.,

$$\text{wave n} = a_n \cos(\omega_n t) + b_n \sin(\omega_n t) \tag{6}$$

and

$$f_T(t) = c + \underbrace{\text{wave 1}}_{a_1 \cos(\omega_1 t) + b_1 \sin(\omega_1 t)} + \underbrace{\text{wave 2}}_{a_2 \cos(\omega_2 t) + b_2 \sin(\omega_2 t)} + \underbrace{\text{wave 3}}_{a_3 \cos(\omega_3 t) + b_3 \sin(\omega_3 t)} + \ldots \tag{7}$$

The resulting issue is to know how to determine constant c and the coefficients a_n and b_n of the various waves, which make it possible to approximate a given function f in a certain interval $[0,T]$.

2.1 Determination of the constant c

In the previous expression, it is possible to determine constant c using the concept of mean value of a function in an interval (figure 3). The mean value of a function in an interval of length T corresponds to the height of a rectangle of base T, of which the area is equal to the area below the function in the mentioned interval

$$v_m = \langle f(t) \rangle_T = \frac{1}{T} \int_0^T f(t)\,dt \tag{8}$$

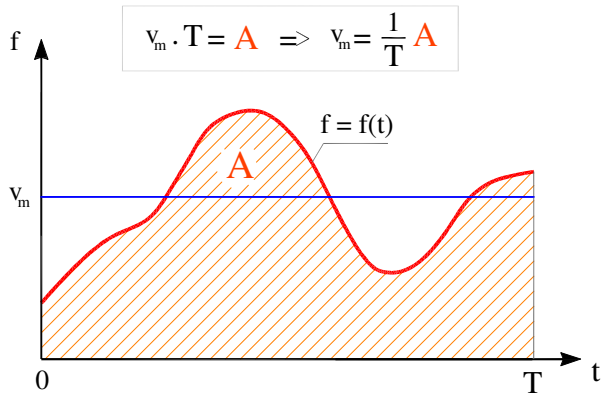


Figure 3: Use of the concept of integral to calculate the mean value of a function in an interval $[0,T]$.

Initially, Fourier observed that due to the periods of the various waves being submultiples of T, the mean value of each "wave n" in interval T was invariably null, i.e., $\langle \text{wave n} \rangle_T = 0$ with $n = 1,2,3 \ldots$

Thus,

$$\langle f_T(t) \rangle_T = \langle c \rangle_T + \underbrace{\langle \text{wave 1} \rangle_T}_{0} + \underbrace{\langle \text{wave 2} \rangle_T}_{0} + \underbrace{\langle \text{wave 3} \rangle_T}_{0} + \ldots \tag{9}$$

Therefore, the value of constant (c) must be exactly equal to the mean value of function f(t) in interval T, i.e.,

$$c = \langle f(t) \rangle_T = \frac{1}{T}\int_0^T f(t)\,dt \tag{10}$$

2.2 Determination of the coefficients a_n and b_n, for each wave n

To determine the coefficients of "wave 1", it is useful to make sure that the mean value in $[0,T]$ of each wave multiplied by $\cos(\omega_1 t)$ is always null, except for the case of the very "wave 1". Thus,

$$\langle f_T(t).\cos(\omega_1.t)\rangle_T = \underbrace{\langle c.\cos(\omega_1.t)\rangle_T}_{0} + \langle wave1.\cos(\omega_1.t)\rangle_T + \underbrace{\langle wave2.\cos(\omega_1.t)\rangle_T}_{0} + \ldots \tag{11}$$

becoming

$$\langle f(t).\cos(\omega_1.t)\rangle_T = \langle wave1.\cos(\omega_1.t)\rangle_T \tag{12}$$

i.e.,

$$\langle f(t).\cos(\omega_1.t)\rangle_T = \langle [a_1.\cos(\omega_1.t) + b_1\sin(\omega_1.t)].\cos(\omega_1.t)\rangle_T \tag{13}$$

which makes it possible to obtain

$$\langle f(t).\cos(\omega_1.t)\rangle_T = \underbrace{\langle a_1.\cos^2(\omega_1.t)\rangle_T}_{a_1/2} + \underbrace{\langle b_1.\sin(\omega_1.t).\cos(\omega_1.t)\rangle_T}_{0} = \frac{a_1}{2} \tag{14}$$

considering that the mean value of function $\cos^2(t)$ in interval $[0,2\pi]$ is precisely 1/2 (see figure 4).

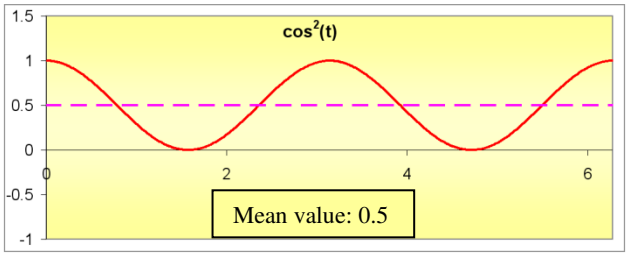


Figure 4: Graphic representation of function $\cos^2(t)$ and of the corresponding mean value in the interval $[0,2\pi]$ ($v_m = 1/2$).

Thus, it is possible to obtain coefficient a_1 as the double of the mean value in [0,T] of function f(t) multiplied by $\cos(\omega_1 t)$, i.e.,

$$a_1 = 2.\langle f(t).\cos(\omega_1.t)\rangle_T = \frac{2}{T}\int_0^T f(t)\,\cos(\omega_1 t)\,dt \tag{15}$$

Likewise, the mean value in [0,T] of each wave multiplied by $\text{sen}(\omega_1 t)$ is also invariably null, except for the case of wave 1, which gives the possibility of determining b_1, in a similar manner as the one used for determining a_1. Therefore, we can conclude that it should be

$$b_1 = 2.\langle f(t).\sin(\omega_1.t)\rangle_T = \frac{2}{T}\int_0^T f(t)\,\sin(\omega_1 t)\,dt \tag{16}$$

By applying this reasoning to the subsequent waves, we can easily conclude that the coefficients a_n and b_n can be determined through expressions similar to the former ones. Fourier concluded thus that the determination of the coefficients of the various waves of the series is simply just a matter of determining the mean values!

In brief, we can conclude that the approximation in a Fourier series of a function f, in a given interval of length T, can be represented (in its trigonometric form) by the following series (summation of infinite waves)

$$f_T(t) = c + \sum_{n=1}^{\infty} \text{wave } n = c + \sum_{n=1}^{\infty}\left(a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right) \quad , \ \omega_n = n\Delta\omega \quad , \ \Delta\omega = \frac{2\pi}{T}$$

of which the coefficients are obtained through the following averages

$$c = \langle f(t)\rangle_T = \frac{1}{T}\int_0^T f(t)\,dt$$

$$a_n = 2.\langle f(t).\cos(\omega_n.t)\rangle_T = \frac{2}{T}\int_0^T f(t)\,\cos(\omega_n t)\,dt \quad , \ n = 1, 2, 3, \ \ldots$$

$$b_n = 2.\langle f(t).\sin(\omega_n.t)\rangle_T = \frac{2}{T}\int_0^T f(t)\,\sin(\omega_n t)\,dt \quad , \ n = 1, 2, 3, \ \ldots$$

2.3 Fourier sums to approach a simple piecewise function. Excel application

Let us consider function f(t) defined in interval $\mathbf{T} = [0, 5]$, of length $\mathbf{T} = 5$

$$f(t) = \begin{cases} t \ , & 0 \le t < 2.5 \\ 1 \ , & 2.5 \le t \le 5 \end{cases} \tag{17}$$

To obtain the approximation of this function through a Fourier series, a computational application can be developed, which will make it possible, for each wave n, to calculate automatically the values of ω_n, a_n e b_n .

Figure 5 shows the application developed in Excel. A table is created with the values of (t, f(t)) and then two columns to calculate $f(t)\cos(\omega_n t)$ and $f(t)\sin(\omega_n t)$. Of note is the fact that initially a cell is reserved to T and another one to $\Delta\omega : \Delta\omega = 2\pi / T$. For each wave n (of which the value is introduced in cell D5), the value of $\omega_n : \omega_n = n\,\Delta\omega$ is calculated. Since $a_n = 2\langle f(t)\cos(\omega_n t)\rangle$ and $b_n = 2\langle f(t)\sin(\omega_n t)\rangle$, we just have to use

function ***Average()*** of Excel, which makes it possible to calculate the mean value of a set of values.

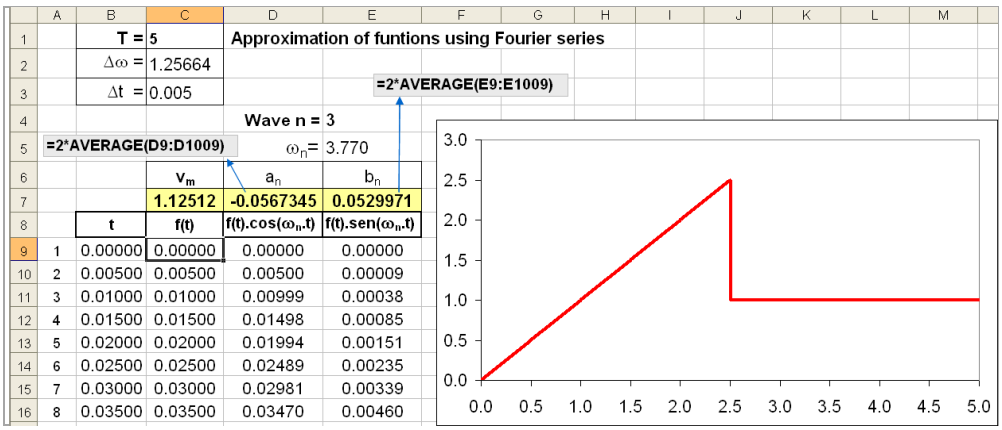


Figure 5: Application developed in Excel: ***Fourier_movie.xls***.

Figure 6 shows the results obtained after using the application above for waves 1, 2, 3 and 10.

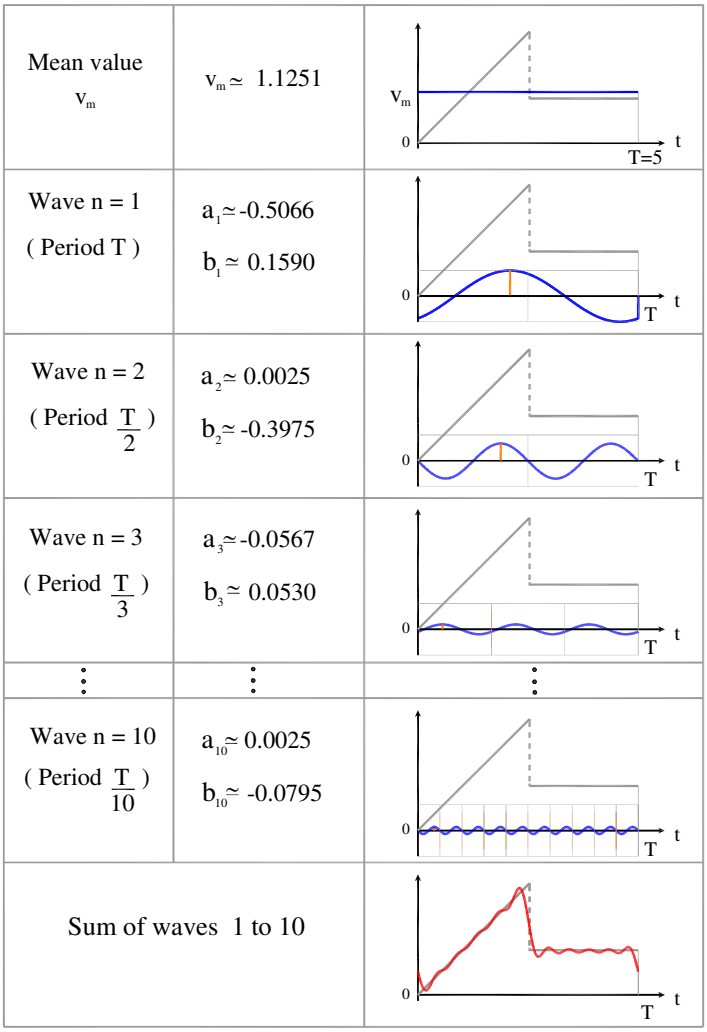


Figure 6: Results obtained by using the application ***Fourier_movie.xls***.

To complete the application, it will be useful to add a command button, which, after clicking, will give the possibility of visualizing (as a movie) the approximation of the function by sum of waves.

Figure 7 shows a table being created after clicking the button "***Fourier Coefficients***". In the last column (column X), a formula is to be introduced for calculating the sum of the values of the previous columns. For the purpose, the function Sum of Excel is used.

By adding the last series of data to the graph, the intended approximation will be obtained, which will make it possible to create an elucidative animation.

```
Private Sub CommandButton1_Click()
'Cleaning
Worksheets("Folha1").Range("I2:BF4").Clear
For n = 1 To 6
Cells(4, 5) = n
Cells(2, 8 + n) = Cells(5, 5) 'wn
Cells(3, 8 + n) = Cells(7, 4) 'an
Cells(4, 8 + n) = Cells(7, 5) 'bn
pausa (0.01)
Next n
```

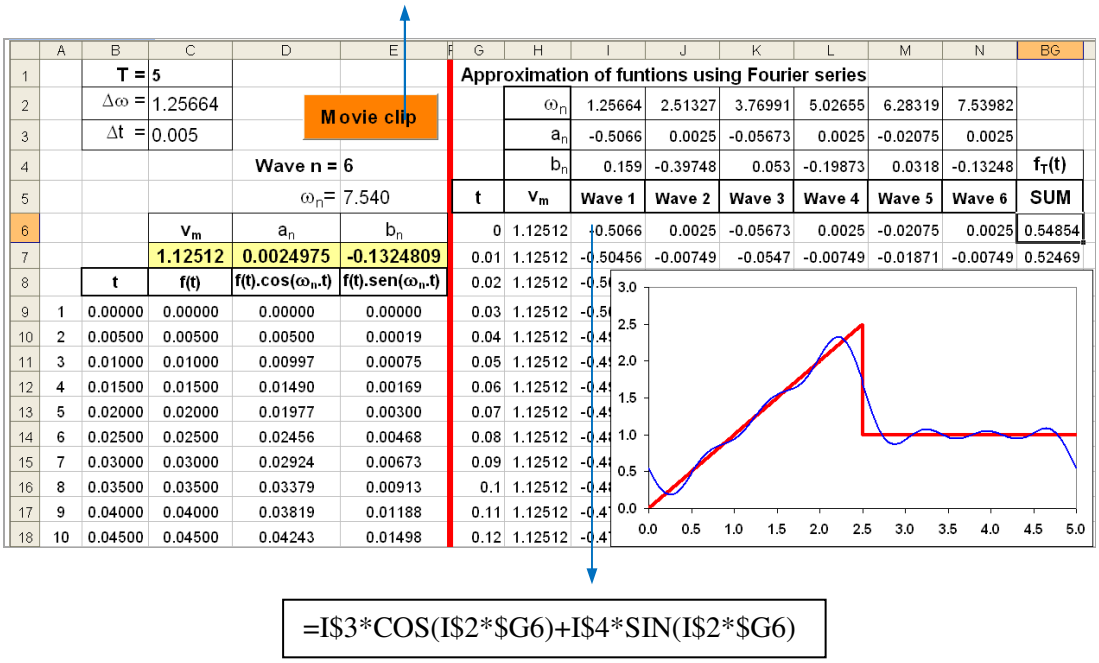


Figure 7: Approximation of functions using Fourier series (***Fourier_movie.xls***).

Figure 8 present the approximation of the function f(t) by partial Fourier sums of 5, 10, 20 and 50 waves.

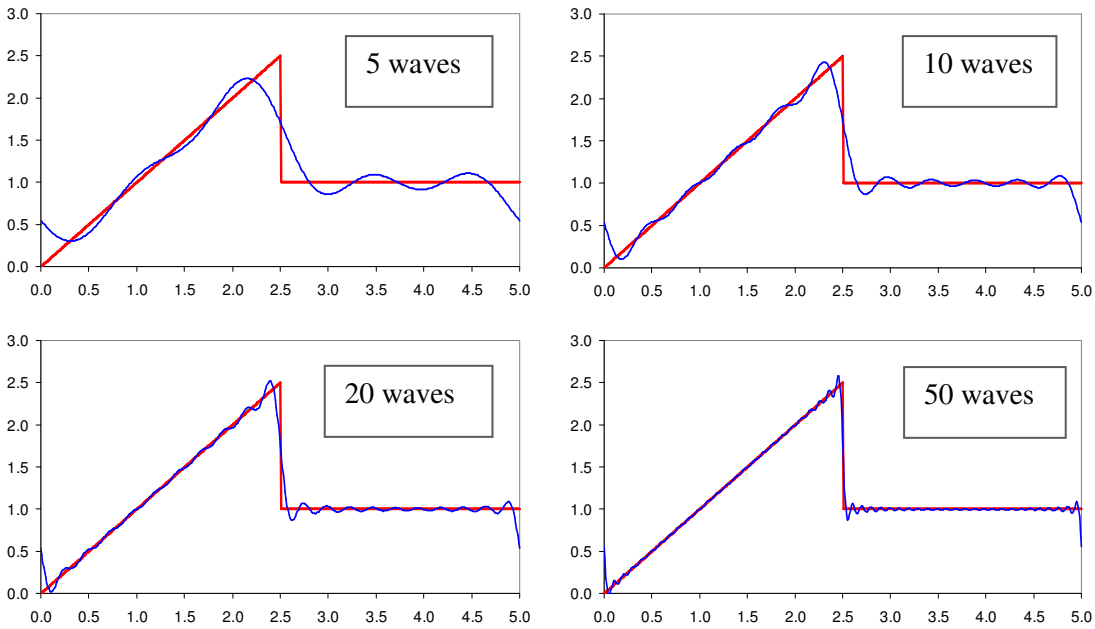


Figure 8: Approximation of the function f(t) by partial Fourier sums (***Fourier_movie.xls***).

3 Fourier series in complex form: Discrete Fourier Transform

3.1 From Fourier series to Fourier Transform

The expression previously deduced and referring to the development of a function in a Fourier series (function defined in an interval of length T)

$$f_T(t) = v_{med} + \text{wave}1 + \text{wave}2 + \text{wave}3 + \ldots = v_{med} + \sum_{n=1}^{+\infty}\left(a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right) \qquad (18)$$

corresponds to the so-called trigonometric form of Fourier series.

At this point, we will observe that this expression can be written in a more compact way, using the complex representation of cosine and sine functions, thus obtaining the so-called representation of Fourier series in its complex form.

Indeed, using Euler formula $e^{ix} = \cos x + i \sin x$, we can write

$$\cos(\omega_n t) = \frac{e^{i\omega_n t} + e^{-i\omega_n t}}{2} \qquad \text{and} \qquad \sin(\omega_n t) = \frac{-ie^{i\omega_n t} + ie^{-i\omega_n t}}{2} \qquad (19)$$

therefore, based on these expressions, we can obtain the intended expression corresponding to the complex form of Fourier series. In fact,

$$f_T(t) = v_m + \sum_{n=1}^{+\infty}\left(a_n \frac{e^{i\omega_n t} + e^{-i\omega_n t}}{2} + b_n \frac{-ie^{i\omega_n t} + ie^{-i\omega_n t}}{2}\right) \qquad (20)$$

$$f_T(t) = v_m + \sum_{n=1}^{+\infty}\left(\frac{a_n - ib_n}{2} e^{i\omega_n t} + \frac{a_n + ib_n}{2} e^{-i\omega_n t}\right) \qquad (21)$$

$$f_T(t) = \frac{a_0 - i\,b_0}{2} e^{i\omega_0 t} + \sum_{n=1}^{+\infty} \frac{a_n - i\,b_n}{2} e^{i\omega_n t} + \sum_{n=-\infty}^{-1} \frac{a_n - i\,b_n}{2} e^{i\omega_n t} \tag{22}$$

which, assuming now that n = … ,-3, -2, -1, 0 ,1, 2, 3, … , it can be simplified as follows:

$$f_T(t) = \sum_{n=-\infty}^{\infty} \frac{a_n - i\,b_n}{2} e^{i\omega_n t}, \qquad -\infty < \omega_n = n.\Delta\omega < +\infty \tag{23}$$

being designated as *complex form of Fourier Series*.

Considering the previous expressions for a_n and b_n, we can observe that

$$\frac{a_n - i\,b_n}{2} = \frac{1}{T}\int_0^T f_T(t) e^{-i\omega_n t} dt \tag{24}$$

Indeed, considering the definition of a_n and b_n, we have

$$\frac{a_n - i\,b_n}{2} = \frac{1}{T}\int_0^T f(t).\cos(\omega_n t)dt - i\frac{1}{T}\int_0^T f(t).\sin(\omega_n t)dt \tag{25}$$

$$\frac{a_n - i\,b_n}{2} = \frac{1}{T}\int_0^T f(t).\underbrace{\left(\cos(\omega_n t) - i.\sin(\omega_n t)\right)}_{e^{-i\omega_n t}} dt \tag{26}$$

i.e.,

$$\frac{a_n - i\,b_n}{2} = \frac{1}{T}\int_0^T f(t)\, e^{-i\omega_n t} dt \tag{27}$$

We usually designate as Discrete Fourier Transform of the function f(t), in the finite interval of length T, the complex function $F_T(\omega_n)$ (function of a discrete real variable, ω_n) given by

$$F_T(\omega_n) = \int_0^T f_T(t) e^{-i\omega_n t} dt\ , \qquad -\infty < \omega_n = n.\Delta\omega < +\infty \tag{28}$$

or

$$F_T(\omega_n) = \frac{a_n - i\,b_n}{2}.T\ , \qquad -\infty < \omega_n = n.\Delta\omega < +\infty \tag{29}$$

So, $F_T(\omega_n)$ is a complex function with real part $a(\omega_n).T/2$, and imaginary part $-b(\omega_n).T/2$.

Hence, the graphic representation of the Discrete Fourier Transform $F_T(\omega_n)$, of a given time function $f_T(t)$, must always be based on two graphs, the representations adopted for the graph of the real part being $\mathrm{Re}\left(F(\omega_n)\right) = a(\omega_n)T/2$ (even function); and for the graph of the imaginary part being $\mathrm{Im}\left(F(\omega_n)\right) = -b(\omega_n)T/2$ (odd function).

3.2 Use of computational modules to compute Discrete Fourier Transforms of functions defined over intervals of finite length T

When the aim is to decompose a given function defined in a time interval of length T into sinusoidal waves (the function is generally defined in a table, considering a regular discretization of the time interval), the usual procedure is to use one of the various programs available on the market (paid programs or free-access programs), with computational modules (based on an algorithm with great computational efficiency, designated as Fast Fourier Transform FFT) to calculate Discrete Fourier Transforms. This will enable us to avoid the most laborious procedure (and less efficient in computational terms), addressed in the previous paragraph, which is based on the direct calculation of the averages included in the original definition of coefficients a_n and b_n of Fourier series.

As demonstrated in 3.1, the DFT of a function incorporates the values of coefficients a_n and b_n of each corresponding constituent wave into a single (complex) number of the form $F_T(\omega_n) = (a_n - i\,b_n).T/2$. Therefore, based on the knowledge of the values of the Discrete Fourier Transform $F_T(\omega_n)$ of a given function f, we can easily obtain the coefficients of each of the various sinusoidal waves integrating the function (waves of frequency $\omega_n = n.(2\pi/T)$, n = 1, 2, 3 …), through the following relations

$$a_n = \frac{2\,\mathrm{Re}\left(F_T(\omega_n)\right)}{T} \qquad \text{and} \qquad b_n = -\frac{2\,\mathrm{Im}\left(F_T(\omega_n)\right)}{T} \tag{30}$$

In many of the mentioned computational modules available on the market to calculate DFT, the user only has to indicate the number of points used in the discretization, NP (sometimes, the programs require the NP to be exactly a power of 2: for instance, 128, 256, 512, 1024 …) and does not need to specify beforehand which was the adopted discretization of domain, i.e., the program instantly assumes $\Delta t = 1$. This means that, in the end, the user must correct the NP values directly indicated for $F_T(\omega_n) = (a_n - i\,b_n).T/2$, by multiplying them by the exact value of Δt (Note: in this modulus, the first half of the complex NP values calculated for $F_T(\omega_n)$ corresponds to positive ω_n (0, $\Delta\omega$, $2\Delta\omega$, $3\Delta\omega$, … , $(NP/2)\Delta\omega$) and the second half to values of ω_n symmetric of the former ones ($-(NP/2)\Delta\omega$, … , $-3\Delta\omega$, $-2\Delta\omega$, $-\Delta\omega$). Hence, the desired information about the constituent waves of the function is entirely integrated in the first half of the complex values calculated for $F_T(\omega_n)$.

Figure 9 shows, as an example, the way a spreadsheet can be organized to determine the Discrete Fourier Transform and, then, it shows the Fourier coefficients corresponding to the various sinusoidal waves constituent of a function f(t) defined in a table.

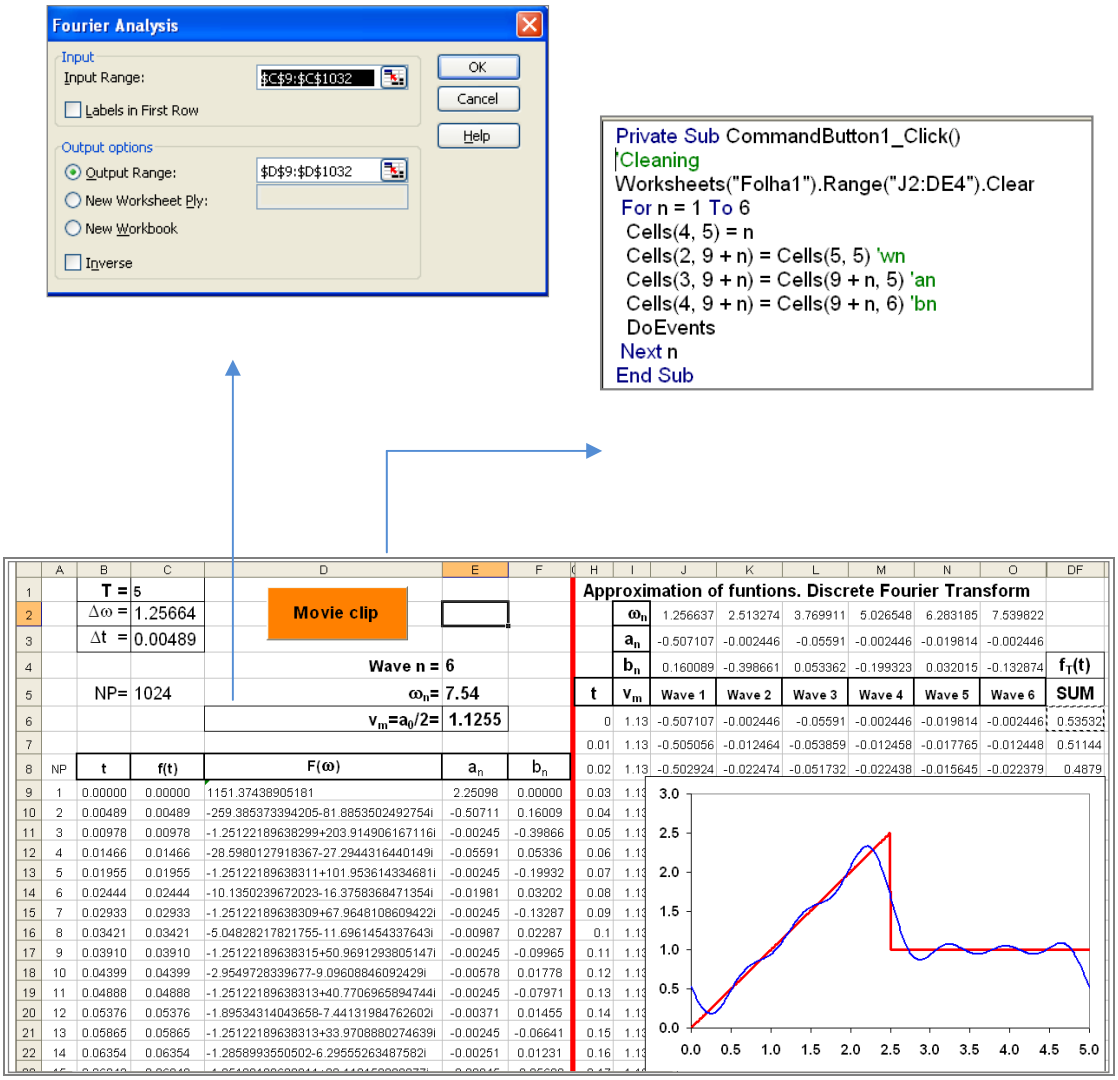


Figure 9: Approximation of functions. Discrete Fourier Transform.

3.3 From Fourier Series to Fourier Integral. Continuous Fourier Transform and Laplace Transform

As a result of the previous definition of Discrete Fourier Transform $F_T(\omega_n)$, the approximation of a function $f_T(t)$, defined in an interval of finite length T, can be represented as a Fourier series in its complex form

$$f_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F_T(\omega_n)e^{i\omega_n t}, \qquad \omega_n = n.\Delta\omega\ , \quad \Delta\omega = \frac{2\pi}{T} \tag{31}$$

Using this notation, and considering that $1/T = \Delta\omega/2\pi$, we can write the approximation of $f_T(t)$ by a complex Fourier series as follows

$$f_T(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F_T(\omega_n)e^{i\omega_n t}\Delta\omega \tag{32}$$

When $T \to \infty$ (we can assume a domain interval of the form $]-T/2,\ T/2[$, which tends to $]-\infty, +\infty[$) the function f(t) is defined in R, and $\Delta\omega$ tends to $d\omega$ ($\Delta\omega \to d\omega$). The discrete variable ω_n tends to a continuous variable ω ($\omega_n \to \omega$), and the summation signal should thus "be replaced" by the integral signal ($\sum \to \int$: "sum of infinite infinitesimal parcels ") becoming then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega).e^{i\omega t}\, d\omega\,, \qquad \text{with} \quad F(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t} dt \tag{33}$$

the so-called Continuous Fourier Transform of the function f(t) defined in R.

Lastly, it is interesting to notice that the Laplace transform currently used to solve differential equations can be seen as a generalization of Fourier Transform, in which the pure imaginary $i\omega$ is replaced by the complex $s = \sigma + i\omega$. On the basis of this conclusion, such a sequence could be adopted in Mathematics curricula for introducing the subject of Laplace transforms, enabling thus the students to easily understand it.

4 Use of Fourier series for the interpretation of experimental results. Decomposition of accelerograms into sinusoidal waves

Know the "waves" that compose a function, besides allowing the development of powerful mathematical tools to solve differential equations, may have a direct application to the analysis of experimental results, as shown below.

Let us assume, for instance, the experimental analysis of the dynamic behaviour of a scale model of a one-storey building with a 5 kg mass and 8600 N/m stiffness (a single-degree dynamic system like the mass-spring system presented in [5]).

With a small accelerometer and a proper acquisition system, we obtained a record of the horizontal accelerations (see figure 10). The accelerations were measured for 15 s using a sample frequency of 51.2 values per second (512 in 10 s), on a model of a one-storey building (horizontal accelerations measured at floor level, according to the lower stiffness direction), being only subject to the action of the so-called white noise or environment excitation (voices, draughts, etc.).

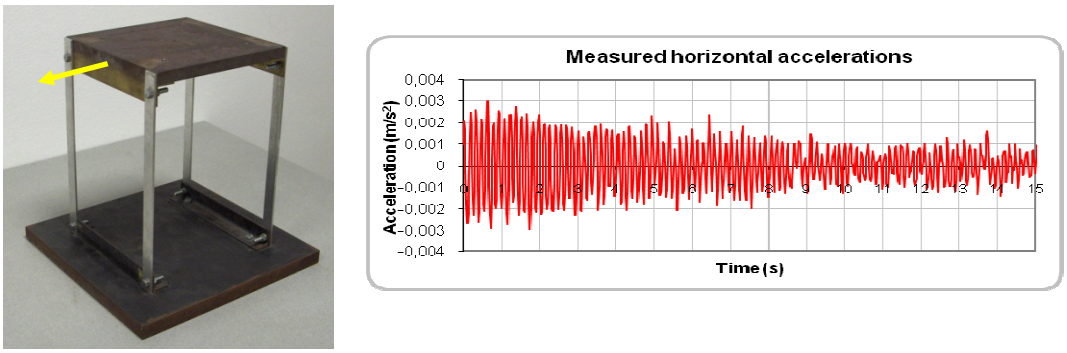


Figure 10: Scale model of a 1-storey building and a record of horizontal accelerations due to environmental excitation.

On the basis of the acceleration graph of the figure above, it is not possible to obtain information about the dynamic characteristics of the structure; we can only state that during the acquisition time (15 s), the structure (scale model) presented horizontal accelerations, with maximum values of about 0.03 ms^{-2}.

For instance, it would be useful to obtain information about the natural frequency ω_N of the structure.

Is it possible to obtain further information in the record to disclose more information about the dynamic characteristics of the structure? Clearly, the measured accelerations are necessarily influenced by the structural characteristics of the building; in another building, under the same excitation conditions, but with different mass characteristics and/or stiffness, we would most certainly obtain a different acceleration record!

This leads us to believe, that this particular record is likely to contain some further information about the structure. The notion is that it might be possible to have access to that information if the accelerogram is decomposed into waves, using Fourier's concept.

First, we included in the spreadsheet the values of the addressed function, in table format (in this case, the function corresponds to the accelerations measured throughout 15 s): in a first column we entered the values of t and in a second column the values of the measured accelerations, which corresponds, in this case, to 768 pairs of measured values since 51.2 acceleration values per second were recorded, for a 15-second period.

Using the application presented in section 2.3, coefficients a_n and b_n, as well as the mean value, are automatically computed (Figure 11). Repeating that procedure for all subsequent waves (a Visual Basic routine can be used) makes it possible to get the values of a_n and b_n for all waves. The wave's amplitudes A_n are also computed in order to get the Amplitude Spectrum (figure 12), that is a function of wave's frequencies (in cycles/s or Hz; 1 Hz = 2π rad/s).

The analysis of the amplitude spectrum, show that, from among the various waves, into which the measured accelerogram can be decomposed, the frequency wave of approximately 6.733 Hz clearly stands out due to its amplitude – the amplitude spectrum presents a very well defined peak in the frequency 6.733 Hz.

Indeed, this result is the intended "new information" about the structure, which could not be obtained through the direct analysis of the measured accelerogram (time field): the observed peak in the Spectrum of amplitudes means that the Fourier discovery about the possibility of decomposing functions into series of sinusoidal waves gives the possibility of identifying experimentally the natural frequency of the model of the one-storey building tested: in this case, the natural frequency identified is $\omega_N = 6{,}733\,\text{Hz}$.

This technique of decomposition of functions into sinusoidal waves is presently a mathematical tool of significant practical interest, which, as observed, has a direct application to civil engineering and to many other areas, such as medicine, chemistry, astronomy, telecommunications, image processing, etc.

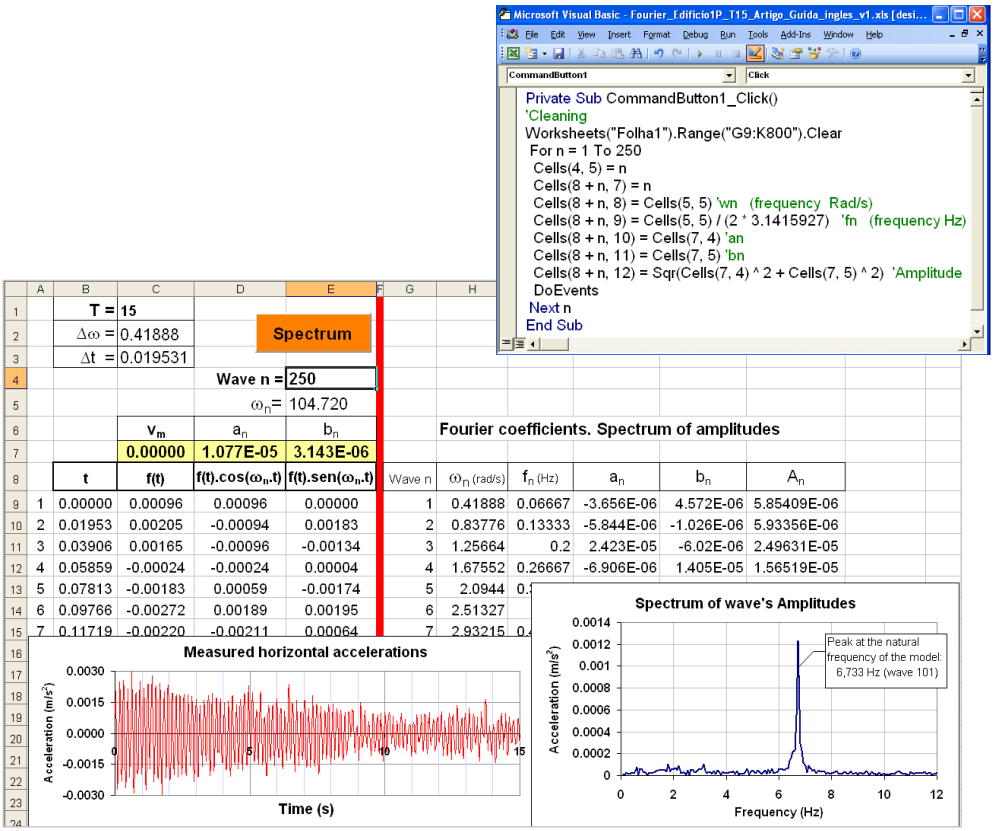


Figure 11: Decomposition of an accelerogram into waves. Organization of a spreadsheet to compute Fourier coefficients using the average function, with a subroutine in VisualBasic to calculate the amplitude spectrum (***Spectrum.xls***).

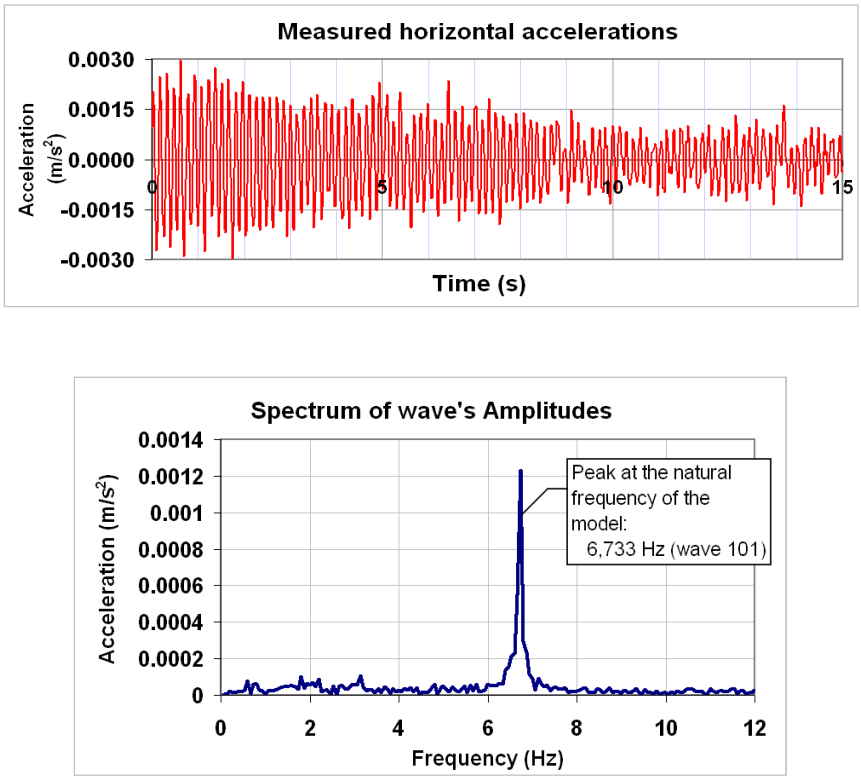


Figure 12: Measured accelerogram and corresponding amplitude spectrum: the wave amplitudes forming the measured accelerogram are represented as a function of the corresponding frequency (cycles/s or Hz).

Once the stiffness and mass values of the building model are determined, we may calculate analytically the value of the natural frequency, being only necessary to determine the value of expression $\sqrt{k/m}$ for k=8600 N/m and m=5 Kg, which corresponds to the value of the natural frequency. The value obtained is 6.60 Hz, which is approximately equal to the one obtained experimentally.

5 Conclusion

The presented innovative approach of the Fourier analysis allows us to show how the computational modelling can be used to facilitate a deep study of a mathematical topic that is generally included on the mathematics subjects of the engineering and science courses. It is a topic which enables students to see interesting connections between technology, science, engineering and many of the main topics studied since high school: trigonometric and exponential functions, integrals, Euler's formula for complex numbers, series, Fourier and Laplace transforms, numerical methods, etc. It is proposed a methodology where both students and teachers should engage themselves on the development of interactive computational applications with command buttons, scroll bars, suggestive graphics and animations.

An innovative graphical representation is presented to illustrate how Fourier series are used to approximate functions defined over a finite interval ("approximation through a sum of infinite waves with increasing frequencies"). The mathematical formalism is gradually introduced.

It is shown that when we get a deeper understanding of the subject we are able to see the beauty and simplicity of the mathematics used: after all, the coefficients of the Fourier series can be computed as simple average values, making use of the ***Average()*** built-in Excel function! That's it!

It is shown in detail how an Excel spreadsheet can be organized to compute (simple VBA programming) the nth first coefficients of the Fourier series that approximates any function (with graphical representation) defined over a real finite interval.

There's an animation included on the suggested computational application which, for any given function, shows how the function can be approximated considering the partial sum of the first n^{th} terms of the corresponding Fourier series.

Lastly it is shown how Fourier analysis can be used over a set of experimental data. The vibrations of a scale model of a 1-floor building were studied. The measured acceleration record (collected using an accelerometer and a data acquisition system) was analysed with the Excel application to obtain the corresponding amplitude spectrum. A peak was clearly identified at a frequency that corresponds to one of the main structural parameters used by civil engineers to design a seismic resistant building: the natural frequency of the first vibration mode of the building.

References

[1] Arganbright, D. (2005). Enhancing Mathematical Graphical Displays in Excel through Animation. *Spreadsheets in Education (eJSiE),* Vol. 2: Iss. 1, Article 8. Available online at http:/epublications.bond.edu.au/ejsie/vol.2/iss1/8/.

[2] DiPrima, R. (2000). Elementary Differential Equations and Boundary Value Problems. *John Wiley & Sons, Inc.*

[3] Sacristán, A.I., Calder, N., Rojano, T., Santos-Trigo, M., Friedlander, A. and Meissner., H., Tabach, M., Moreno, L., Perrusquía, E. (2010). The Influence and Shaping of Digital Technologies on the Learning - and Learning Trajectories - of Mathematical Concepts, in *Mathematics Education and Technology - Rethinking the terrain, The 17th ICMI Study.* 179-226, Springer 2010, Celia Hoyles and Jean-Baptiste Lagrange (eds).

[4] Oliveira, S. (2007). Mathematics applied to Civil Engineering (in Portuguese). http://sites.google.com/site/maecisel2/fichas.

[5] Oliveira, M., Nápoles, S. (2010). Using a Spreadsheet to study the oscillatory movement of a mass-spring system. *Spreadsheets in Education (eJSiE)*, Vol. 3: Iss. 3, Article 2. Available online at http:/epublications.bond.edu.au/ejsie/vol3/iss3/2/.

[6] Robinson, G., Jovanoski, Z. (2011). The use of Microsoft Excel to Illustrate Wave Motion and Fraunhofer Diffraction in First Year Physics Courses. *Spreadsheets in Education (eJSiE),* Vol. 4: Iss. 3, Article 5. Available online at http:/epublications.bond.edu.au/ejsie/vol4/iss3/5/.

[7] Sugden, S. (2011). The slumbarumba jumbuck problem. *Spreadsheets in Education (eJSiE),* Vol. 5: Iss. 1, Article 2. Available online at http:/epublications.bond.edu.au/ejsie/vol5/iss1/2/.

[8] Walkenbach, J. (2004). Excel 2003 Power Programming with VBA. *Wiley Publishing Inc. USA, John Wiley & Sons.*

[9] Wischniewsky, W.A.L. (2008). Movie-like Animation with Excel's Single Step Iteration Exemplified by Lissajous Figures. *Electronic Journal of Spreadsheets in Education.* Available online, http:/epublications.bond.edu.au/ejsie/vol3/iss1/4/.