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Bond Duration: A Pedagogic Illustration

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Abstract
In view of the importance of reliable duration measures for bond immunization strategies, this paper considers, from a pedagogic perspective, the duration concept. Both a basic bond model and a more realistic bond model, which accounts for the accrued interest, are considered. The use of graphical features and scroll bars in Microsoft Excel^{TM} allows the duration concept to be delivered via an interactive approach. This paper also addresses an unresolved issue on bond duration. Specically, it explains why a saw-toothed time pattern of traditionally-defined duration exists and provides a corrective measure, which is easy to implement on Excel. Analytical materials pertaining to bond duration, as well as illustrative examples based on actual bond quotation data, are provided.

Keywords
bond, duration, accrued interest

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1 Introduction

The concept of bond duration dates back to the seminal work of Frederick Macaulay (1938). Macaulay's duration is a well-known quantitative measure for use in bond immunization against interest-rate changes. By definition, it is a weighted average of arrival times of cash flows from a bond investment, with the individual weights based on the present values of the corresponding cash flows. It measures the sensitivity of the price of a bond in response to interest-rate changes. The greater the measure, the greater is the bond's price sensitivity.

Justification of Macaulay's duration in standard investment textbooks, if provided, is typically confined to a basic bond model. In the model, the valuation date is either the issue date of a new bond or a subsequent date at which a coupon has just been paid; in either situation, there is a full period before the next coupon payment. In the case of a semi-annual coupon bond, for example, a full period is six months and the valuation date in the model is six months prior to the next coupon date. The use of differential calculus allows the corresponding proportional changes in bond price and one-plus-interest-rate each period (in opposite directions) to be related analytically. The positive proportionality constant turns out to be Macaulay's duration.

In practice, however, Macaulay's duration has been computed routinely for dates that a basic bond model cannot accommodate. So has its well-known variant, called modified duration, which measures the percentage price change corresponding to a yield change of 100 basis points (or, equivalently, of 1.00%). Currently, some subscription-based daily bond quotations also include these duration measures to assist investment decisions.[1] Further, these duration measures are among the financial functions in Microsoft Excel^{TM} and various other spreadsheets.[2]

As illustrated pedagogically in Feng and Kwan (2011), if the valuation date does not match any of the coupon dates of a bond, the accrued interest must be accounted for in bond-price determination. Here, the accrued interest is the proportion of the next coupon that the seller of a bond is entitled to receive from the buyer. Accordingly, the cash flows from a bond investment

[1] See, for example, the GlobeInvestor Gold website (http://gold.globeinvestor.com/) of The Globe and Mail, a Canadian national newspaper.

[2] Macaulay's duration and modified duration can be computed by using Excel functions DURATION and MDURATION, respectively. Analogous functions of the same names are available in Google Docs^{TM} as well. The corresponding functions in OpenOffice.org Spreadsheet^{TM} are DURATION_ADD and MDURATION; in Apple Numbers^{TM}, they are BONDDURATION and BONDMDURATION instead. Although these spreadsheets, among others, are also suitable for the pedagogic purposes of this paper, Excel is selected in view of its popularity among many spreadsheet users, including students in investment courses at our univerities.

also include the accrued interest, which is an immediate cash outflow. As this cash outflow seems to be absent from the expressions of Macaulay's duration and modified duration, its impact on their computed values, if any, is not immediately obvious. Neither is whether the two duration measures, as covered in standard investment textbooks and implemented in Excel, are versatile enough to be used for dates that a basic bond model cannot accommodate.

The following is what is known thus far: Macaulay's duration has a peculiar feature, if computed for different valuation dates including those that a basic bond model cannot accommodate. The feature can be inferred from three separate studies in the 1980's. Specifically, Caks et al. (1985) and Chua (1988) showed that, in the absence of any changes in the discount rate as time passes by, Macaulay's duration always declines linearly from one coupon date to the next. While confirming the same analytical feature, Babcock (1986) also illustrated graphically a saw-toothed time pattern of Macaulay's duration, with a sudden upward jump at each coupon date. As modified duration differs from Macaulay's duration only by a multiplicative factor (which is interest-rate dependent), it will also exhibit a saw-toothed time pattern.

The above time pattern is at odds with the smooth graphs of duration versus the term to maturity as illustrated in investment textbooks, where the term to maturity — or, more succinctly, maturity — is the number of periods before a bond matures.[3] In textbook illustrations, duration increases smoothly with maturity for each coupon bond selling at a premium or at par; however, a maximum in duration can occur for a coupon bond selling at a deep discount. Here, a bond is deemed to sell at a premium, at par, and at a discount, if its price is above, equal to, and below its face value, respectively.

To reconcile the smooth and saw-toothed graphs of duration versus maturity requires that each smooth graph be the result of interpolations of cases pertaining to a basic bond model; that is, cases where maturity is confined to integers. The presence of a saw-toothed time pattern implies that the computed duration for each non-integer maturity is always below the corresponding interpolated value. The one-day jump in duration at each coupon date — which is typically an upward jump — appears to be the eventual correction over a period between two adjacent coupon dates. This feature inevitably raises the concern of whether duration computed directly for a valuation date other than a coupon date is valid.

Adding to the above concern is the lack of internal consistency in the available proofs, such as those by Hawawini (1984) and Pianca (2006), in support of textbook illustrations of the smooth

[3] See, for example, Bodie et al. (2004, Chapter 10), Bodie et al. (2011, Chapter 15), and Reilly et al. (2010, Chapter 12).

graph of duration versus maturity. This is also a concern, from a pedagogic perspective, because such proofs are calculus-based, involving the first derivative of the analytical expression of Macaulay's duration with respect to maturity. As maturity cannot be confined to integer values in a calculus-based proof, a key assumption in the basic bond model involved must have been violated.

This paper is intended to address the above concerns, with Excel playing an important pedagogic role. The Excel file accompanying this paper, which contains some of its illustrative examples, can be downloaded from the journal website. Such examples will enable students to gain some hands-on experience with various relevant issues pertaining to bond duration, thus making the corresponding analytical materials less abstract. To achieve pedagogic effectiveness and to avoid technical digressions, the Excel tools as utilized in this paper are generally those that are already familiar to business students.

The remainder of this paper is organized as follows: It starts in Section 2 with a derivation of duration — which includes Macaulay's duration and modified duration — for a basic bond model. A non-calculus version of the same derivation is provided in Appendix A. Also shown in Section 2 is a closed-form expression of duration, as well as an Excel example that illustrates its computational convenience. Its derivation is provided in Appendix B. Section 3 examines the impacts on duration by various underlying parameters. The impact of maturity on duration, which is tedious to verify analytically without relying on calculus tools, is provided in Appendix C.[4]

Further Excel illustrations for a basic bond model are provided in Section 4. The use of graphical features in Excel allows representative smooth graphs of duration versus maturity to be generated, like those in textbook illustrations, for various combinations of the underlying parameters in bond valuation. The use of scroll bars in Excel also allows the impact on each graph by changes in the underlying parameters to be explored interactively.

Section 5 extends the bond model to accommodate valuation dates that fall anywhere within a period between two adjacent coupon dates. Duration is derived for such a model. The cause of the saw-toothed time pattern is revealed in Section 6. The same section also includes an Excel example, which utilizes scroll bars to enable students to explore interactively the saw-toothed graphs of duration versus maturity. Guidance is also provided to help students recognize the connection between each underlying parameter in bond valuation and the saw-tooth problem.

A simple remedy, which allows the saw-tooth problem to be eliminated, is proposed in Section 7. A nice feature of each revised duration measure is that the same idea of using a weighted average of arrival times of cash flows for its computation can be retained. Also provided in Section 7 is a

[4] To the best of our knowledge, the algebraic proof in Appendix C is an original contribution.

closed-form expression for each revised duration measure. A derivation is provided in Appendix D.

The potential severity of the saw-tooth problem in practice is assessed with actual daily bond quotations in Section 8, with Excel playing an important role in the assessment. Notice that maturity and yield to maturity — which are two key parameters in bond valuation, with the latter parameter affected by bond market conditions — do change over time. Then, it is not immediately obvious whether the combined effect of these changes can conceal the saw-tooth problem, thus rendering the whole issue irrelevant in practice.

However, as it turns out, the use of Excel functions DURATION and MDURATION (for computing Macaulay's duration and modified duration, respectively) on actual bond data reveals clearly the existence of a saw-toothed time pattern. This revelation is important, from both practical and pedagogic perspectives, as the two duration measures have been widely accepted by bond investors as price sensitivity measures and covered in standard investment textbooks. Given the simplicity of the remedy, to revise the two Excel-based duration measures is straightforward. Section 8 includes this aspect of the Excel illustrations as well. Finally, Section 9 provides some concluding remarks.

2 Duration Measures According to a Basic Bond Model

Consider a bond that has a face value F and pays coupon C each period until maturity. Depending on the bond in question, a period can be a year, six months, three months, or others. A basic bond model is where, as of the valuation date, there is a full period before the next coupon payment. Suppose that the bond has n remaining coupon payments until maturity. For a required return r each period, also known of the yield to maturity, the price of the bond is

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^n} + \frac{F}{(1+r)^n}, \tag{1}$$

In bond quotations in practice, the quoted annual yield and annual coupon are mr and mC, respectively, with m being the annual frequency of coupon payments. In the case of a semi-annual coupon bonds, for example, m is 2.

Equation (1) reveals an inverse relationship between the price P of a bond and investors' required yield r for holding it. Given how bond price is affected by the underlying parameters, however, price sensitivity in response to a given yield change differs among bonds. This calls for an examination of the price-sensitivity issue. The examination requires the use of a price-sensitivity measure that captures the joint effect of the underlying parameters. A derivation of such a measure that requires differential calculus is presented below. To make the corresponding materials accessible to more students, an algebraic derivation is also provided in Appendix A.

Analytically, to see the impact of changes of r on P, for a bond with given C, F, and n, we take the first derivative of P with respect to r based on equation (1). The result is

$$\frac{dP}{dr} = -\frac{1}{1+r}\left[\frac{C}{1+r} + \frac{2C}{(1+r)^2} + \cdots + \frac{nC}{(1+r)^n} + \frac{nF}{(1+r)^n}\right]. \quad (2)$$

Combining equations (1) and (2) leads to

$$\frac{dP}{P} = -D\frac{dr}{1+r}, \quad (3)$$

which relates, in proportional terms, price changes and the corresponding one-plus-yield changes, with the negative sign capturing their inverse relationship. Here, the proportionality constant

$$D = \left[\frac{C}{1+r} + \frac{2C}{(1+r)^2} + \cdots + \frac{nC}{(1+r)^n} + \frac{nF}{(1+r)^n}\right] \Big/ \left[\frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^n} + \frac{F}{(1+r)^n}\right] \quad (4)$$

is a price-sensitivity measure. Notice that, for a pure discount bond where $C = 0$, equation (4) always leads to $D = n$ regardless of r.

The expression in equation (4) is commonly known as Macaulay's duration or, simply, duration, as measured in periods. When measured in years, it is D/m instead. The greater the measure, the more sensitive is the bond price in response to a given yield change. As the cash flows from the bond investment will arrive at times $1, 2, \ldots, n$, with the present-value factors for the corresponding cash flows being $1/(1+r), 1/(1+r)^2, \ldots, 1/(1+r)^n$, what equation (4) represents is the weighted average of arrival times of all cash flows.

A closed-form expression of duration is

$$D = \frac{\left[(1+r) - (1+r+nr)(1+r)^{-n}\right]c + nr^2(1+r)^{-n}}{r\left[1-(1+r)^{-n}\right]c + r^2(1+r)^{-n}}, \quad (5)$$

where $c = C/F$, the coupon rate each period. Its computational advantage over the expression in equation (4) will be noticeable if n is large. Its derivation is provided in Appendix B. The expression of D in equation (5) is equivalent to those in Chua (1984, 1985).

Notice that, if r is exactly zero, equation (5) will fail to provide a value of D. As equation (5) is intended for practically relevant cases, for which r is strictly positive, its being unable to accommodate $r = 0$ is not a concern. However, this does not mean that D is undefined for $r = 0$. Rather, if $r = 0$, equation (4) reduces to

$$D = \frac{c(1+2+\cdots+n)+n}{cn+1} = \frac{cn(n+1)/2+n}{cn+1}. \quad (6)$$

Of interest is that equation (6) implies $D = n$ if $c = r = 0$. In contrast, as noted earlier, equation (4) always leads to $D = n$ if $c = 0$. Therefore, if $c = 0$, we always have $D = n$, regardless of whether $r = 0$ or $r > 0$.

Modified duration, measured in periods and defined as

$$D^{(M)} = \frac{D}{1+r}, \tag{7}$$

is a more natural measure of bond price sensitivity. It allows equation (3) to be written as

$$\frac{dP}{P} = -D^{(M)} dr, \tag{8}$$

which shows a proportional price change in response to a yield change, without involving the scaling factor $1 + r$. When measured in years, it is $D^{(M)}/m$ instead.

2.1 An Excel Example

Standard textbook illustrations of the computational aspect of Macaulay's duration are typically based on an expression equivalent to that in equation (4). The idea is to compute a weighted average of arrival times of individual cash flows, with each weight based on the present value of the corresponding cash flow. In a spreadsheet setting, the arrival time of each cash flow and its weighting factor are conveniently placed, so that a weighted average can easily be obtained via simple spreadsheet operations.

As Figure 1 illustrates, it is straightforward to set up an Excel worksheet to compute D according to equation (4) for a given maturity. The example here is based on a semi-annual coupon bond (where $m = 2$), for which all input parameters for bond valuation are shaded. Given $C = \$5$ each period, $F = \$100$, $n = 10$ periods, and $r = 3\%$ each period, each term in the numerator and the denominator of the expression of D in equation (4) can easily be computed, as displayed in F17:G26. Notice that the arrival times of cash flows in B17:B26, with B17 being 1, have been generated by pasting the formula of B18, which is =B17+1, to B18:B26. The computed D, as a ratio of the sum of the terms in the numerator to the sum of the terms in the denominator, is displayed in F34. With D known, $D^{(M)}$ follows directly; so do the two duration measures in terms of years. They are displayed in F35:F37.

A change in n can be accommodated in a spreadsheet setting. For example, if $n = 8$ periods instead, the required adjustments to the Excel worksheet include only the following: change of E5 to 4 (which is n/m, the bond's maturity in years), copy of C26 (which contains $C + F$) to C24, and deletion of the current rows 25 and 26. If $n = 16$ periods instead, a simple way to accommodate

	A	B	C	D	E	F	G
1	m: number of annual coupon payments				2		
2	mC: annual coupon				$10		
3	C: coupon each period				$5		E3 =E2/E1
4	F: face value				$100		
5	n/m: maturity in years				5		
6	n: maturity in periods				10		E6 =E5*E1
7	mc: annual coupon rate				10%		E7 =E2/E4
8	c: coupon rate each period				5%		E8 =E7/E1
9	mr: annual yield				6%		
10	r: yield each period				3%		E10 =E9/E1
11							
12	h: number of periods				(CF): cash flow		
13	D: Macaulay's duration in periods				(DM): modified duration in periods		
14	D/m: Macaulay's duration in years				(DM)/m: modified duration in years		
15							
16		h	(CF)	h*(CF)	(1+r)^(-h)	(CF)*(1+r)^(-h)	h*(CF)*(1+r)^(-h)
17		1	5	5	0.9709	4.8544	4.8544
18		2	5	10	0.9426	4.7130	9.4260
19		3	5	15	0.9151	4.5757	13.7271
20		4	5	20	0.8885	4.4424	17.7697
21		5	5	25	0.8626	4.3130	21.5652
22		6	5	30	0.8375	4.1874	25.1245
23		7	5	35	0.8131	4.0655	28.4582
24		8	5	40	0.7894	3.9470	31.5764
25		9	5	45	0.7664	3.8321	34.4888
26		10	105	1050	0.7441	78.1299	781.2986
27							
28	B18	=B17+1	copied to B18:B26			(sum 1): sum of	(sum 2): sum of
29	C17	=E$3	copied to C17:C25			above	above
30	C26	=E$3+E$4				117.0604	968.2889
31	F30	=SUM(F17:F26)					
32	G30	=SUM(G17:G26)				based on	based on closed-
33						(sum 2)/(sum 1)	form expression
34	F34	=G30/F30			D	8.2717	8.2717
35	F35	=F34/$E1			D/m	4.1359	4.1359
36	F36	=F34/(1+$E10)			(DM)	8.0308	8.0308
37	F37	=F36/$E1			(DM)/m	4.0154	4.0154
38	F35:F37	copied to G35:G37					
39							
40	G34	=((1+E10-(1+E10+E6*E10)*(1+E10)^(-E6))*E8+E6*E10*E10*(1+E10)^(-E6))/					
41		(E10*(1-(1+E10)^(-E6))*E8+E10*E10*(1+E10)^(-E6))					

Figure 1 An Excel Worksheet Showing Two Approaches to Compute Macaulay's Duration for a Basic Bond Model.

the change, after entering 8 to E5, is to insert 6 new rows, after the current row 18 and before the current row 26. The current row 18 is then pasted to these 6 new rows. To complete the task, the current cell formula of B18, which is =B17+1, will have to be pasted to column B in the expanded block, starting from B18. This will ensure that consecutive integers from 1 to 16, for the arrival times of cash flows, be displayed in column B.

Although the above adjustments are already easy to implement, a more convenient way to compute D is to use equation (5) directly, especially if n is large. The result of the computation based on equation (5) is displayed in G34. The corresponding cell formula is displayed in rows 40 and 41. Equation (5) can directly accommodate any change in n; all that is required is to change the corresponding number of n/m in E5. As expected, the results from the two alternative approaches are exactly the same, for various combinations of the input parameters attempted. From a pedagogic perspective, it is important to explain to students why equations (4) and (5) are equivalent. The Excel example in Figure 1 is intended to help in this regard.

3 Impacts on Duration by Each of the Underlying Parameters in Bond Valuation

In bond quotations, prices and coupons are always scaled to have a common face value F, such as $F = \$100$ in U.S. markets. For conciseness of various algebraic expressions below, we use $c = C/F$, the coupon each period by scaling the face value of the bond to a dollar. To complement the textbook materials pertaining to a basic bond model, the impact of each of c, r, and n on D is examined analytically below. For such tasks, we first write equation (4) as

$$D = \frac{c\sum_{h=1}^{n} h/(1+r)^h + n/(1+r)^n}{c\sum_{h=1}^{n} 1/(1+r)^h + 1/(1+r)^n}. \tag{9}$$

This is equivalent to substituting C and F in equation (4) with c and 1, respectively, and to using summation signs for series expressions.

3.1 The Impact of Coupon

According to equation (9), if $n = 1$, the numerator and the denominator in the expression of D are both $(c+1)/(1+r)^n$, and thus we always have $D = 1$ regardless of c. To assess the impact of changes of c on D analytically, for any given $r > 0$ and any given integer $n \geq 2$, we first establish the weighted average of arrival times for the n coupons only, denoted as q, according to the same weighting scheme. That is,

$$q = \frac{c\sum_{h=1}^{n} h/(1+r)^h}{c\sum_{h=1}^{n} 1/(1+r)^h} = \frac{\sum_{h=1}^{n} h/(1+r)^h}{\sum_{h=1}^{n} 1/(1+r)^h}, \tag{10}$$

which is independent of c. Being a weighted average of $1, 2, \ldots, n$, with all positive weights, q is always less than n.

Next, we write equation (9) equivalently as

$$D = \frac{\left[c \sum_{h=1}^{n} 1/(1+r)^h\right] q + [1/(1+r)^n]\, n}{c \sum_{h=1}^{n} 1/(1+r)^h + 1/(1+r)^n}, \tag{11}$$

which is a weighted average of q and n. The higher the coupon rate c each period, the higher is the weight that q receives, and the lower is the weight that n receives. Accordingly, with $q < n$, the bond's duration will be lower. Likewise, a lower coupon rate will lead to a higher duration. That is, there is an inverse relationship between D and c for any given n and r. With $D^{(M)} = D/(1+r)$, which differs from D by a scaling factor, there is also an inverse relationship between $D^{(M)}$ and c.

3.2 The Impact of Yield

According to equation (9), if $n = 1$, we always have $D = 1$ regardless of r. To assess the impact of changes in r on D for any given $n > 1$, we can use equations (10) and (11) again. In equation (10), a higher r will make the present values of distant coupons progressively lower. As a result, the earlier coupons will receive progressively higher weights to establish q. The end result is a lower q. Subsequently, when D is expressed as a weighted average of q and n in equation (11), the weight that n receives is

$$\frac{1/(1+r)^n}{c \sum_{h=1}^{n} 1/(1+r)^h + 1/(1+r)^n} = \frac{1}{c \sum_{h=1}^{n} (1+r)^{n-h} + 1}. \tag{12}$$

The higher the yield, the lower the weight that n receives. Thus, a higher r not only makes q lower, but also provides a higher weight for q in the weighted average of q and n to establish D. Accordingly, a higher r will result in a lower D. Likewise, a lower r will result in a higher D. That is, there is an inverse relationship between D and r for any given c and n. Notice that, as $D^{(M)} = D/(1+r)$, the inverse relationship extends to $D^{(M)}$ and r as well.

3.3 The Impact of Maturity

To explore how duration varies with maturity for a bond, let us attach a subscript n to the symbol D, indicating that there are n periods before the bond matures. The use of such a subscript allows equation (9) to be written as $D_n = w_n/p_n$, with w_n and p_n being the numerator and the denominator of the expression, respectively. For each zero-coupon bond, where $c = 0$, we have $D_n = n$. Accordingly, the graph of D_n versus n for each zero-coupon bond is a 45^o line when adjacent points (D_n, n), for $n = 1, 2, 3, \ldots$, are connected.

For each coupon bond, where $c > 0$, we divide both the numerator and the denominator of the expression of $D_{n-1} = w_{n-1}/p_{n-1}$ with $(1 + r)$ and simplify the result to obtain

$$D_{n-1} = \frac{w_n - p_n}{p_n - c/(1+r)}. \tag{13}$$

This is an expression of the duration of a bond that matures in $n-1$ periods, in terms of information about the bond with n periods before maturity. It follows that

$$D_n - D_{n-1} = \frac{(1+r)p_n - D_n c}{(1+r)p_n - c}, \tag{14}$$

for any integer $n > 1$. Notice that, with $c < p_n$ for a bond with practical relevance, the denominator, $(1 + r)p_n - c$, is always positive. As $D_n > 1$ for $n > 1$ and then $(1 + r)p_n - D_n c < (1 + r)p_n - c$, we must have $D_n - D_{n-1} < 1$ for each coupon bond. Thus, when adjacent points (D_n, n), for $n = 1, 2, 3, \ldots$, are connected, the resulting smooth graph of D_n versus n for each coupon bond must be below the 45^o line, as established earlier for a zero-coupon bond.

Also of interest are the signs of $D_n - D_{n-1}$, which capture the relationships between D and n for various combinations of the input parameters in bond valuation. Typical textbook illustrations of such relationships for some selected values of c and r, are graphical or numerical. For analytical convenience, available analytical proofs, such as those by Hawawini (1984) and Pianca (2006), require differential calculus. Specifically, the sign of $\partial D/\partial n$ is examined. Although the corresponding analytical results are consistent with textbook illustrations, the fact that n is an integer in a basic bond model does question the validity of performing the partial derivative of D with respect to n.

An alternative proof, which is algebraic and explicitly recognizes that n can only have integer values, is provided in Appendix C. It is shown that, if $c \geq r$, duration always increases with maturity. Otherwise, a monotonic relationship between duration and maturity is not assured; that is, if the yield is much higher than the coupon rate, the graph of duration versus maturity can have a single maximum. Again, with $D^{(M)}$ being different from D only by the scaling factor $1 + r$, the relationships between D and n, as noted above, also extend to those between $D^{(M)}$ and n.

4 Graphs of Duration versus Maturity for a Basic Bond Model and Interactive Illustrations with Excel

Figure 2 shows some representative graphs of Macaulay's duration versus maturity, for semi-annual coupon bonds. To generate these graphs, the valuation dates are confined to those that can be accommodated by a basic bond model. The computations are based on equation (5). Each bond

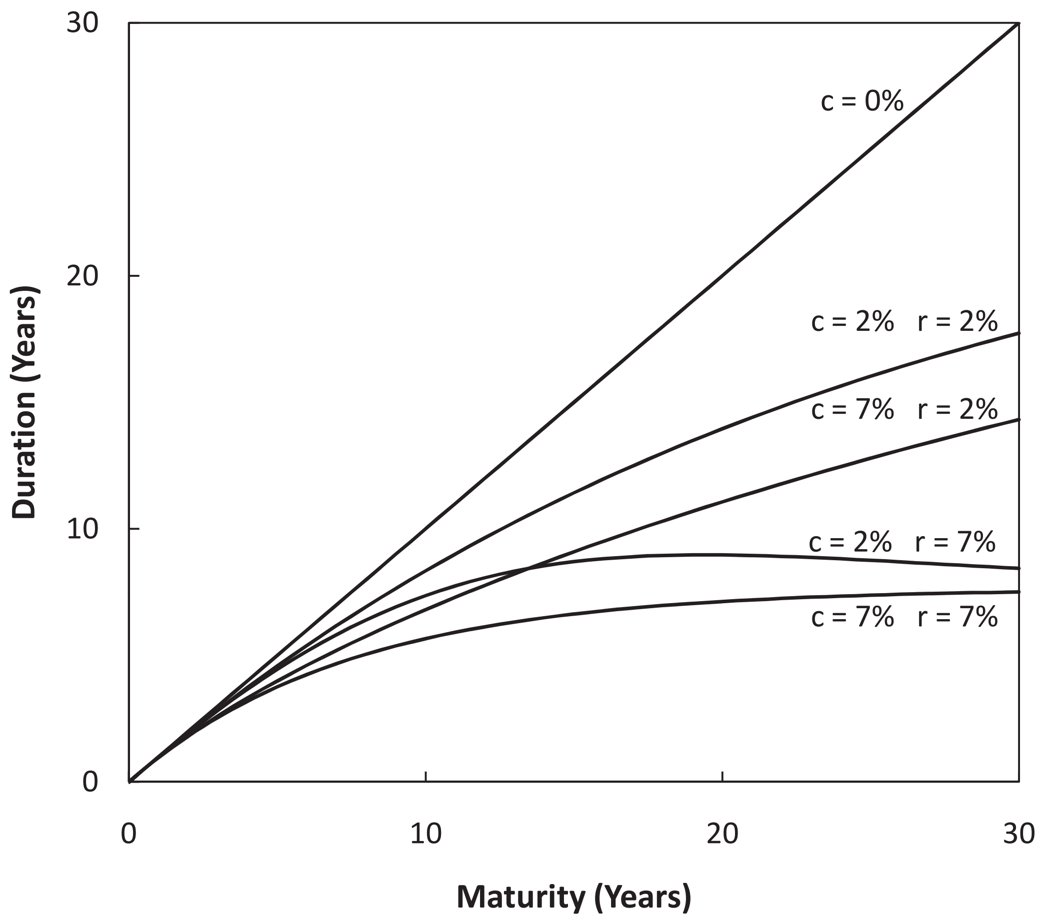

Figure 2   Graphs of Macaulay's Duration versus Maturity for a Basic Bond Model, with Various Combinations of Coupon Rates and Yields to Maturity.

either carries no coupon or has a 4% or 14% annual coupon rate, with the annual yield to maturity being either 4% or 14%. With $m = 2$ being the frequency of periodic coupon payments each year, the underlying parameters of equation (5) for these graphs are various combinations of $c = 0\%$, 2%, and 7% and $r = 2\%$ and 7%. However, instead of displaying directly D versus n, with the number of semi-annual periods being the measurement unit as implied by equation (5), the graphs in Figure 2 are shown in terms of years, with the vertical and horizontal axes being D/m and n/m, respectively.

The graphs are generated by inserting an X Y (Scatter) Chart to an Excel worksheet. Smooth curves are drawn between adjacent data points. Each data point, which is not explicitly shown, captures the corresponding values of D and n, for a pre-determined combination of c and r. Such graphs are similar to those found in standard investment textbooks. As expected from equation (4), the 45^o line in Figure 2, where $D = n$, pertains to zero-coupon bonds, regardless of the bond yield. For a given n, D decreases with increasing c and with increasing r. For a given combination of c and r, D always increases with increasing n only if $c \geq r$; the presence of a single maximum in D requires that $c < r$.

Figure 3 illustrates, with year being the measurement unit, how a representative graph of Macaulay's duration versus maturity for each semi-coupon bond can be generated in Excel. Two scroll bars are utilized, separately, for the coupon rate and the yield to maturity. Each scroll bar is generated via the *Developer* tab on the *Menu* bar. Selecting *Insert* in the *Controls* group there leads to *Form Controls*, in which a scroll bar is available. Adjustments to the properties of each scroll bar are made via its *Format Controls*. Besides its appearance, the range of values, the incremental changes within the range, and the cell in the Excel worksheet to be directly linked, can be specified.

For the purpose of the graph in Figure 3, each scroll bar covers the range of 0 to 1000, with increments of 1. The corresponding values are linked to C2 and C6. With $m = 2$ for a semi-annual coupon bond, as stored in C10, the minimum and the maximum of each of mc and mr are set to be 0% and 20%, respectively. They are stored in C3:C4 and C7:C8. All pre-determined values in Figure 3 are shaded. As displayed in C11:C12, mc and mr are based on C3+(C4-C3)*C2/1000 and C7+(C8-C7)*C6/1000, respectively, from which c and r in C13:C14 are deduced for use in equation (5). Figure 3 shows the graph of D/m versus n/m for $c = 2\%$ and $r = 7\%$. The graph corresponds to the case of $c < r$ in Figure 2.

The interactive nature of using the two scroll bars to change the underlying parameters of bond valuation in generating a graph of D/m versus n/m will allow students to see immediately

	A	B	C	D	E	F	G
1							
2	scroll bar (0-1000)		200				
3	annual coupon rate (min)		0%				
4	annual coupon rate (max)		20%				
5							
6	scroll bar (0-1000)		700				
7	annual yield (min)		0%				
8	annual yield (max)		20%				
9							
10	m: number of annual coupon payments		2				
11	mc: annual coupon rate		4.000%		C11	=C3+(C4-C3)*C2/1000	
12	mr: annual yield		14.000%		C12	=C7+(C8-C7)*C6/1000	
13	c: coupon rate each period		2.000%		C13	=C11/C$10	
14	r: yield each period		7.000%		C14	=C12/C$10	
15	n: maturity in periods						
16	n/m: maturity in years						
17	D: Macaulay's duration in periods						
18	D/m: Macaulay's duration in years						
19							
20	n	n/m	D/m				
21	0	0	0				
22	1	0.5	0.5				
23	2	1	0.989725				
24	3	1.5	1.468432				
25	4	2	1.935388				
26	5	2.5	2.389884				
27	6	3	2.831236				
28	7	3.5	3.258795				
29	8	4	3.671956				
30	9	4.5	4.07016				
31	10	5	4.452905				
32	11	5.5	4.819748				
33	12	6	5.170314				
34	13	6.5	5.504297				
35	14	7	5.821465				
36	15	7.5	6.121665				
37	16	8	6.404817				
38	17	8.5	6.670922				
39	18	9	6.92006				
40	19	9.5	7.152383				
41	20	10	7.368118				

Figure 3 An Excel Worksheet Showing the Computations of Macaulay's Duration for Different Maturities and the Resulting Graph for a Basic Bond Model, with Coupon Rates and Yields to Maturity Generated via Two Scroll Bars.

A22	=A21+1 copied to A22:A81
B21	=A21/C$10 copied to B21:B81
C21	=IF(C$14=0,(C$13*A21*(A21+1)/2+A21)/(C$13*A21+1)/C$10, ((1+C$14-(1+C$14+$A21*C$14)*(1+C$14)^(-$A21))*C$13 +$A21*C$14*C$14*(1+C$14)^(-$A21))/(C$14*(1-(1+C$14)^(-$A21))*C$13 +C$14*C$14*(1+C$14)^(-$A21))/C$10) copied to C21:C81

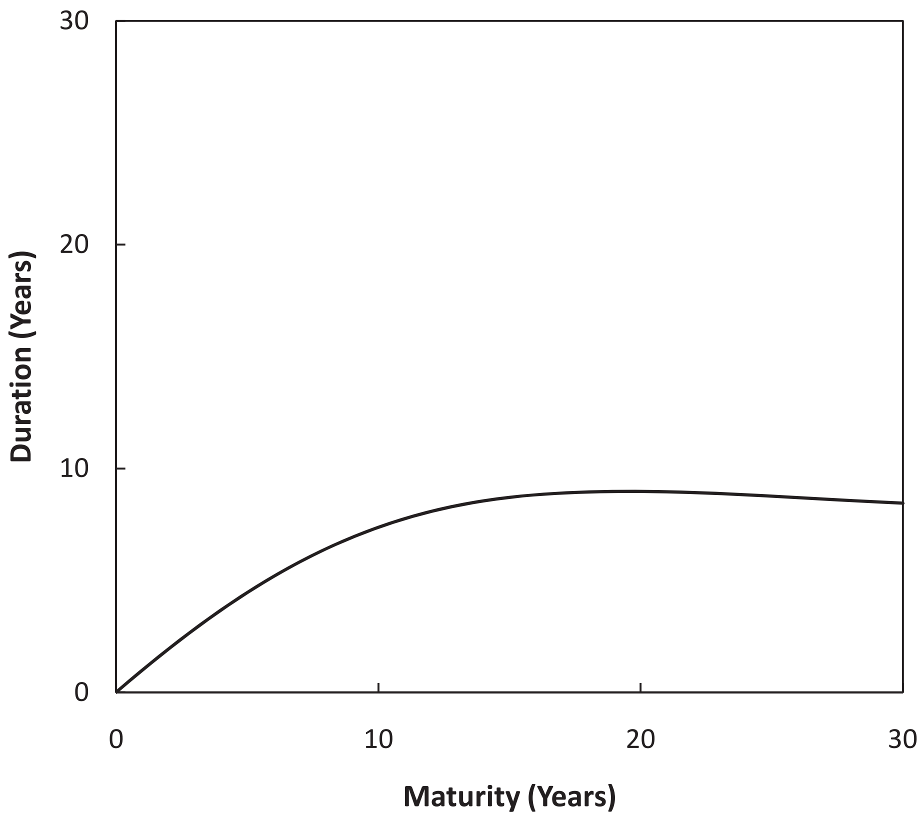

Figure 3 An Excel Worksheet Showing the Computations of Macaulay's Duration for Different Maturities and the Resulting Graph for a Basic Bond Model, with Coupon Rates and Yields to Maturity Generated via Two Scroll Bars (Continued).

the impact of each parameter on the graph, as implied by the analytical materials in Section 3. Given prevailing bond market conditions, students themselves can also enter minimum and maximum values of mc and mr to C3:C4 and C7:C8. This interactive feature will enhance students' understanding of the corresponding textbook materials on bond duration.

To accommodate $r \geq 0$ for computational convenience, each cell formula in C21:C81 uses an IF statement to choose between equations (5) and (6) for the computations involved. For each integer n in A21:A81 over the range of 0 to 60, the corresponding n/m and D/m are stored in B21:C81. Notice that the part of the Excel worksheet containing the data in A42:C81, for which $n = 21$ to 60, is not displayed. In addition to the cell formulas to generate c and r as described earlier, the representative cell formulas to generate n and to compute n/m and D/m, pertaining to A21:C81, are displayed, along with the corresponding graph in Figure 3.

5 Duration Measures Based on a More Realistic Bond Model

Consider a bond that has a face value F and pays coupon C each period until maturity. Implicitly, the bond pays m periodic coupons each year. Let us label each day in terms of the time elapsed since the most recent coupon date, as proportion α of a full period, for $0 \leq \alpha < 1$. Suppose that, as of the day corresponding to the proportion α, the bond has n remaining coupon payments until maturity. For a required return r each period, as of that day, the full price of the bond is

$$P + \alpha C = \frac{C}{(1+r)^{1-\alpha}} + \frac{C}{(1+r)^{2-\alpha}} + \cdots + \frac{C}{(1+r)^{n-\alpha}} + \frac{F}{(1+r)^{n-\alpha}}, \tag{15}$$

where αC is the accrued interest. The price P is commonly called the clean price.

Equation (15) captures the idea that, when an investor buys a bond with the settlement date not at the start of a full period, the investor is required to pay also the accrued interest, which is part of the coupon that the seller is entitled to receive. The proportion α can be deduced from the coupon dates and the valuation date, which is the settlement date for any buy-sell transactions. Although the exact value of α depends on the day-counting conventions of the markets involved, it does not affect the analytical feature of equation (15).

As the way the accrued interest is determined is independent of r, the first derivative of the full price of the bond or its clean price with respect to r is the same; that is,

$$\frac{dP}{dr} = -\frac{1}{1+r}\left[\frac{(1-\alpha)C}{(1+r)^{1-\alpha}} + \frac{(2-\alpha)C}{(1+r)^{2-\alpha}} + \cdots + \frac{(n-\alpha)C}{(1+r)^{n-\alpha}} + \frac{(n-\alpha)F}{(1+r)^{n-\alpha}}\right]. \tag{16}$$

Combining equations (15) and (16) leads to

$$\frac{dP}{P + \alpha C} = -D\frac{dr}{1+r} = -D^{(M)}dr. \tag{17}$$

Here,

$$D = \left[\frac{(1-\alpha)C}{(1+r)^{1-\alpha}} + \frac{(2-\alpha)C}{(1+r)^{2-\alpha}} + \cdots + \frac{(n-\alpha)C}{(1+r)^{n-\alpha}} + \frac{(n-\alpha)F}{(1+r)^{n-\alpha}}\right] \Big/ \left[\frac{C}{(1+r)^{1-\alpha}} + \frac{C}{(1+r)^{2-\alpha}} + \cdots + \frac{C}{(1+r)^{n-\alpha}} + \frac{F}{(1+r)^{n-\alpha}}\right] \tag{18}$$

is Macaulay's duration and $D^{(M)} = D/(1+r)$ is modified duration, as measured in the number of periods. When measured in years, they are D/m and $D^{(M)}/m$ instead.

To give equation (18) an intuitive interpretation, let the day corresponding to the proportion α of a period be time 0. Accordingly, the cash flows from the bond investment will arrive at the following times: $1-\alpha, 2-\alpha, \ldots, n-\alpha$. With the present-value factor for each cash flow at time $t-\alpha$ being $1/(1+r)^{t-\alpha}$, for $t = 1, 2, \ldots, n$, what equation (18) represents is the weighted average of arrival times of all cash flows.

6 A Saw-toothed Time Pattern

To explore the time pattern of Macaulay's duration, let us attach a subscript $n-\alpha$ to the symbol D, indicating that there are $n-\alpha$ periods before the bond matures. Let us also multiply both the numerator and the denominator of the expression in equation (18) with $(1+r)^{-\alpha}$ to obtain

$$D_{n-\alpha} = \left[\frac{(1-\alpha)C}{1+r} + \frac{(2-\alpha)C}{(1+r)^2} + \cdots + \frac{(n-\alpha)C}{(1+r)^n} + \frac{(n-\alpha)F}{(1+r)^n}\right] \Big/ \left[\frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^n} + \frac{F}{(1+r)^n}\right]. \tag{19}$$

Separating α from the remaining terms in the above expression leads to

$$D_{n-\alpha} = D_n - \alpha, \tag{20}$$

where D_n represents Macaulay's duration of the bond, as measured in the number of periods, when there are n full periods before maturity.

If there are no changes in r as time passes by, equation (20) implies the following: On the day immediately before the next coupon date, as $\alpha \approx 1$, the corresponding duration $D_{n-\alpha}$ has declined to a level that is almost one period lower than D_n, before it is reset at D_{n-1}. If each $D_{n-\alpha}$ differs from D_{n-1} by a nontrivial amount, then the presence of a saw-toothed time pattern of Macaulay's duration will be noticeable, with a one-day jump at each coupon date or, more precisely, from the day before each coupon date to the coupon date in question.

6.1 The Magnitude and the Direction of the Jump in Duration at Each Coupon Date

As $c = C/F$ and $p_n = P_n/F$, combining equations (14) and (20) leads to

$$D_{n-1} - D_{n-\alpha} = \alpha - \frac{(1+r)p_n - D_n c}{(1+r)p_n - c} = \alpha - 1 + \frac{(D_n - 1)C}{(1+r)P_n - C}. \quad (21)$$

Does this equation imply a jump in duration, from the day before a coupon date to the coupon date? If so, is the jump always upward? To answer the two questions, let us first explore what the term $(1+r)P_n - C$ in this equation represents. Once we multiply both sides of equation (1) with $1+r$ and re-arrange the resulting terms, it will become obvious that $(1+r)P_n - C = P_{n-1}$. With C/P_{n-1} being the current yield of the bond with $n-1$ periods before maturity, the final term on the right-hand side of equation (21) is the current yield scaled by the factor $D_n - 1$.

Graphically, equation (14) provides points (n, D_n) and $(n-1, D_{n-1})$ on the maturity-duration plane. The slope of the connecting line is strictly less than 1. Equation (20) provides instead a family of points on a different line segment, with $(n-\alpha, D_{n-\alpha})$ representing each point, for $0 \leq \alpha < 1$. The slope of the latter line segment is exactly 1. Of interest are the two end points, which are (n, D_n) and $(n-\alpha, D_{n-\alpha})$, with α corresponding to the day immediately prior to the next coupon date. The two line segments meet at point (n, D_n).

Should $\alpha = 1$ be allowed for the purpose of computing $D_{n-\alpha}$, a vertical line drawn from the corresponding point $(n-\alpha, D_{n-\alpha})$ based on equation (20) would pass the point $(n-1, D_{n-1})$ based on equation (14). With the former point being lower, $D_{n-1} - D_{n-\alpha}$ would be positive. However, given that α is strictly less than 1, a non-positive $D_{n-1} - D_{n-\alpha}$ cannot be ruled out. Its occurrence requires either $(D_n - 1)$ or C/P_{n-1}, or both, be numerically close to zero.

For a semi-annual coupon bond, for example, the proportion of a period, from the day before a coupon date to the coupon date, which $1-\alpha$ represents, is $1/180 = 0.00556$ for a 360-day year. For bonds with practically relevant coupon rates, the corresponding current yields do exceed $0.00556 = 0.556\%$ by some wide margins. If such bonds are not near their maturity dates, the scaling factor $D_n - 1$ will also help in making $D_{n-1} - D_{n-\alpha}$ positive. Thus, in practice, we can normally expect an upward jump in duration, from the day before a coupon date to the coupon date.

For a bond selling at a premium or at par, as D_n always increases with n, the jump that $D_{n-1} - D_{n-\alpha}$ represents is always less than 1 period. For a bond selling at a discount, if n is small enough to ensure that $D_n - D_{n-1}$ be positive, the corresponding jump is still less than 1 period. For a deep-discount bond, however, the graph of D_n versus n can have a single maximum. For such a bond, if n is large enough to make $D_n - D_{n-1}$ negative, the corresponding jump will be greater than 1 period.

Upon dividing $D_{n-1} - D_{n-\alpha}$ with $D_{n-\alpha}$, we have $(D_{n-1} - D_{n-\alpha})/D_{n-\alpha}$, a term that captures the jump in percentage terms.[5] As α approaches 1, $D_{n-\alpha}$ and $D_{n-1} - D_{n-\alpha}$ reduce to $D_n - 1$ and $(D_n - 1)C/P_{n-1}$, respectively. Thus, $(D_{n-1} - D_{n-\alpha})/D_{n-\alpha}$ approaches C/P_{n-1} or simply C/P, with the subscript omitted, the current yield at the coupon date where the jump occurs. Notice that the percentage jump in modified duration is also $(D_{n-1} - D_{n-\alpha})/D_{n-\alpha}$. Further, regardless of whether Macaulay's duration or modified duration is computed, the percentage jump is unaffected by the choice of measurement units.

To gauge the severity of the saw-tooth problem in terms of the current yield C/P, let us examine the impact on it by each of the underlying parameters in bond valuation, by holding all other parameters constant. As there is an inverse relationship between r and P, C/P increases with r. For a bond selling at a premium, the premium declines as time passes by. For such a bond, as P increases with n, an increase in n will lead to a decrease in C/P. In contrast, for a bond selling at a discount, the discount becomes smaller as time passes by. For such a bond, given the inverse relationship between P and n, C/P increases with n. For a bond selling at par, as $P = F$, C/P does not vary with n.

To see how C/P varies with C, let us write equation (15) equivalently as

$$\frac{P}{C} = \frac{1}{(1+r)^{1-\alpha}} + \frac{1}{(1+r)^{2-\alpha}} + \cdots + \frac{1}{(1+r)^{n-\alpha}} - \alpha + \frac{F}{C(1+r)^{n-\alpha}}. \tag{22}$$

This equation confirms that there is an inverse relationship between P/C and C. Thus, C/P increases with C.

6.2 An Interactive Excel Illustration of the Saw-tooth Problem with Some Guidance on the Choice of Input Parameters

In view of equation (20), the Excel worksheet used for Figure 3 can easily be revised for the purpose of computing duration of a semi-annual coupon bond with both integer and non-integer maturities. Figure 4 shows such a worksheet. The idea is to use equation (5) for coupon dates and equation (20) for other dates. As duration varies linearly with maturity within each period, the computations for Figure 4 pertain to each coupon date and the day preceding it.

Figure 4 shows the case where the annual coupon rate is 16% and the annual yield is 12%. The maturities considered are between 0 and 8 years. Again, all pre-determined values are shaded.

[5] Strictly speaking, $(D_{n-1} - D_{n-\alpha})/D_{n-\alpha}$ is a proportion, rather than a percentage. Only when written equivalently as $[100(D_{n-1} - D_{n-\alpha})/D_{n-\alpha}]\%$, it is in percentage terms.

	A	B	C	D	E	F	G
1							
2	scroll bar (0-1000)		800				
3	annual coupon rate (min)		0%				
4	annual coupon rate (max)		20%				
5							
6	scroll bar (1-1000)		600				
7	annual yield (min)		0%				
8	annual yield (max)		20%				
9							
10	m: number of annual coupon payments		2				
11	mc: annual coupon rate		16.000%				
12	mr: annual yield		12.000%				
13	c: coupon rate each period		8.000%				
14	r: yield each period		6.000%				
15	n: maturity in periods						
16	n/m: maturity in years						
17	D: Macaulay's duration in periods						
18	D/m: Macaulay's duration in years						
19	alpha: proportion of a period elapsed		0.994444		C19	=1-1/(360/C$10)	
20							
21	n	n/m	D/m		C11:C14, B22:C22, B23, and		
22	16	8	5.067536		B24:C24		
23	15.005556	7.502778	4.570314			same as corres-	
24	15	7.5	4.873506			ponding cells in	
25	14.005556	7.002778	4.376284			Figure 3	
26	14	7	4.66854				
27	13.005556	6.502778	4.171317		A23	=A22-C$19	
28	13	6.5	4.45186		C23	=C22-C$19/C$10	
29	12.005556	6.002778	3.954638		A24	=A22-1	
30	12	6	4.22261				
31	11.005556	5.502778	3.725388		A23:C23	copied to A25:C25	
32	11	5.5	3.979844		A24:C25	copied to A24:C53	
33	10.005556	5.002778	3.482621				
34	10	5	3.72251				
35	9.005556	4.502778	3.225288				
36	9	4.5	3.449441				
37	8.005556	4.002778	2.952219				
38	8	4	3.159329				
39	7.005556	3.502778	2.662107				
40	7	3.5	2.850708				
41	6.005556	3.002778	2.353486				

Figure 4 An Excel Worksheet Showing the Computations of Macaulay's Duration for Different Maturities and the Resulting Saw-toothed Graph, with Coupon Rates and Yields to Maturity Generated via Two Scroll Bars.

	A	B	C	D	E	F	G
42	6	3	2.521926				
43	5.005556	2.502778	2.024704				
44	5	2.5	2.171112				
45	4.005556	2.002778	1.673889				
46	4	2	1.796136				
47	3.005556	1.502778	1.298914				
48	3	1.5	1.394565				
49	2.005556	1.002778	0.897343				
50	2	1	0.963599				
51	1.005556	0.502778	0.466377				
52	1	0.5	0.5				
53	0.005556	0.002778	0.002778				
54							
55							
56							
57							
58							
59							
60							
61							
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63							
64							
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82							

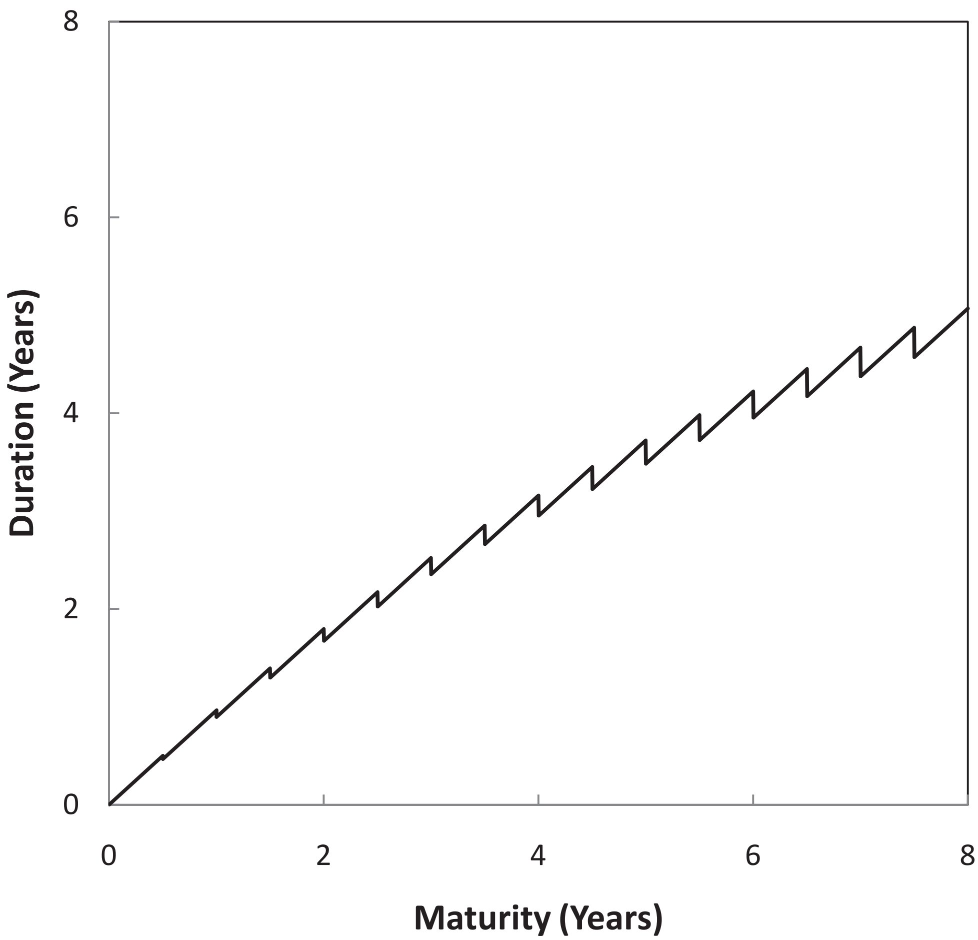

Figure 4 An Excel Worksheet Showing the Computations of Macaulay's Duration for Different Maturities and the Resulting Saw-toothed Graph, with Coupon Rates and Yields to Maturity Generated via Two Scroll Bars (Continued).

Some representative cell formulas are indicated. The graph of duration versus maturity uses Excel's graphical feature that displays the connecting line between each pair of adjacent points. This graphical feature works particular well for the purpose of illustrating the saw-tooth problem that the combination of equations (20) and (5) implies.

For the specific combination of the input parameters as shown in Figure 4, the saw-toothed pattern of the graph of duration versus maturity is clearly noticeable. The interactive nature of the two scroll bars will allow students to explore the severity of the problem for various combinations of the input parameters. Again, the range of values that each scroll bar covers can be adjusted, so that a wider range of each input parameter can be attempted.

Notice that a longer or shorter maturity can easily be accommodated. For example, to reduce n/m to 6 years, A22 will have to be changed to 12. As the change will make each cell in A46:A53 negative, A46:C53 ought to be deleted. The axes of the graph will have to be adjusted as well. Likewise, to extend to $n/m = 10$ years instead, A22 will have to be changed to 20. To generate additional data for the graph, A52:C53 will have to be copied to A52:C61. The data series for the graph will have to be expanded to cover B22:C61. In addition, the axes of the graph will also have to be adjusted accordingly. Alternatively, instead of covering maturities from 0 to n/m years in the graph, the coverage can be between any two specific maturities. In such cases, changes to the worksheet and the axes of the graph in Figure 4 are analogous.

As explained in Subsection 6.1, the percentage jump in duration at each coupon date is connected directly to the current yield C/P which, in turn, is determined by the underlying parameters in bond valuation. In view of this analytical feature, Table 1 shows numerically how various combinations of the underlying parameters affect the jumps in Macaulay's duration at some selected coupon dates. Such information is also intended to provide some guidance for the choice of parameter values for the interactive Excel exercise as illustrated in Figure 4.

For a 360-day year, $\alpha = 179/180$ represents the proportion of a semi-annual period elapsed since the previous coupon date. With $m = 2$, the annual coupon rates (mc) and the annual yields to maturity (mr) considered are 4%, 8%, and 12%. The maturities $[(n-1)/m]$ considered are 5, 15, and 25 years.

As expected, the percentage jump increases with the coupon rate and with the yield to maturity. For example, in the case of a 4% annual yield to maturity and 5 years to maturity, as shown in the first row of the displayed figures, the 1.9%, 3.3%, and 4.3% jumps correspond to 4%, 8%, and 12% annual coupon rates, respectively. Likewise, in the case of a 12% annual coupon and 5 years to maturity, as shown in the last column of the displayed figures, the 4.3%, 5.1%, and 5.9% jumps correspond to 4%, 8%, and 12% annual yields, respectively. The same patterns can be noted for other combinations of such given parameters as well.

Table 1 Values of Macaulay's duration on the day before each coupon date, measured in years, and its one-day jump, measured in years and in percentage terms (in parentheses), for various combinations of annual coupon rates, annual yields to maturity, and years to maturity.

Years to Maturity	Yield to Maturity	Coupon Rate 4%		Coupon Rate 8%		Coupon Rate 12%	
		Duration	Jump	Duration	Jump	Duration	Jump
5	4%	4.49	0.09 (1.9%)	4.15	0.14 (3.3%)	3.91	0.17 (4.3%)
	8%	4.43	0.10 (2.3%)	4.06	0.16 (3.9%)	3.80	0.19 (5.1%)
	12%	4.36	0.12 (2.8%)	3.96	0.18 (4.6%)	3.68	0.22 (5.9%)
15	4%	11.20	0.22 (2.0%)	9.79	0.27 (2.7%)	9.05	0.28 (3.1%)
	8%	10.10	0.31 (3.0%)	8.65	0.34 (4.0%)	7.95	0.35 (4.4%)
	12%	8.89	0.39 (4.4%)	7.50	0.41 (5.5%)	6.89	0.41 (6.0%)
25	4%	15.71	0.31 (2.0%)	13.65	0.33 (2.4%)	12.74	0.34 (2.6%)
	8%	12.44	0.43 (3.5%)	10.74	0.43 (4.0%)	10.07	0.42 (4.2%)
	12%	9.47	0.51 (5.4%)	8.31	0.48 (5.8%)	7.88	0.47 (6.0%)

Table 1 also shows that, as expected, for cases where the coupon rate is below the yield to maturity, percentage jump increases with maturity. Further, as expected, for cases where the coupon rate exceeds the yield to maturity, the longer it takes for a bond to mature, the lower is the percentage jump. For example, the set of displayed figures in parentheses in the middle columns of Table 1, where the annual coupon rate is 8%, confirms such analytical features. Specifically, for the case of a 12% yield, the 4.6%, 5.5%, and 5.8% jumps correspond to 5, 15, and 25 years to maturity, respectively. For the case of a 4% annual yield, the corresponding jumps are 3.3%, 2.7%, and 2.4% instead.

As Table 1 shows, given the underlying parameters for bond valuation, the percentage jump ranges from 1.9% to 6.0%. The former figure corresponds to a 4% annual coupon and a 4% annual yield; the latter, a 12% for the same underlying parameters. Thus, for a given term to maturity, the combination of a high coupon rate and a high yield is a recipe for a high percentage jump.

7 A Corrective Measure

In the absence of any changes in r, as the maturity date of the bond gets closer, the magnitude of any premium or discount, which $P - F$ represents, becomes smaller. However, as time passes by, the accrued interest αC exhibits a periodic, saw-toothed pattern that resets itself to zero at all coupon dates. Thus, a saw-toothed time pattern in the full price, $P + \alpha C$, is an inevitable outcome.

As a corrective measure, let us combine equations (15) and (16) into

$$\frac{dP}{P} = -D^{(R)}\frac{dr}{1+r} = -D^{(RM)}dr, \tag{23}$$

which differs from equation (17) in that the clean price of the bond is used instead. Here, the corresponding proportionality constants,

$$\begin{aligned} D^{(R)} \quad = \quad & \left[\frac{(1-\alpha)C}{(1+r)^{1-\alpha}} + \frac{(2-\alpha)C}{(1+r)^{2-\alpha}} + \cdots + \frac{(n-\alpha)C}{(1+r)^{n-\alpha}} + \frac{(n-\alpha)F}{(1+r)^{n-\alpha}}\right] \Big/ \\ & \left[\frac{C}{(1+r)^{1-\alpha}} + \frac{C}{(1+r)^{2-\alpha}} + \cdots + \frac{C}{(1+r)^{n-\alpha}} + \frac{F}{(1+r)^{n-\alpha}} - \alpha C\right] \end{aligned} \tag{24}$$

and $D^{(RM)} = D^{(R)}/(1+r)$, can be viewed as revised duration measures; they are the counterparts of D and $D^{(M)}$, respectively. These duration measures are related by $D^{(R)} = D(P+\alpha C)/P$ and $D^{(RM)} = D^{(M)}(P+\alpha C)/P$. Again, both $D^{(R)}$ and $D^{(RM)}$ are measured in the number of periods. When measured in years instead, they are $D^{(R)}/m$ and $D^{(RM)}/m$, respectively, where m is the annual frequency of coupon payments.

In the case of a pure discount bond, where $C = 0$, equation (24) reduces to $D^{(R)} = n - \alpha$. Thus, just like what is shown in standard investment textbooks, the graph of a revised duration measure versus maturity for such a bond is also a 45^o line. The graph can be either $D^{(R)}$ versus $n - \alpha$ or $D^{(R)}/m$ versus $(n-\alpha)/m$, depending on whether the measurement unit is period or year.

7.1 A Weighted Average of Arrival Times of Cash Flows

Can the revised duration measure, as defined by equation (24), still be interpreted as a weighted average of arrival times of cash flows from a bond investment? To answer this question, recall that we have treated the settlement date of the buy-sell transactions as time 0. In order to receive the full amount of the next coupon at time $1 - \alpha$, the buyer pays at time 0 not only the clean price P, but also the accrued interest αC. The accrued-interest payment can be viewed as a negative cash flow of $-\alpha C$ at time 0. The cash flows $-\alpha C, C, C, \ldots, C, C + F$ occur at times $0, 1-\alpha, 2-\alpha, \ldots, n-1-\alpha, n-\alpha$, respectively. The cash flow of $-\alpha C$ at time 0 has no contribution to the numerator in the expression of $D^{(R)}$ in equation (24), where the present value of each cash flow is multiplied by the corresponding time of occurrence. In such a context, $D^{(R)}$ is still a weighted average of arrival times of all cash flows from the bond investment.

7.2 A Connection to the Current Yield

As established earlier in this section, the multiplicative correction factor for either D or $D^{(M)}$ is $(P+\alpha C)/P = 1+\alpha C/P$. Here, C/P is the current yield per period of the bond, as of the valuation date at which a proportion α of a period has elapsed. In the case of D, the percentage correction at such a date is $[D(1+\alpha C/P)-D]/D = \alpha C/P$. Likewise, with the symbol D here substituted by $D^{(M)}$, the corresponding percentage correction pertaining to modified duration is $\alpha C/P$ as well.

The above analytical feature allows us to understand better the one-day jump in duration at each coupon date, as described in Subsection 6.1. Specifically, as the accrued interest αC — which is an immediate cash outflow for the bond purchaser involved — is ignored, a consequence is an understatement of the corresponding duration measure. As α increases, the problem worsens. On the day immediately prior to each coupon date, as $\alpha \approx 1$, the percentage understatement in either duration measure is almost at the level of C/P. This is consistent with the one-day jump in duration at each coupon date being nearly C/P, as established in Subsection 6.1. Notice that, as the way C/P varies with each of C, n, and r has already been established in that subsection, there is no need to repeat the same results here.

7.3 A Closed-form Expression

As shown in Appendix D, equation (24) can be equivalently written as

$$D^{(R)} = \frac{[(1+r-\alpha r)(1+r)^n - (1+r-\alpha r+nr)]\,c + (n-\alpha)r^2}{r\,[(1+r)^n - 1 - r(1+r)^{n-\alpha}\alpha]\,c + r^2}. \qquad (25)$$

This is an explicit expression in terms of c, n, α, and r. With $D^{(R)}$ known, $D^{(RM)} = D^{(R)}/(1+r)$ can be deduced directly. Notice that, for the special case where $\alpha = 0$, once a common factor $(1+r)^{-n}$ is multiplied to both the numerator and the denominator of the expression of $D^{(R)}$, equation (25) reduces to equation (5) for a basic model.

Notice also that, like equation (5), equation (25) is intended for practically relevant cases where r is strictly positive. If $r = 0$, equation (25) will fail to provide a value of $D^{(R)}$. Likewise, this does not mean that $D^{(R)}$ is undefined if $r = 0$. In such a case, equation (24) reduces to

$$D^{(R)} = \frac{c(1 + 2 + \cdots + n) - \alpha cn + n - \alpha}{c(n - \alpha) + 1} = \frac{cn[(n + 1)/2 - \alpha] + n - \alpha}{c(n - \alpha) + 1}. \quad (26)$$

Equation (26) implies that $D^{(R)} = n - \alpha$ if $c = r = 0$. However, as indicated earlier, equation (24) leads to the same result if $c = 0$. Thus, regardless of whether $r = 0$ or $r > 0$, cases of $c = 0$ always correspond to $D^{(R)} = n - \alpha$.

8 Duration Functions in Excel: An Assessment of the Saw-tooth Problem with Bond Quotations and a Corrective Measure

As indicated in the introductory section, Excel has specific functions to compute Macaulay's duration and modified duration, which are DURATION and MDURATION, respectively. Each function has six arguments, including the settlement date, the maturity date, the annual coupon rate, the annual yield to maturity, the frequency of coupon payments each year, and a code for the day-counting convention involved. The settlement date is the date to settle any buy-sell transactions; this is also the valuation date for the bond in question. The two Excel functions are intended to be used for any valuation dates, regardless of whether such dates can be accommodated by a basic bond model.

From a practical perspective, a relevant question is whether the daily yield fluctuations in bond markets are severe enough to render the saw-tooth problem in duration practically irrelevant. If so, the two Excel functions can still serve their intended purposes. Otherwise, there are both practical and pedagogical implications.

To assess the situation, we first rely on quotation data of two Canadian federal government bonds. Daily quotations until July 5, 2011 were accessed from the *GlobeInvestor Gold* website of *The Globe and Mail*, a Canadian national newspaper.[6] Both are semi-annual coupon bonds. The day-counting convention for the accrued interest is based on the actual calendar days elapsed. The quoted price and yield data are on the bid side. The bond that matures on June 1, 2020, labeled as Bond A, pays \$3.50 coupons annually for a \$100 face value has data since September 2, 2009. The bond that matures on March 15, 2021, labeled as Bond B, pays \$10.50 coupons annually has available data commencing on March 26, 2009 instead. The coupon dates are June 1 and December 1 for Bond A, and March 15 and September 15 for Bond B. As the settlement date is unavailable from this data source, a three-business-day settlement is assumed.

[6] Data access from the website http://gold.globeinvestor.com/ is subscription-based.

As an illustration, let us consider the daily quotations of Bond B for Thursday, September 9, 2010 and Friday, September 10, 2010. The quoted annual yields for the two days are 3.11% and 3.12%, respectively. Under the assumption of a three-business-day settlement, the corresponding valuation dates are Tuesday, September 14, 2010 and Wednesday, September 15, 2010, with the latter date being a coupon date. Although the quoted annual yields for the two consecutive dates in question differ in only 1 basis point, the results from the Excel function DURATION are 7.29404 and 7.52106 years, for September 14, 2010 and September 15, 2010, respectively. Likewise, the corresponding results from the Excel function MDURATION are 7.18236 and 7.40554 years. With a one-day jump in either Macaulay's duration or modified duration exceeding 0.22 years, the adequacy of these Excel functions seems questionable.

Indeed, some saw-toothed time patterns are clearly noticeable in Macaulay's duration and modified duration, based on daily quotations for the two bonds over the sample period. As the latter is smaller in magnitude than the former because of the multiplicative factor $1/(1+r)$, its saw-toothed time pattern would tend to be easier to conceal, for given yield fluctuations in the bond market. Thus, only the latter is shown in Figure 5. Analytically, when measured in years, rather than in periods, what is shown is $D^{(M)}/m$, where $m = 2$. Any day-to-day yield fluctuations notwithstanding, there is a clear one-day jump in duration at each coupon date.

The two lighter-shade graphs, also shown in Figure 5, pertain to a revised duration measure in response to the saw-tooth problem. Such graphs are based on $D^{(RM)}/m$; the revision is by applying directly the multiplicative factor $(P + \alpha C)/P$ to $D^{(M)}/m$. The results are the same as those from computing $D^{(RM)}/m$ directly, where $D^{(RM)} = D^{(R)}/(1 + r)$, with $D^{(R)}$ based on equation (25). This revised version has apparently removed the jumps that are associated with the traditional duration measure $D^{(M)}/m$.

As Canadian corporate bonds are much riskier than Canadian government bonds, daily fluctuations of their annual yields have a better potential to mask the saw-tooth problem in the traditional duration measures. Thus, to gauge the potential severity of the problem empirically, we accessed some Canadian corporate bond quotations from the same data source for Figure 5. The data collection process started with identifying bonds with annual coupon rates that exceed 9%. To net out any maturity effects, we considered additional bonds with matching maturities within ±2 months of the each of these high coupon bonds, for two groups of lower annual coupon rates. The lowest coupon group had annual coupon rates up to 4.5%. Each of the three groups had 33 bonds, for a total of 99 matching bonds in the sample.

Macaulay's duration and modified duration were computed by using Excel functions DURA-

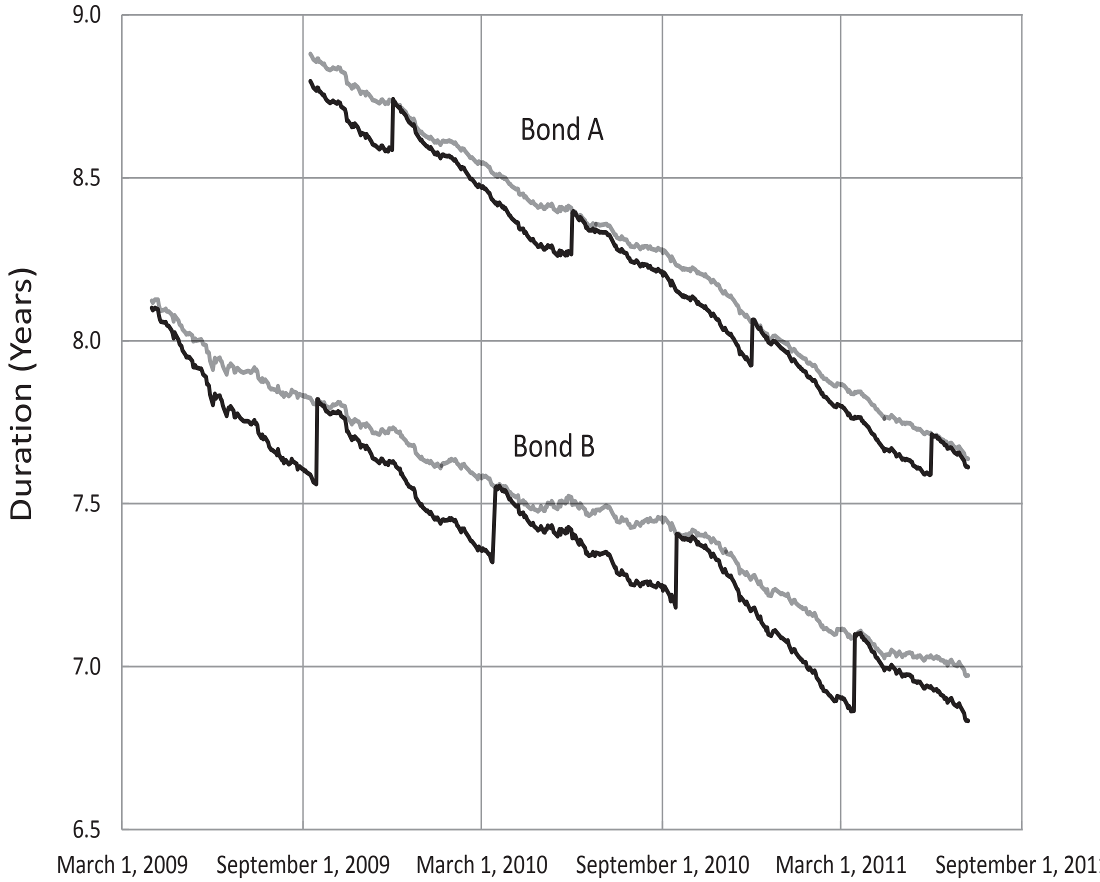

Figure 5 The Saw-toothed Time Pattern of Modified Duration of Two Canadian Government Bonds. [Bonds A and B Pay $3.50 and $10.50 Annual Coupons, Respectively, for a $100 Face Value; the Corresponding Lighter-shade Graphs Provide Revised Duration Measures.]

TION and MDURATION, respectively. For either duration measure, the average one-day jumps at the most recent coupon date of each bond, prior to July 2011, for the low, mid, and high coupon groups, were 1.7%, 2.6%, and 4.0%, respectively. As such results indicate, even for low coupon bonds in today's market that is characterized by low bond yields, the observed jumps in duration are far from being trivially small.

9 Concluding Remarks

Macaulay's duration and its popular variant, called modified duration, can be stated in terms of a weighted average of arrival times of cash flows from a bond investment. To complement the coverage on bond duration in standard investment textbooks, this paper has considered analytically both a basic model and a more realistic model for bond valuation. The difference between the two models is the absence or the presence of an accrued interest. The accrued interest is the part of the coupon that the seller of a bond is entitled to receive, if the settlement date of the transaction is not on a coupon date.

For a basic bond model, for which the issue of accrued interest is not a concern, this paper has utilized both Excel's graphical features and the interactive nature of scroll bars to enhance students' understanding of the duration concept. Although textbook justification of the duration concept pertains to a basic bond model, its applications, including its Excel implementation, seem to be more versatile in practice. In particular, to assist investment decisions, the Excel functions for computing Macaulay's duration and modified duration — the two traditionally-defined duration measures — are intended to accommodate any settlement dates.

As this paper has shown, the saw-toothed time pattern of Macaulay's duration and modified duration can be traced to the accrued interest in bond valuation. The accrued interest, which increases linearly with the passage of time between two adjacent coupon dates, is reset to zero once the next coupon date is reached. Once it is established that these duration measures imply the use of the full price of a bond, which includes the accrued interest, what underlies the saw-toothed time pattern there becomes obvious. This paper has also used Excel's graphical features to illustrate the saw-tooth problem. The interactive nature of scroll bars is particularly useful, as it allows students to assess the severity of the problem for various combinations of the input parameters in bond valuation.

Given the direct connection between the accrued interest and the saw-tooth problem in duration, it is easy to rectify the situation. All it takes is to use the clean price instead when establishing the percentage price changes in response to interest-rate changes. For investors who are relying on canned programs to provide duration measures, a revised version can be reached by applying directly a multiplicative factor, which is the ratio of the full price and the clean price of the bond in question. From a pedagogic perspective, a nice feature of this corrective measure is that, with the accrued interest treated as an immediate cash flow from a bond investment, the idea of a weighted average of arrival times of cash flows is retained.

For analytical convenience, a flat yield curve has been assumed in this paper; that is, as in standard investment textbooks, the graph of yield versus maturity has been treated as a horizontal line. However, the simple idea in this paper still holds when Macaulay's duration is used for other shapes of the yield curve. Regardless of how future cash flows from a bond investment are discounted, the accrued interest, which is unaffected by the shape of the yield curve, can still be treated as an immediate cash flow. Again, to implement the idea, all it takes is to apply a multiplication factor that is the ratio of the bond's full price to its clean price. This simple corrective measure will contribute to improve the accuracy of duration measures in bond immunization strategies.

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Appendix A: Algebraic Derivation of Duration for a Basic Bond Model

Suppose that there is a small change in the yield to maturity from r to $r+\triangle r$. The corresponding change in the bond price is from P to $P+\triangle P$. As the bond price given by equation (1) in the main text is the sum of the present values of future coupons and the face value upon maturity, the price change $\triangle P$ is the sum of the changes of the individual present values. For the cash flow at time h, where $h = 1, 2, \ldots, n$, the change in the present-value factor is

$$\frac{1}{(1+r+\triangle r)^h} - \frac{1}{(1+r)^h} = \frac{(1+r)^h - (1+r+\triangle r)^h}{(1+r+\triangle r)^h (1+r)^h}, \qquad \text{(A1)}$$

which can be simplified via a binomial expansion of $(1+r+\triangle r)^h$. Provided that $\triangle r$ is small as compared to r, the term $(1+r+\triangle r)^h$ in the numerator can be approximated as $(1+r)^h + h(1+r)^{h-1} \triangle r$, with terms containing any of $(\triangle r)^2, (\triangle r)^3, \ldots, (\triangle r)^h$ ignored.[7] Noting that the denominator does not involve the difference of two numerically similar terms, we can approximate the term $(1+r+\triangle r)^h$ there simply as $(1+r)^h$. Thus, the change in the present-value factor for time h can be approximated as $-[h(1+r)^{h-1} \triangle r]/(1+r)^{2h}$ or, simply, $-[h/(1+r)^{h+1}] \triangle r$.

[7] According to the binomial theorem, $(a+b)^h$, where h is any positive integer, can be expressed as the sum of $h+1$ terms, with each term being $\frac{h!}{(h-k)!k!} a^{h-k} b^k$, for $k = 0, 1, 2, \ldots, h$. With $a = 1+r$, and $b = \triangle r$, the term $(1+r+\triangle r)^h$ can be expressed as $(1+r)^2 + 2(1+r)\triangle r + (\triangle r)^2$, for $h = 2$; $(1+r)^3 + 3(1+r)^2\triangle r + 3(1+r)(\triangle r)^2 + (\triangle r)^3$, for $h = 3$; $(1+r)^4 + 4(1+r)^3\triangle r + 6(1+r)^2(\triangle r)^2 + 4(1+r)(\triangle r)^3 + (\triangle r)^4$, for $h = 4$; and so on. Thus, the second term in the expansion of $(1+r+\triangle r)^h$ is $h(1+r)^{h-1}\triangle r$.

Accordingly, we can write

$$\triangle P = -\frac{1}{1+r}\left[\frac{C}{1+r} + \frac{2C}{(1+r)^2} + \cdots + \frac{nC}{(1+r)^n} + \frac{nF}{(1+r)^n}\right] \triangle r. \tag{A2}$$

Combining equations (1) and (A2) leads to

$$\frac{\triangle P}{P} = -D\frac{\triangle r}{1+r} = -D^{(M)} \triangle r. \tag{A3}$$

Here, the proportionality constants D and $D^{(M)}$ are as defined in equations (4) and (7), respectively.

Appendix B: Derivation of a Closed-form Expression of Macaulay's Duration for a Basic Bond Model

To express equation (4) in an equivalent but computationally convenient form, let

$$A = \frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^n}. \tag{B1}$$

As the expression of A is a geometric series, we can write

$$(1+r)A = 1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^{n-1}}. \tag{B2}$$

Differencing the two expressions leads to

$$rA = 1 - \frac{1}{(1+r)^n} \tag{B3}$$

and, equivalently,

$$A = \frac{1}{r}\left[1 - \frac{1}{(1+r)^n}\right], \tag{B4}$$

provided that $r \neq 0$.

Likewise, letting

$$B = \frac{1}{1+r} + \frac{2}{(1+r)^2} + \cdots + \frac{n}{(1+r)^n}, \tag{B5}$$

we have

$$(1+r)B = 1 + \frac{2}{1+r} + \cdots + \frac{n}{(1+r)^{n-1}}. \tag{B6}$$

Differencing the two expressions gives us

$$rB = 1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^{n-1}} - \frac{n}{(1+r)^n}, \tag{B7}$$

which contains a geometric series as well. In view of equation (B4), equation (B7) can be written as

$$B = \frac{1}{r}\left[1 + \frac{1-(1+r)^{-n}}{r} - \frac{n+1}{(1+r)^n}\right] = \frac{1}{r^2}\left[(1+r) - (1+r+nr)(1+r)^{-n}\right]. \tag{B8}$$

As equation (4) is equivalent to

$$D = \frac{Bc + n(1+r)^{-n}}{Ac + (1+r)^{-n}}, \tag{B9}$$

equation (5) follows directly upon substituting the expressions of A and B from equations (B4) and (B8).

Appendix C: The Relationship between Macaulay's Duration and Maturity for a Basic Bond Model

Given $D_n = w_n/p_n$, for $n = 1, 2, 3, \ldots$, as defined in Subsection 3.3 of the main text, we can also write $D_{n+1} = w_{n+1}/p_{n+1}$. It follows that[8]

$$D_{n+1} - D_n = \frac{c + p_n + w_n}{c + p_n} - \frac{w_n}{p_n} = 1 - \frac{cw_n}{p_n(c + p_n)}. \tag{C1}$$

Thus, if $p_n(c + p_n) - cw_n > 0$, increasing the maturity from n to $n+1$ full periods will correspond to an increase in Macaulay's duration; that is, $D_{n+1} > D_n$.

To establish the sign of $p_n(c + p_n) - cw_n$, let us start with

$$\begin{aligned} p_n(c + p_n) &= \left[c\sum_{h=1}^{n} \frac{1}{(1+r)^h} + \frac{1}{(1+r)^n}\right]\left[c\sum_{k=0}^{n} \frac{1}{(1+r)^k} + \frac{1}{(1+r)^n}\right] \\ &= c^2 \sum_{h=1}^{n}\sum_{k=0}^{n} \frac{1}{(1+r)^{h+k}} \\ &\quad + \frac{2c}{(1+r)^n}\sum_{h=1}^{n} \frac{1}{(1+r)^h} + \frac{c}{(1+r)^n} + \frac{1}{(1+r)^{2n}}. \end{aligned} \tag{C2}$$

The double summation $\sum_{h=1}^{n}\sum_{k=0}^{n} 1/(1+r)^{h+k}$, with n cases of h and $n+1$ cases of k, consists of $n(n+1)$ terms. They can be combined into the sum of the following $2n$ terms: $(1+r)^{-1}, 2(1+r)^{-2}, \ldots, n(1+r)^{-n}$ and $n(1+r)^{-(n+1)}, (n-1)(1+r)^{-(n+2)}, \ldots, (1+r)^{-2n}$.[9] As

$$\begin{aligned} \sum_{h=1}^{n}\sum_{k=0}^{n} \frac{1}{(1+r)^{h+k}} &= \sum_{h=1}^{n} \frac{h}{(1+r)^h} + \sum_{h=n+1}^{2n} \frac{2n-h+1}{(1+r)^h} \\ &= \sum_{h=1}^{n} \frac{h}{(1+r)^h} + \frac{1}{(1+r)^n}\sum_{h=1}^{n} \frac{n-h+1}{(1+r)^h}, \end{aligned} \tag{C3}$$

[8] In Subsection 3.3, the use of the difference $D_n - D_{n-1}$ allows the analytical expression involved to be applied directly to Section 6, where the saw-tooth problem is examined analytically. For the purpose of the algebraic proof here, the use of the difference $D_{n+1} - D_n$ instead, with D_n and its underlying parameters being the reference point, is for notational convenience.

[9] To illustrate, suppose that $n = 3$. As $h = 1, 2, 3$ and $k = 0, 1, 2, 3$, expanding the double summation will result in the sum of 12 individual terms among $(1+r)^{-1}, (1+r)^{-2}, \ldots, (1+r)^{-6}$. The only way to reach $(1+r)^{-1}$ is with $h = 1$ and $k = 0$. There are two ways to reach $(1+r)^{-2}$; that is, with $h = 2$ and $k = 0$ or with $h = 1$ and $k = 1$. There are three ways to reach $(1+r)^{-3}$; that is, with $h = 3$ and $k = 0$, with $h = 2$ and $k = 1$, or with $h = 1$ and $k = 2$. There is also three ways to reach $(1+r)^{-4}$; that is, with $h = 3$ and $k = 1$, with $h = 2$ and $k = 2$, or with $h = 1$ and $k = 3$. There are two ways to reach $(1+r)^{-5}$; that is, with $h = 3$ and $k = 2$, or with $h = 2$ and $k = 3$. The only way to reach $(1+r)^{-6}$ is with $h = 3$ and $k = 3$. Thus, the double summation can be written as $(1+r)^{-1} + 2(1+r)^{-2} + 3(1+r)^{-3} + 3(1+r)^{-4} + 2(1+r)^{-5} + (1+r)^{-6}$. As $1+2+3+3+2+1 = 12$, all cases of h and k in the double summation are accounted for.

we have

$$p_n(c+p_n) - cw_n = \frac{1}{(1+r)^n}\left[\sum\nolimits_{h=1}^{n} \frac{c^2(n-h+1)+2c}{(1+r)^h} - (n-1)c + \frac{1}{(1+r)^n}\right]. \tag{C4}$$

Given equation (C4), we now proceed to show by induction that $D_{n+1} - D_n > 0$, for $n = 1, 2, 3, \ldots,$ if $c \geq r$. As $D_1 = 1$ and $1 < D_2 < 2$, we have $D_2 - D_1 > 0$. Suppose that $D_n - D_{n-1} > 0$. Equation (C4) with n substituted by $n-1$ gives us

$$p_{n-1}(c+p_{n-1}) - cw_{n-1} = \frac{a_{n-1}}{(1+r)^{n-1}}, \tag{C5}$$

where

$$a_{n-1} = \sum\nolimits_{h=1}^{n-1} \frac{c^2(n-h)+2c}{(1+r)^h} - (n-2)c + \frac{1}{(1+r)^{n-1}} \tag{C6}$$

is positive.

To show that $D_{n+1} - D_n > 0$, we write equation (C4) as

$$\begin{aligned} p_n(c+p_n) - cw_n &= \frac{1}{(1+r)^n}\left\{a_{n-1} + c\left[c\sum\nolimits_{h=1}^{n} \frac{1}{(1+r)^h} + \frac{1}{(1+r)^n} - 1\right]\right. \\ &\quad \left. + \frac{c}{(1+r)^n} + \frac{1}{(1+r)^n} - \frac{1}{(1+r)^{n-1}}\right\} \\ &= \frac{1}{(1+r)^n}\left[a_{n-1} + c(p_n - 1) + \frac{c-r}{(1+r)^n}\right]. \end{aligned} \tag{C7}$$

As $c = r$ implies $p_n = 1$ and $c > r$ implies $p_n > 1$, the above expression is positive if $c \geq r$. This completes the proof by induction.

If $c < r$ instead, we have $p_n < 1$. If so, an increase in maturity does not always correspond to an increase in duration. To see this, suppose that $p_1(c+p_1) - cw_1$, $p_2(c+p_2) - cw_2$, $\ldots$, $p_{n-1}(c+p_{n-1}) - cw_{n-1}$ are all positively or, simply, that $a_1, a_2, \ldots, a_{n-1}$ are all positive. As soon as the combined negative effect of $c(p_n - 1)$ and $(c-r)/(1+r)^n$ is strong enough to offset completely the positive a_{n-1} in equation (C7), a_n will not be positive. Once this happens, we must have $D_{n+1} \leq D_n$. When equation (C7) is used again recursively, $p_{n+1}(c+p_{n+1}) - cw_{n+1}, p_{n+2}(c+p_{n+2}) - cw_{n+2}, \ldots,$ as well as $a_{n+1}, a_{n+2}, \ldots,$ must all be negative. It follows that $D_1 < D_2 < \ldots < D_n$ and $D_n \geq D_{n+1} > D_{n+2} > D_{n+3} > \ldots$ Thus, there will be a single maximum in the graph of duration versus maturity.

Appendix D: Derivation of Closed-form Expressions of Revised Duration Measures

The revised duration measures as proposed in Section 7 in the main text can be obtained by revising their conventional counterparts via $D^{(R)} = D(P + \alpha C)/P$ and $D^{(RM)} = D^{(R)}/(1+r)$. However, they can also be computed directly in terms of c, n, α, and r, where $c = C/F$ is the coupon rate each period. For this task, let us multiply both the numerator and the denominator of the expression in equation (24) with $(1+r)^{-\alpha}$ to obtain

$$\begin{aligned} D^{(R)} &= \left[\frac{(1-\alpha)C}{1+r} + \frac{(2-\alpha)C}{(1+r)^2} + \cdots + \frac{(n-\alpha)C}{(1+r)^n} + \frac{(n-\alpha)F}{(1+r)^n}\right] \Bigg/ \\ &\quad \left[\frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^n} + \frac{F}{(1+r)^n} - \frac{\alpha C}{(1+r)^\alpha}\right]. \end{aligned} \tag{D1}$$

We can write equation (D1) as

$$D^{(R)} = \left[cB - \alpha c A + \frac{n-\alpha}{(1+r)^n}\right] \Bigg/ \left[cA + \frac{1}{(1+r)^n} - \frac{\alpha c}{(1+r)^\alpha}\right]. \tag{D2}$$

Here, A and B are as defined in equations (B1) and (B5), respectively, in Appendix B. Once the expressions of A and B in equations (B4) and (B8), respectively, in Appendix B are substituted into equation (D2), equation 25 in Section 7 of the main text follows directly.