Spreadsheets in Education (eJSiE)

Volume 5 | Issue 3 Article 2

July 2012

Connecting Binomial and Black-Scholes Option Pricing Models: A Spreadsheet-Based Illustration

Yi Feng
Ryerson University, yi.feng@ryerson.ca

Clarence C. Y. Kwan
McMaster University, kwanc@mcmaster.ca

Connecting Binomial and Black-Scholes Option Pricing Models: A Spreadsheet-Based Illustration

Abstract
The Black-Scholes option pricing model is part of the modern financial curriculum, even at the introductory level. However, as the derivation of the model, which requires advanced mathematical tools, is well beyond the scope of standard finance textbooks, the model has remained a great, but mysterious, recipe for many students. This paper illustrates, from a pedagogic perspective, how a simple binomial model, which converges to the Black-Scholes formula, can capture the economic insight in the original derivation. Microsoft Excel™ plays an important pedagogic role in connecting the two models. The interactivity as provided by scroll bars, in conjunction with Excel's graphical features, will allow students to visualize the impacts of individual input parameters on option pricing.

Keywords
binomial option pricing model, Black-Scholes formula, Bernoulli trials

Cover Page Footnote
The authors wish to thank the anonymous reviewers for helpful comments and suggestions.

Connecting Binomial and Black-Scholes Option Pricing Models: A Spreadsheet-Based Illustration

1 Introduction

Call option is a financial instrument that gives its holder the right, not the obligation, to purchase from its seller one unit of the underlying security, at a predetermined price, at or before an expiry date. American and European versions differ in that the latter can be exercised only at the expiry date. The seminal work of Black and Scholes (1973) on the pricing of European call options of stocks has led to many innovations in the financial world. The most noticeable are the phenomenal growth of markets for trading various derivative securities and the corresponding research activities on creating and pricing such securities.[1] Given its academic and practical significance, the Black-Scholes option pricing model has been part of the finance curriculum for decades now, even at the introductory level.

The derivation of the model, which requires advanced mathematical tools including those for solving partial differential equations, is well beyond the scope of most finance textbooks. Standard textbook coverage tends to be confined to the derived result — which is the well-known Black-Scholes formula — and its implications, along with some numerical and graphical illustrations.[2] The concepts of financial options, in contrast to those pertaining to more traditional financial instruments such as bonds and stocks, are already quite abstract for many students. Therefore, of relevance to finance instructors is how best to present the model to students, if its derivation is not to be bypassed completely. As students can learn more about option pricing if the Black-Scholes formula is not merely a mysterious recipe to them, the challenge is to find a mathematical language for its derivation that they can understand. With such a constraint in mind, the scope of this paper is limited to the pricing of European call options.

Cox, Ross, and Rubinstein (1979) have derived instead a binomial option model, which converges to the Black-Scholes version. The binomial approach shares the same statistical idea of finding the probability of some specific numbers of heads and tails from repeatedly tossing a biased coin. What is attractive from a pedagogic perspective is that, while being purely algebraic and much easier for students to follow, the derivation of the binomial option pricing model still retains the same economic insight of the Black-Scholes version. Thus, a good understanding of the binomial

[1] As MacKenzie (2006) indicates, derivative contracts have grown from virtually nothing in 1970 to 273 trillion U.S. dollars outstanding worldwide by 2004.

[2] See, for example, Sharpe, Alexander, Bailey, Fowler, and Domian (2000, Chapter 19) and Copeland, Weston, and Shastri (2005, Chapter 7).

model will make the Black-Scholes model less mysterious to students. The binomial model has an additional advantage; it provides a foundation to develop numerical approaches to price derivative securities for which closed-form solutions are unavailable.[3]

The matching of numerical results of option prices from the two models may simply require appropriate choices of the values of their underlying parameters. In contrast, statistical convergence of the two models also requires that the number of binomial periods — or the number of tosses in the coin-toss analogy — approaches infinity, as this is about the conversion of a discrete-time model to a continuous-time case. However, the analytical materials involved are quite complicated; a crucial step to connect the two models also requires some advanced statistical knowledge. Indeed, even in financial theory textbooks, the final connection between the two models has not been provided explicitly. Rather, readers are referred to the Cox, Ross, and Rubinstein article for analytical details.[4]

To complicate matters further, the binomial option pricing model is actually a family of models, which under some conditions can all converge to the Black-Scholes model. Chance (2008) has evaluated 11 of such models in the finance literature, including the Cox-Ross-Rubinstein version. In essence, these models differ in their characterizations of the up-down movements of the underlying stock price and of the probabilities of such movements. The finding of Chance is that none of the 11 models consistently outperform all remaining models, in terms of convergence to the Black-Scholes model for various combinations of input parameters. However, the Cox-Ross-Rubinstein model, though being the simplest among all binomial models considered and being one of the two earliest versions of such models, has received the best overall performance score.[5] For ease of exposition in what follows, we treat the Cox-Ross-Rubinstein version of binomial models simply as the binomial model, unless noted otherwise.

As the connection of the binomial model to the Black-Scholes model is nearly complete by relying on algebraic tools alone, of interest is whether option prices based on the two models are similar or very different, for some combinations of input parameters. If it turns out that they are similar, then the binomial model not only can help students reduce the mystery in option pricing, but also can serve as a good proxy for the original Black-Scholes model. In this paper, we use Microsoft Excel^{TM} for a pedagogic illustration.[6]

[3] See, for example, Hull and White (1988) for a lattice approach — which is an extension of the binomial approach — to price American put options on dividend and non-dividend paying stocks, with an efficient numerical method called the control variate technique.

[4] See, for example, financial theory textbooks such as Copeland, Weston, and Shastri (2005, Chapter 7) and Tucker (1991, Chapter 13), which provide derivations of the binomial model.

[5] The other pioneering binomial model, which was subsequently refined in the literature, was the work of Rendleman and Bartter (1979).

[6] Hereafter, the commercial software Microsoft Excel^{TM} is referred to as Excel, for simplicity. It is implicit that

In particular, the Excel functions NORMSDIST and BINOMDIST are applied directly to the standardized normal distribution in the Black-Scholes formula and to the binomial distribution in the binomial case, respectively.[7] The use of Excel's scroll bars to vary the common input parameters, in conjunction of its graphical features, will allow students to explore interactively the option pricing results for various combinations of input parameters. Thus, Excel will play an important role in connecting the two models and in reducing the perceived mystery of the Black-Scholes formula.

This paper is organized as follows: Section 2 reproduces the Black-Scholes formula. An algebraic derivation of the binomial option pricing model is provided in Section 3. Its connection to the Black-Scholes version is considered next in Section 4. The pedagogic emphasis of Section 4 is its use of only algebraic tools to connect the two models. Section 5 presents some interactive Excel illustrations for comparing the two models, both numerically and graphically. Convergence of the two models is further analyzed and tabulated there. Section 6 provides some concluding remarks.

Readers who are already familiar with Black-Scholes and binomial option pricing models can skip Sections 2 and 3, as these two sections are intended to provide the background material. For other readers, especially business students who require guidance to understand the algebraic and statistical tools involved, the detailed derivation in Section 3 is still useful. The key pedagogic contributions of this paper, however, are in Sections 4 and 5.

2 The Black-Scholes Option Pricing Model

Consider a European call option on an underlying stock of price S, in a market where the continuously compounded annual risk-free interest rate is r. The option, which expires in T years (with T typically being a proportion), has an exercise price of X. Before introducing the Black-Scholes option pricing model, let us establish below feasible values of the option, which is independent of the model itself. Such information will also be used for the Excel illustrations in Section 5.

2.1 Feasible Option Values (Boundary Conditions)

First, with C being the option price, we must have $C \leq S$. That is, the price of a call option cannot be higher than the price of the underlying stock. The reason is as follows: The option gives the holder the right to buy a share of the stock at a predetermined price. Since the stock

the software trademark is recognized whenever the name Excel is mentioned in this paper.

[7] Notice that the functions NORMSDIST and BINOMDIST, though still supported, are renamed NORM.S.DIST and BINOM.DIST, respectively, in Excel 2010.

can also be purchased directly in the market, no rational investor would pay more than the share price for just the right to buy the stock. Second, we must have $C \geq 0$. This is because, as a call option is a right, not an obligation, for the holder, it is at worst worthless.

Third, we must have $C \geq S - Xe^{-rT}$. That is, the call price cannot be lower than the difference between the underlying stock price and the present value of the exercise price, which is Xe^{-rT}. To see this, let us compare the current values of the following investment plans: (i) an investment in the option and X risk-free bonds, each of which pays a dollar on the expiry date of the option, for a total of $C + Xe^{-rT}$, and (ii) an investment in the underlying stock alone. Let C^* and S^* be the option and stock prices, respectively, on the expiry date of the option. We have $C^* = 0$ if $S^* < X$, and $C^* = S^* - X$ otherwise. Accordingly, on the expiry date of the option, plan (i) is worth $C^* + Xe^{-r \cdot 0} = X$ if $S^* < X$, and worth S^* otherwise, and plan (ii) is worth S^*. Thus, the current value of plan (i) cannot be lower; that is, $C + Xe^{-rT} \geq S$ or, equivalently, $C \geq S - Xe^{-rT}$.

The above feasible values of the option can be stated succinctly as

$$S \geq C \geq \max(0, S - Xe^{-rT}). \tag{1}$$

Thus, the graph of C versus S for a given combination of the input parameters pertaining to a call option pricing model, including X, r, T, and any others, must be in the open area on the (S, C)-plane, bounded by the following lines: $C = S$, $C = 0$, and $C = S - Xe^{-rT}$. The first and the third are parallel lines, with each slope being 1. The area narrows as T increases; it collapses to the line $C = S$ in the first quadrant of the (S, C)-plane, as T approaches infinity. In contrast, the area widens as T decreases; the third line becomes $C = S - X$ if $T = 0$.

2.2 The Black-Scholes Formula

The well-known Black-Scholes option pricing formula is

$$C = S\,\mathbf{N}(d_1) - X\ e^{-rT}\,\mathbf{N}(d_2), \tag{2}$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S}{X}\right) + rT\right] + \frac{1}{2}\sigma\sqrt{T} \tag{3}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}. \tag{4}$$

Here, besides the various symbols previously defined, σ is the standard deviation of annual returns of the underlying stock and $\mathbf{N}(\cdot)$ is the cumulative standardized normal distribution. In the normal distribution function $f(z)$, which has a zero mean and a unit standard deviation, $\mathbf{N}(d_1)$, for example, is the area under the symmetric bell-shaped curve from $z = -\infty$ to $z = d_1$; that is,

$$\mathbf{N}(d_1) = \int_{z=-\infty}^{d_1} f(z)dz. \tag{5}$$

Notice that $\mathbf{N}(-\infty) = 0$, $\mathbf{N}(0) = 0.5$, and $\mathbf{N}(\infty) = 1$.

As revealed in equations (2)-(4), C depends explicitly on S, X, T, r, and σ. It can be proven that $\partial C/\partial X$ is negative and all of $\partial C/\partial S$, $\partial C/\partial T$, $\partial C/\partial r$, and $\partial C/\partial \sigma$ are positive. Given the signs of these first partial derivatives, the impact of each of the underlying parameters on the option price can be explained.[8]

A crucial economic insight in the derivation of the Black-Scholes formula is the formation of hedged portfolios instantaneously. Such portfolios, which are based on the call option and its underlying stock, are risk-free. The Black-Scholes formula is the solution of the corresponding partial differential equation.

However, without knowing the derivation or at least the idea underlying the derivation, many business students would simply view the Black-Scholes formula as a great recipe. How the recipe was generated and why it worked so well would remain a mystery to them. As a result, they would not have the foundation to understand the pricing of any derivative securities. By using an algebraic approach, the next two sections of this paper are intended to dispel the perceived mystery of the Black-Scholes formula, thus building a better foundation for such students.

3 The Binomial Option Pricing Model

As shown below, the binomial option pricing model in a single-period setting — which is like a single toss of a biased coin in the coin-toss analogy — is able to reveal the crucial idea underlying the Black-Scholes derivation. To see this, let us consider an option that can be exercised at the end of the period with an exercise price X. Suppose that the underlying stock has a beginning-of-period price S and that its end-of-period price will be either uS or dS. Here, u and d, with $u > 1 > d > 0$, are given multiplicative factors, capturing the up-down movements of S with probabilities q and $1 - q$, respectively.

Suppose also that the beginning-of-period option price is c. Let c_u and c_d be the two potential end-of-period option prices, corresponding to the up-down price movements of the underlying stock.

[8] Intuitively, the lower the exercise price of a call option or the higher the underlying stock price, the greater is the potential gain from exercising the option. The higher the risk-free interest rate or the longer the time to expiry, the lower is the present value of the exercise price. Further, the greater the volatility of the underlying stock, the more upside potential is available from exercising the option while the maximum downside loss is limited. They all have positive impacts on the option price. See, for example, Hull (2002, Chapter 14) for more detail.

As option prices can never be negative, we have $c_u = \max(0, uS - X)$ and $c_d = \max(0, dS - X)$. At the beginning of the period, an investor buys one unit of the stock and writes (sells), as a hedge against any possible losses, m units of the call option that the stock underlies. The net investment is $S - mc$. A risk-free hedged portfolio requires $uS - mc_u = dS - mc_d$, from which the hedge ratio

$$m = \frac{(u-d)S}{c_u - c_d} \tag{6}$$

can be deduced. Notice that the use of a risk-free hedged portfolio also underlies the derivation of the Black-Scholes model.

Letting r_f be the one-period risk-free interest rate, we also have

$$(1 + r_f)(S - mc) = \ uS - mc_u. \tag{7}$$

Combining equations (6) and (7) to eliminate m leads to

$$\begin{aligned} c &= \frac{pc_u + (1-p)c_d}{1 + r_f} \\ &= \frac{[p \max(0, uS - X) + (1-p)\max(0, dS - X)]}{1 + r_f}, \end{aligned} \tag{8}$$

where

$$p = \frac{1 + r_f - d}{u - d}. \tag{9}$$

An interesting feature of equation (8) is the absence of the probabilities q and $1 - q$ of the up-down price movements.

Further, the beginning-of-period option price turns out to be the present value of the expected end-of-period option values, with p and $1 - p$ taking the roles of the probabilities for c_u and c_d, respectively. As $d < 1 < u$, the condition for $0 < p < 1$ is $1 + r_f < u$. The condition, if violated, would provide arbitrage opportunities to exploit the inferior risk-return trade-off in the underlying stock, given the availability of a higher risk-free return.

With the condition for $0 < p < 1$ satisfied, p and $1 - p$ are commonly called risk-neutral probabilities. The term originates from a special case where the investor involved is risk neutral. In the above analytical setting, a risk-neutral investor is indifferent between investing risk-free a dollar amount equal to S to achieve $(1 + r_f)S$ at the end of the period and investing in the stock instead for an expected end-of-period value of $quS + (1 - q)dS$. Equating the two dollar amounts leads to $q = p$; that is, the expression of q in terms of r_f, u, and d is the same as that in equation (9). This is why p can be viewed as a risk-neutral probability.

However, although p has the properties of a probability in a mathematical sense, it is not an actual probability of occurrence. This distinction is crucial, as the beginning-of-period option price c that equation (8) provides does not require risk neutrality of investors. In fact, p is a compounded price of a pure security under the state preference framework, with $1+r_f$ being the compounding factor. To avoid digressions, the concept of pure securities and its connection to risk-neutral probabilities is covered in the Appendix.[9]

3.1 An Extension to a Multi-period Setting

Equation (8) can be written equivalently as

$$c = \left[\sum\nolimits_{n=0}^{1} p^n (1-p)^{1-n} \max\left(0, u^n d^{1-n} S - X\right)\right] \frac{1}{1+r_f}. \tag{10}$$

Although the use of a summation sign here may appear unnecessary at first glance, it actually facilitates an extension of the same model to a multi-period setting. In a two-period setting, with up-down stock price movements captured by the multiplicative factors u and d each period, the eventual stock price is among u^2S, udS, and d^2S. The corresponding risk-neutral probabilities are p^2, $2p(1-p)$, and $(1-p)^2$, respectively. Likewise, in a three-period setting, the eventual stock price is among u^3S, u^2dS, ud^2S, and d^3S, corresponding to probabilities p^3, $3p^2(1-p)$, $3p(1-p)^2$, and $(1-p)^3$, respectively. As the coefficients for the probability terms are from Pascal's triangle, we can easily extend the same idea to a general N-period setting. Students who have learned the statistical materials pertaining to coin-toss experiments should be able to understand the concepts involved.

In an N-period setting, we can extend equation (10) to

$$c = \left[\sum\nolimits_{n=0}^{N} \binom{N}{n} p^n (1-p)^{N-n} \max\left(0, u^n d^{N-n} S - X\right)\right] \frac{1}{(1+r_f)^N}, \tag{11}$$

where

$$\binom{N}{n} = \frac{N!}{(N-n)!n!}, \tag{12}$$

with $n! = n(n-1)(n-2)\cdots(2)(1)$, for $n \geq 1$, and $0! = 1$. The positive terms among cases of $\max\left(0, u^n d^{N-n} S - X\right)$ all have $u^n d^{N-n} S > X$. We can establish the lowest n that ensures positive values of $\max\left(0, u^n d^{N-n} S - X\right)$ by solving this inequality or, equivalently, $(u/d)^n > X/(d^N S)$ for n. The result is the lowest integer n, in the range of 0 to N, satisfying the condition of

$$n > \frac{\ln[X/(d^N S)]}{\ln(u/d)}. \tag{13}$$

Let us label this specific integer n as a.

[9] See, for example, Copeland, Weston, and Shastri (2005, Chapter 4) for a description of the state preference framework.

Then, equation (11) reduces to

$$c = \left[\sum\nolimits_{n=a}^{N} \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n} \left(u^n d^{N-n} S - X\right)\right] \frac{1}{(1+r_f)^N}. \tag{14}$$

Letting

$$p' = \frac{pu}{1+r_f}, \tag{15}$$

we can write equation (14) as

$$c = S\ B(n \geq a|N, p') - \frac{X}{(1+r_f)^N}\ B(n \geq a|N, p), \tag{16}$$

where

$$B(n \geq a|N, p) = \sum\nolimits_{n=a}^{N} \frac{N!}{(N-n)!n!} p^n (1-p)^{N-n}. \tag{17}$$

The functions $B(n \geq a|N, p')$ and $B(n \geq a|N, p)$, with each being a complementary binomial distribution function, differ only in the probabilities involved.[10]

Notice that, under the condition of $1 + r_f < u$, which already ensures that $0 < p < 1$, we also have $0 < p' < 1$. To see this, let us write

$$p' = \frac{1+r_f-d}{u-d} \cdot \frac{u}{1+r_f} = \frac{(1+r_f)u - ud}{(1+r_f)u - (1+r_f)d}. \tag{18}$$

As $p' > p > 0$, the condition of $1 + r_f < u$ ensures that $0 < p' < 1$.

4 A Connection between Binomial and Black-Scholes Formulas

Analytically, the Black-Scholes model is a continuous-time model, and the binomial model can be viewed as its discrete-time version. This analytical difference, however, does not affect the crucial idea that underlies the derivation of each model; risk-free hedging is equally important in both cases. Being a continuous-time case, the Black-Scholes model relies on continuous risk-free hedging by revising the required hedge ratio instantaneously. The binomial model, in contrast, allows changes in the hedge ratio from one period to the next, thus allowing students to follow the analytical process involved.

A comparison of equations (2) and (16) reveals the similarities between the two pricing formulas. Let us first compare the present value factors there. In the Black-Scholes case, the option expires in T years; in the binomial case, it expires in N periods instead. With r being the continuously compounded risk-free interest rate each year, the present value factor for the exercise price is e^{-rT}

[10] The two functions are complementary distribution functions of their corresponding cumulative distribution functions, which are $B(n < a|N, p')$ and $B(n < a|N, p)$, respectively. As $B(n < a|N, p') + B(n \geq a|N, p') = 1$ and $B(n < a|N, p) + B(n \geq a|N, p) = 1$, each pair of such functions fully covers all potential outcomes as characterized by the corresponding distribution.

in the Black-Scholes case. In the binomial version, as the risk-free interest rate each period is r_f, the present value factor for the exercise price is $(1+r_f)^{-N}$ instead. Thus, r and r_f are related via[11]

$$e^{rT} = (1+r_f)^N. \tag{19}$$

For the two pricing formulas to match, we require both the equality of $B(n \geq a|N, p')$ and $\mathbf{N}(d_1)$ and the equality of $B(n \geq a|N, p)$ and $\mathbf{N}(d_2)$. The binomial distribution, which is the distribution of the number of successes from repeated Bernoulli trials with given probabilities of success and failure for each attempt, can be viewed as the sum of the individual trial results.[12] When the number of Bernoulli trials increases, we can rely on the central limit theorem in statistics to get a better approximation of each binomial distribution function in equation (16) with a normal distribution function.

However, no matter how close to normality that each approximation can achieve, there is no assurance that it approximates the corresponding cumulative normal distribution function in equation (2). To achieve a good numerical match requires that the underlying parameters in the two models be connected in a specific way. We now follow Cox-Ross-Rubinstein to find the required connection, with more algebraic details provided for the pedagogic purpose of this paper.

Let S^* be the random price of the underlying stock at the end of the N periods. For n upward price movements over the N periods, we have, from $S^* = u^n d^{N-n} S$,

$$\ln\left(\frac{S^*}{S}\right) = n\ln(u) + (N-n)\ln d = n\ln\left(\frac{u}{d}\right) + N\ln d. \tag{20}$$

Here, $\ln(S^*/S)$ is the natural logarithm of one-plus-return for holding the stock over the N periods; it is equivalent to a continuously compounded return over the N periods.

For a binomial distribution with the probability of each upward movement being q, the expected value and the variance of n are $E(n) = Nq$ and $Var(n) = Nq(1-q)$, respectively.[13] Accordingly,

[11] Equation (19) can also be established by recognizing that each of the N periods is equivalent to the proportion T/N of a year. A dollar that is invested risk-free at the beginning of each of the N periods will become $1+r_f$ dollars at the end of the same period. Over the proportion T/N of a year, the use of a continuously compounded annual risk-free interest rate r instead will allow the same dollar to become $(e^r)^{T/N}$ dollars. Given the equivalence of the two risk-free investment outcomes, equation (19) follows directly.

[12] A Bernoulli trial is a statistical term in honour of Jacob Bernoulli (1654-1705). It pertains to an experiment that has two random outcomes with given probabilities, commonly known as success or failure. Repeated Bernoulli trials are independent repetitions of such an experiment.

[13] Given $\sum_{n=0}^{N} \binom{N}{n} q^n(1-q)^{N-n} = 1$, we can establish, after some simple algebraic steps, that $E(n) = \sum_{n=0}^{N} n \binom{N}{n} q^n(1-q)^{N-n} = Nq$. To show that $Var(n) = Nq(1-q)$, a simple way is via $Var(n) = E(n^2) - [E(n)]^2$, where $E(n^2) = \sum_{n=0}^{N} n^2 \binom{N}{n} q^n(1-q)^{N-n}$.

we have

$$E\left[\ln\left(\frac{S^*}{S}\right)\right] = Nq\ln\left(\frac{u}{d}\right) + N\ln d = [q\ln u + (1-q)\ln d]\,N \tag{21}$$

and

$$Var\left[\ln\left(\frac{S^*}{S}\right)\right] = q(1-q)\left[\ln\left(\frac{u}{d}\right)\right]^2 N. \tag{22}$$

We can capture the above expressions more succinctly by defining

$$\widehat{\mu} = q\ln u + (1-q)\ln d \tag{23}$$

and

$$\widehat{\sigma}^2 = q(1-q)\left[\ln\left(\frac{u}{d}\right)\right]^2. \tag{24}$$

As a total of N periods represents the time interval of T years, $\widehat{\mu}N$ and $\widehat{\sigma}^2 N$ when stated as annual figures are $\widehat{\mu}N/T$ and $\widehat{\sigma}^2 N/T$, respectively. Suppose that μ and σ^2 are the stock's annual expected return and variance of returns, respectively, which can be estimated with empirical data. Suppose also that this σ is the same as the parameter σ in the Black-Scholes model. The convergence of the two models requires that $\widehat{\mu}N/T$ and $\widehat{\sigma}^2 N/T$ converge to μ and σ^2, respectively, as N approaches infinity.

Before proceeding to look for some specific expressions of u, d, and q in terms of some or all of μ, σ, N, and T for the convergence to occur, let us clarify what N approaches infinity entails in the context of the analytical setting here. The idea is to divide the entire time interval, between the time when we price the option and the time when the option expires, into an increasingly larger number of shorter periods. In doing so, we allow the stock price to have increasingly more, but individually smaller, up-down movements over the entire time interval.

Analytically, the definitions of $\widehat{\mu}$ and $\widehat{\sigma}^2$ give us two equations to use. In addition, we have two conditions for convergence, which is, $\widehat{\mu}N/T \to \mu$ and $\widehat{\sigma}^2 N/T \to \sigma^2$ as $N \to \infty$. However, we have seven parameters — which include u, d, q, N, μ, σ^2, and T — to connect. Thus, the solution will not be unique.

For simplicity, it is reasonable to impose $u = 1/d$, as this condition ensures that the original stock price be retained after an equal number of upward and downward price movements in any up-down sequences. For algebraic convenience that will soon be clear, let us write $q = 1/2 + b$, where the parameter b has yet to be determined. Accordingly, we can write

$$\widehat{\mu} = (2q-1)\ln u = 2b\ln u \tag{25}$$

and

$$\widehat{\sigma}^2 = 4q(1-q)(\ln u)^2 = (1-4b^2)(\ln u)^2 = (\ln u)^2 - \widehat{\mu}^2. \tag{26}$$

If we express u as an exponential function of the form $u = e^h$, equations (25) and (26) reduce to $\widehat{\mu} = 2bh$ and $\widehat{\sigma}^2 = h^2(1-4b^2)$, respectively. There are various combinations of b and h that can satisfy the condition of $\mu = \widehat{\mu}N/T = 2bhN/T$. One of such combinations is revealed by $\widehat{\sigma}^2 N/T = (h^2N/T)(1-4b^2)$. Specifically, by equating h^2N/T and σ^2 for all N, we have $h = \sigma\sqrt{T/N}$. With $b = \mu\sqrt{T}/(2\sigma\sqrt{N})$, it follows that $4b^2 = \mu^2T/(\sigma^2N)$ decreases as N increases; that is, the condition of

$$\lim_{n\to\infty}\left(\widehat{\sigma}^2\frac{N}{T}\right) = \sigma^2 \tag{27}$$

is assured.

Finally, we have

$$u = \frac{1}{d} = e^{\sigma\sqrt{T/N}} \tag{28}$$

and

$$q = \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\frac{T}{N}} \tag{29}$$

to connect the binomial model and the Black-Scholes model. Of interest now is whether the specifications of u, d, and q in equations (28) and (29) always lead to $0 < p < 1$ and $0 < p' < 1$, as required for equation (16) to work. As established earlier, the feasibility of p and p' requires the condition of $1 + r_f < u$ to be satisfied.

Given equation (28), the condition becomes

$$e^{\sigma\sqrt{T/N}} > 1 + r_f. \tag{30}$$

In view of equation (19), this inequality is equivalent to

$$e^{\sigma\sqrt{T/N}} > e^{rT/N} \tag{31}$$

or, simply,

$$\sigma > r\sqrt{\frac{T}{N}}. \tag{32}$$

For any given σ, r, and T, the greater the number of binomial periods, the easier is for inequality (32) to hold. Then, provided that a large number of binomial periods is involved in the implementation of equation (16), the potential violation of inequality (32) is not a concern.

To conclude this section, we acknowledge that the above is as far as we can go by using only algebraic tools.[14] No attempt has been made here to seek formal convergence of the two models — that is, $B(n \geq a|N,p') \to \mathbf{N}(d_1)$ and $B(n \geq a|N,p) \to \mathbf{N}(d_2)$ as $N \to \infty$ — by converting the

[14] See, for example, Chance (2008) for more sophisticated characterizations of u, d, and q, as well as the corresponding references to the binomial models involved.

discrete-time binomial model into a continuous-time setting. To do so will require advanced statistical knowledge.[15] Therefore, instead of relying on a formal approach to make the final connection between the two models, we illustrate their numerical convergence via some Excel examples in the next section.

5 Interactive Excel Illustrations

We start our Excel illustrations in Figure 1 to compare the option prices from the two models. As r_f, u, d, p, and p' in the binomial model with N binomial periods can be expressed directly in terms of r, T, and σ from the Black-Scholes model, we place the latter set of input parameters, along with X and N, in B2:B6. Predetermined values of S, from \$5 to \$100 in increments of \$5, are placed in A15:A34. The choice of the combination of r, T, σ, X, and N is via five scroll bars, which are linked individually to E2:E6.[16] The values in B2:B6 are connected to those in E2:E6 via some simple cell formulas. For example, with the scroll bar for σ set to cover the range of 10 to 100, the formula in B4 is =E4/200, so that the value 70 in E4, as provided by the scroll bar, corresponds to $\sigma = 35\%$ in B4.

The range of values covered by each scroll bar can easily be revised by changing the corresponding settings via its format control. The input parameters as required for the computations, along with those cells with assigned zero values in row 14 of the worksheet for graphical convenience, are shaded. All representative cell formulas are displayed in the worksheet. The various computed values, pertaining to each S as shown in Figure 1, are based on $r = 5\%$, $T = 0.75$ years, $\sigma = 35\%$, $X = \$40$, and $N = 60$ periods.

The computational results of u, d, p, and p' are stored in B8:B11. These parameters, in conjunction with other required parameters, allow a and c to be computed, as displayed in B15:C34. Notice that, for the computation of c pertaining to the binomial model, we have relied on differences in cumulative probabilities,

$$B(n \geq a|N, p') = B(n \leq N|N, p') - B(n < a|N, p') \tag{33}$$

and

$$B(n \geq a|N, p) = B(n \leq N|N, p) - B(n < a|N, p), \tag{34}$$

[15] Leisen and Reimer (1996) have considered the convergence properties of various binomial models including the Cox-Ross-Rubinstein version. Such properties pertain to both the behaviour and the speed of conversion. That study has also revised the binomial formulation to achieve improvements in convergence to the Black-Scholes model.

[16] The use of scroll bars, which is more convenient than manual data entry, is intended to facilitate the interactivity of the Excel illustrations in this paper. Further, students who have limited prior experience in computations involving option pricing models can benefit from some guidance as to what parameter values are reasonable. The range of values that each scroll bar initially provides can serve such a purpose.

	A	B	C	D	E	F	G	H	I
1									
2	r	5.00%			50				
3	T	0.75			75				
4	sigma	35.00%			70				
5	X	40			40				
6	N	60			60				
7									
8	u	1.03991							
9	d	0.96162							
10	p	0.4982							
11	p'	0.51776							
12									
13	S	a	c(Binom)	d1	d2	c(BS)	c(max)	c(min)	c(min0)
14	0		0			0	0	0	0
15	5	56	3.3E-14	-6.58511	-6.88821	4.8E-12	5	0	0
16	10	47	2.6E-06	-4.29831	-4.60142	5.2E-06	10	0	0
17	15	42	0.00158	-2.96062	-3.26373	0.00184	15	0	0
18	20	38	0.04277	-2.01152	-2.31463	0.04523	20	0	0
19	25	36	0.31162	-1.27534	-1.57845	0.32237	25	0	0
20	30	33	1.18097	-0.67383	-0.97694	1.17619	30	0	0
21	35	31	2.89295	-0.16527	-0.46838	2.88328	35	0	0
22	40	30	5.48197	0.27527	-0.02784	5.50177	40	1.47222	0
23	45	28	8.92414	0.66386	0.36075	8.90665	45	6.47222	5
24	50	27	12.8983	1.01146	0.70835	12.8995	50	11.4722	10
25	55	25	17.2812	1.3259	1.02279	17.2908	55	16.4722	15
26	60	24	21.9311	1.61296	1.30985	21.9345	60	21.4722	20
27	65	23	26.7274	1.87703	1.57392	26.7306	65	26.4722	25
28	70	22	31.6116	2.12153	1.81842	31.6157	70	31.4722	30
29	75	21	36.5467	2.34914	2.04603	36.5517	75	36.4722	35
30	80	21	41.514	2.56206	2.25896	41.5162	80	41.4722	40
31	85	20	46.4957	2.76207	2.45897	46.4965	85	46.4722	45
32	90	19	51.4848	2.95065	2.64754	51.4857	90	51.4722	50
33	95	18	56.4786	3.12902	2.82591	56.4797	95	56.4722	55
34	100	18	61.476	3.29825	2.99514	61.4764	100	61.4722	60
35									
36	Cell Formulas								
37		B2 =E2/1000							
38		B3 =E3/100							
39		B4 =E4/200							
40		B5 =E5							
41		B6 =E6							

Figure 1 An Excel Worksheet Illustrating, Numerically and Graphically, the Option Prices Based on the Binomial Model and the Black-Scholes Model.

	J	K	L	M	N	O	P	Q	R
1	Cell Formulas								

```
B8       =EXP(B$4*SQRT(B$3/B$6))
B9       =1/B8
B10      =(EXP(B$2*B$3/B$6)-B9)/(B8-B9)
B11      =B8*B10*EXP(-B$2*B$3/B$6)

A15      =A14+5
B15      =MIN(B$6,MAX(0,INT(LN(B$5/A15/(B$9^B$6))/LN(B$8/B$9))))
C15      =A15*(BINOMDIST(B$6,B$6,B$11,TRUE)-BINOMDIST(B15,B$6,B$11,TRUE))
             -B$5*EXP(-B$2*B$3)*(BINOMDIST(B$6,B$6,B$10,TRUE)
             -BINOMDIST(B15,B$6,B$10,TRUE))
D15      =(LN(A15/B$5)+B$2*B$3)/B$4/SQRT(B$3)+B$4*SQRT(B$3)/2
E15      =D15-B$4*SQRT(B$3)
F15      =A15*NORMSDIST(D15)-B$5*EXP(-B$2*B$3)*NORMSDIST(E15)
G15      =A15
H15      =MAX(0,A15-B$5*EXP(-B$2*B$3))
I15      =MAX(0,A15-B$5)

A15:I15   copied to A15:I34
```

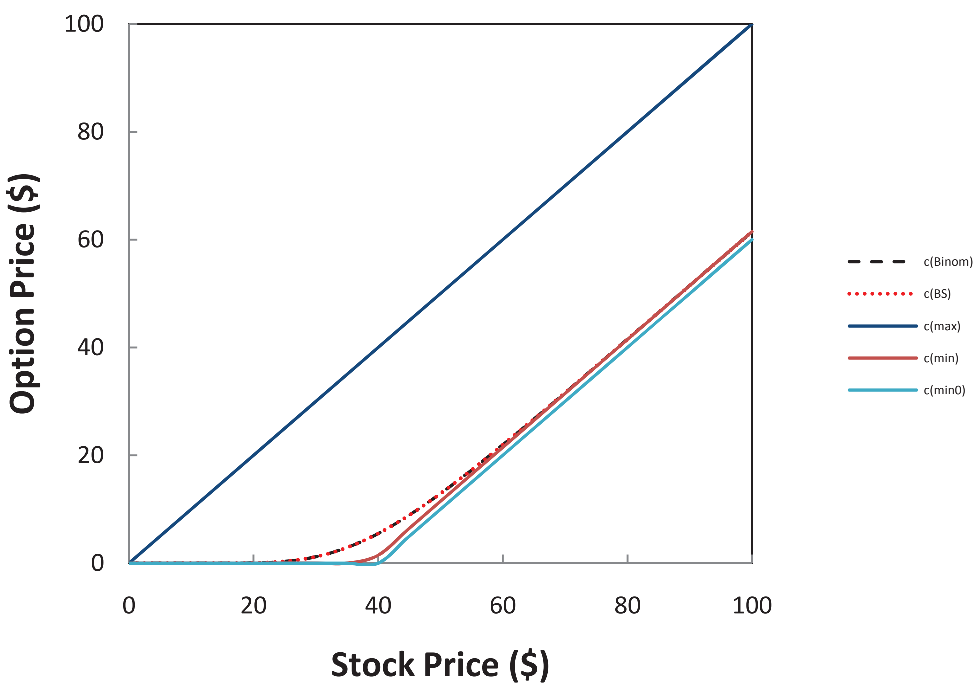

Figure 1 An Excel Worksheet Illustrating, Numerically and Graphically, the Option Prices Based on the Binomial Model and the Black-Scholes Model (Continued).

in order to use Excel's BINOMDIST. The function BINOMDIST has four arguments, namely, the number of successful trials, the total number of trials, the probability of a successful trial, and a true/false indicator of whether a cumulative distribution is intended. Thus, for example, the difference between BINOMDIST(B$6,B$6,B$11,TRUE) and BINOMDIST(B15,B$6,B$11,TRUE), which is part of the formula for C15, allows $B(n \geq a|N, p')$ for the specific case of a in B15 to be computed.

The computational results of d_1, d_2, and C pertaining to the Black-Scholes model are displayed in D15:F34. For the computations of C, the Excel function NORMSDIST is used directly for all cases of $\mathbf{N}(d_1)$ and $\mathbf{N}(d_2)$, by indicating the cells where the corresponding d_1 and d_2 are stored. The upper and lower bounds of feasible option values as described in Subsection 2.1 are displayed in G15:H34, with the lower bound for cases where $T = 0$ also shown in I15:I34.

The results in A14:A34, C14:C34, and F14:I34 are captured graphically in Figure 1 as well. The graphical part of Figure 1 — which is drawn by using Excel's XY (Scatter) Chart — is just like typical textbook illustrations of the Black-Scholes model. The only difference here is that the results from the binomial model based on the same set of input parameters are also included. The computed values of option prices for the two models are similar for each value of S ranging from \$5 to \$100 in increments of \$5. The greatest difference occurs at $S = X = \$40$. With $c = \$5.4820$ and $C = \$5.5018$, the binomial model understates the Black-Scholes result by \$0.0198. Although such a difference is too small to be noticeable graphically, this nearly 2-cent difference in price is not a trivial amount.

To improve the match of the option prices from the two models for the same set of input parameters, many more binomial periods are required. For example, if we let $N = 120$ (by setting the scroll bar that links to E6 at 120), the greatest difference between the two computed option prices still occurs at $S = X = \$40$. With $c = \$5.4919$ and $C = \$5.5018$, the Black-Scholes result is understated by only \$0.0099.

If we attempt only a few binomial periods instead, the mismatch between the option prices from the two models will be easily noticeable. In fact, for the same set of input parameters, but with $N < 10$, the computed c according to equation (16) will fall below the line $C = S - Xe^{-rT}$ if the option is deep in the money (that is, if S is much greater than X). As c cannot be less than $S - Xe^{-rT}$ regardless of S, such an option price is unacceptable. In view of this undesirable feature, considered next is how the mismatch of the two computed option prices varies with N.

5.1 The Number of Binomial Periods and the Mismatch in Option Prices

The Excel worksheet for Figure 2 is primarily the result of a re-arrangement of various cells in the Excel worksheet for Figure 1, intended for graphical convenience. The computations leading to c and C remain the same. Instead of showing how each of c and C varies with S, for a common set of input parameters and for a given N, we now show how the difference between c and C varies with N, for the same set of input parameters. For the illustration in Figure 2, we use $r = 3.5\%$, $T = 0.4$ years, $\sigma = 30\%$, $X = \$60$, and $S = \$55$, with N ranging from 1 to 160 periods. Just like Figure 1, all input parameters (in B2:B6) and values of N (in A13:A172) are shaded.

As d_1, d_2, and C, which pertain to the Black-Scholes model, remain the same regardless of N, they are shown in B8:B10. The results of u, d, p, p', a, $B(n \geq a|N, p')$, $B(n \geq a|N, p)$, and c corresponding to $N = 1$ to 160 are placed in B13:I172. To accommodate the width reduction of columns G and H in the worksheet, the headings for $B(n \geq a|N, p')$ and $B(n \geq a|N, p)$ in G12:H12 are abbreviated as $B(p')$ and $B(p)$, respectively. The price differences, as captured by $c - C$, are stored in J13:J172. For graphical convenience, $\mathbf{N}(d_1)$ and $\mathbf{N}(d_2)$ are shown in K13:L172 as well. The part of the worksheet below row 36, which contains all computational results for $N > 24$, is omitted from Figure 2.

The graphs in Figure 2 are also drawn by using Excel's XY (Scatter) Chart. The graph of price difference, $c - C$, versus N shows a high level of fluctuations for $N < 10$. Such fluctuations for greater values of N are still noticeable graphically. For the magnitudes of the fluctuations, as measured by $|c - C|$, to be consistently within \$0.010 for $N, N+1, N+2, \ldots$, the minimum N pertaining to the set of input parameters in Figure 2 is 84.

The graphs of binomial distributions, $B(n \geq a|N, p')$ and $B(n \geq a|N, p)$, versus N show that they oscillate in a synchronized pattern, over the corresponding cumulative normal distributions, $\mathbf{N}(d_1)$ and $\mathbf{N}(d_2)$, in the entire range of N considered. Although the noise in each graph is gradually subsided as N increases, the synchronized oscillation pattern remains visible. Such a pattern seems to have caused the computed values of c to deviate from C in opposite signs, as N alternates between adjacent even and odd numbers. Therefore, it seems that the average of the computed values of c based on N and $N-1$ binomial periods approximates C better than does the computed value of c for the same N without taking an average.

Figure 3 is intended for illustrating the issue. For the illustration in Figure 3, we use $r = 5\%$, $T = 0.25$ years, $\sigma = 30\%$, $X = \$50$, and $S = \$45$, with N ranging from 1 to 500 periods. The worksheet for Figure 3 is a variant of that for Figure 2. In order to leave adequate space for displaying the cell formulas beyond those in Figure 2, the five scroll bars, columns J, K, and L, and

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2	r	3.5%			35							
3	T	0.4			40							
4	sigma	30.0%			60							
5	X	60			60							
6	S	55			55							
7												
8	d1	-0.290										
9	d2	-0.480										
10	c(BS)	2.546										
11												
12	N	u	d	p	p'	a	B(p')	B(p)	c(Bin)	c diff	N(d1)	N(d2)
13	1	1.209	0.827	0.490	0.584	0	0.584	0.490	3.134	0.588	0.386	0.316
14	2	1.144	0.874	0.493	0.559	1	0.313	0.243	2.854	0.308	0.386	0.316
15	3	1.116	0.896	0.494	0.549	1	0.573	0.491	2.448	-0.098	0.386	0.316
16	4	1.100	0.909	0.495	0.542	2	0.378	0.305	2.771	0.225	0.386	0.316
17	5	1.089	0.919	0.495	0.538	3	0.238	0.182	2.346	-0.199	0.386	0.316
18	6	1.081	0.925	0.496	0.534	3	0.410	0.336	2.698	0.152	0.386	0.316
19	7	1.074	0.931	0.496	0.532	4	0.282	0.220	2.490	-0.056	0.386	0.316
20	8	1.069	0.935	0.496	0.530	4	0.430	0.355	2.643	0.097	0.386	0.316
21	9	1.065	0.939	0.496	0.528	5	0.312	0.247	2.553	0.007	0.386	0.316
22	10	1.062	0.942	0.497	0.527	5	0.444	0.369	2.601	0.055	0.386	0.316
23	11	1.059	0.944	0.497	0.525	6	0.334	0.267	2.581	0.036	0.386	0.316
24	12	1.056	0.947	0.497	0.524	6	0.454	0.379	2.568	0.022	0.386	0.316
25	13	1.054	0.949	0.497	0.523	7	0.352	0.283	2.594	0.048	0.386	0.316
26	14	1.052	0.951	0.497	0.523	7	0.463	0.387	2.540	-0.006	0.386	0.316
27	15	1.050	0.952	0.497	0.522	8	0.366	0.296	2.598	0.052	0.386	0.316
28	16	1.049	0.954	0.497	0.521	8	0.469	0.394	2.517	-0.029	0.386	0.316
29	17	1.047	0.955	0.497	0.520	9	0.377	0.307	2.597	0.051	0.386	0.316
30	18	1.046	0.956	0.498	0.520	9	0.475	0.399	2.497	-0.049	0.386	0.316
31	19	1.044	0.957	0.498	0.519	10	0.387	0.316	2.594	0.048	0.386	0.316
32	20	1.043	0.958	0.498	0.519	11	0.309	0.245	2.500	-0.046	0.386	0.316
33	21	1.042	0.959	0.498	0.518	11	0.396	0.324	2.588	0.043	0.386	0.316
34	22	1.041	0.960	0.498	0.518	12	0.320	0.255	2.520	-0.026	0.386	0.316
35	23	1.040	0.961	0.498	0.518	12	0.403	0.331	2.583	0.037	0.386	0.316
36	24	1.039	0.962	0.498	0.517	13	0.330	0.264	2.535	-0.011	0.386	0.316

Figure 2 An Excel Worksheet Illustrating, Numerically and Graphically, the Differences between Option Prices from the Binomial Model and the Black-Scholes Model, and between the Probability Distributions Involved, for Different Numbers of Binomial Periods.

Cell Formulas

B2	=E2/1000
B3	=E3/100
B4	=E4/200
B5	=E5
B6	=E6
B8	=(LN(B$6/B$5)+B$2*B$3)/B$4/SQRT(B$3)+B$4*SQRT(B$3)/2
B9	=B8-B$4*SQRT(B$3)
B10	=B$6*NORMSDIST(B8)-B$5*EXP(-B$2*B$3)*NORMSDIST(B9)
B13	=EXP(B$4*SQRT(B$3/A13))
C13	=1/B13

B13:C13 copied to B13:C172

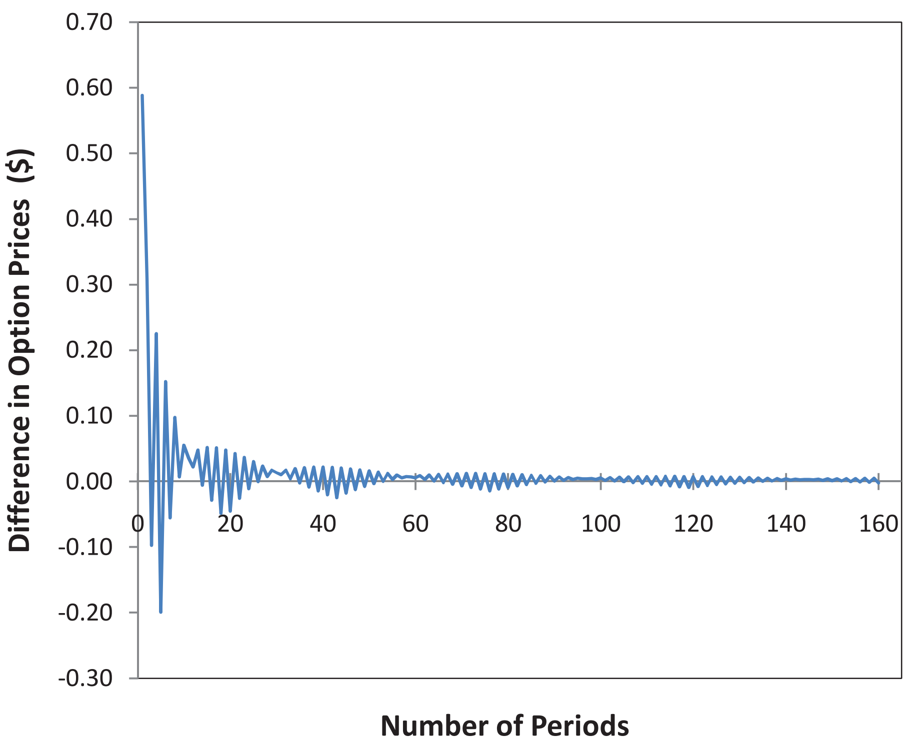

Figure 2 An Excel Worksheet Illustrating, Numerically and Graphically, the Differences between Option Prices from the Binomial Model and the Black-Scholes Model, and between the Probability Distributions Involved, for Different Numbers of Binomial Periods (Continued).

	M	N	O	P	Q	R	S	T	U	V
37	Cell Formulas									
38		D13	=(EXP(B$2*B$3/A13)-C13)/(B13-C13)							
39		E13	=B13*D13*EXP(-B$2*B$3/A13)							
40		F13	=MIN(A13,MAX(0,INT(LN(B$5/B$6/(C13^A13))/LN(B13/C13))))							
41		G13	=BINOMDIST(A13,A13,E13,TRUE)-BINOMDIST(F13,A13,E13,TRUE)							
42		H13	=BINOMDIST(A13,A13,D13,TRUE)-BINOMDIST(F13,A13,D13,TRUE)							
43		I13	=B$6*G13-B$5*EXP(-B$2*B$3)*H13							
44		J13	=I13-B$10							
45		K13	=NORMSDIST(B$8)			D13:J13	copied to D13:J172			
46		L13	=NORMSDIST(B$9)			K13:L13	copied to K13:L172			
47		A14	=A13+1			A14	copied to A14:A172			
48										
49										
50										
51										
52										
53										
54										
55										
56										
57										
58										
59										
60										
61										
62										
63										
64										
65										
66										
67										
68										
69										
70										
71										
72										

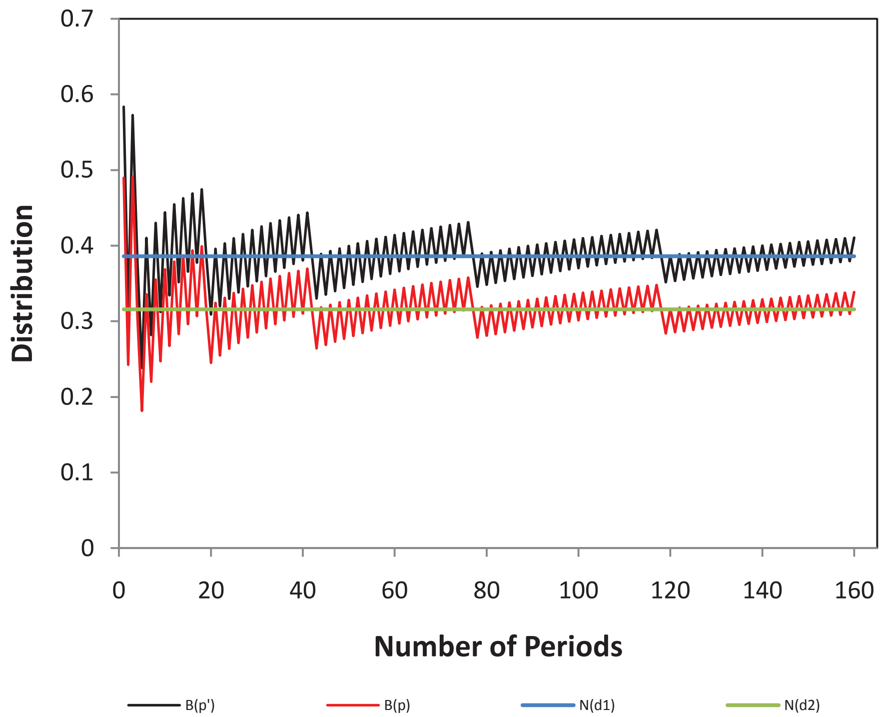

Figure 2 An Excel Worksheet Illustrating, Numerically and Graphically, the Differences between Option Prices from the Binomial Model and the Black-Scholes Model, and between the Probability Distributions Involved, for Different Numbers of Binomial Periods (Continued).

both XY (Scatter) Charts there, along with the original descriptions of the cell formulas have been deleted. Two additional rows have been inserted after row 10. The headings for $B(n \geq a|N, p')$ and $B(n \geq a|N, p)$ in G14:H14 are also abbreviated as $B(p')$ and $B(p)$, respectively. Not displayed in Figure 3 is the part of the worksheet below row 72, which contains all numerical results for $N > 58$. The absolute difference, $|c - C|$ (under the heading of "|diff1|"), for all cases of N, are placed in J15:J514, with K15:K514 (under the heading of "<filter") showing only those cases under \$0.010. For example, with $|c-C|$ in J15 being $0.017 > 0.010$, K15 is left black in the display; with $|c - C|$ in J26 being $0.003 < 0.010$, the number 0.003 is retained in K26.

Each cell in L15:L514 (under the heading of "N1") is intended to show the corresponding N for the cell in K15:K514 from the same row where, starting from it, none of the remaining cells in column K until K514 are blank. For example, although the number 0.003 is shown in K26, many cells in K26:K514 have no displayed numbers. Thus, L26 is left blank. In contrast, the number 0.007 is shown in K70 and none of the cells in K70:K514 are blank, L70 shows the number in A70, which is $N = 56$. For $N \geq 56$, the departure of the corresponding c from C is always under \$0.010. The minimum among the cells in L15:L514, which is 52, as displayed in B11, is the lowest N to ensure the imposed accuracy of c for the current set of input parameters.

The same idea for J15:L514 is repeated for M16:O514 (with analogous headings, "|diff2|, <filter," and "N2"). The difference is that, instead of using the c for N periods in establishing $|c - C|$, we now use the average of the c for N periods and the c for $N - 1$ periods. The minimum among the cells in O16:O514 is displayed in B12. For the current set of input parameters, the number is 15, which is much lower. For the sake of clarity, the two blocks of cells, as well as B11:B12, are shaded in different colours.

5.2 Numerical Convergence of the Two Models

Although it has been illustrated above that using the average of two adjacent values of c for N and $N - 1$ periods allows numerical convergence of the two models to be achieved with far fewer binomial periods, whether this is generally true has yet to be confirmed or refuted. For this task, we follow the general idea of Chance (2008) as intended for comparing different binomial models. As applied to the current setting, where there is only the Cox-Ross-Rubinstein model to consider, we repeat the same computations as illustrated in Figure 3, for $X = \$50$ and all combinations of the following input parameters: $r = 1\%$, 5%, and 10%; $T = 0.25$, 1, and 4 years; $\sigma = 10\%$, 30%, and 50%; and $S = \$45$, \$50, and \$55.

With three different values for each of the four input parameters, there are $3^4 = 81$ combinations

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1				Cell		J15 =ABS(I15-B$10)									
2	r	5%		Formulas		K15 =IF(J15>0.01,"",J15)									
3	T	0.25				L15 =IF(K15="","",IF(COUNT(K15:K$514)=ROW(K$514)									
4	sigma	30%				-ROW(K14),A15,""))									
5	X	50								J15:L15	copied to J15:L514				
6	S	45				M16 =ABS((I15+I16)/2-B$10)									
7						N16 =IF(M16>0.01,"",M16)									
8	d1	-0.54				O16 =IF(N16="","",IF(COUNT(N16:N$514)=ROW(N$514)									
9	d2	-0.69				-ROW(N15),A16,""))									
10	c(BS)	1.154								M16:O16	copied to M16:O514				
11	minN1	52				B11 =MIN(L15:L514)									
12	minN2	15				B12 =MIN(O16:O514)									
13															
14	N	u	d	p	p'	a	B(p')	B(p)	c(Bin)	|diff1|	<filter	N1	|diff2|	<filter	N2
15	1	1.16	0.86	0.50	0.58	0	0.58	0.50	1.137	0.017					
16	2	1.11	0.90	0.50	0.56	1	0.31	0.25	1.408	0.253			0.118		
17	3	1.09	0.92	0.50	0.55	2	0.16	0.13	1.046	0.108			0.073		
18	4	1.08	0.93	0.50	0.54	2	0.37	0.32	1.243	0.088			0.010	0.010	
19	5	1.07	0.94	0.50	0.54	3	0.23	0.19	1.192	0.038			0.063		
20	6	1.06	0.94	0.50	0.53	3	0.41	0.35	1.140	0.014			0.012		
21	7	1.06	0.94	0.50	0.53	4	0.28	0.23	1.215	0.061			0.023		
22	8	1.05	0.95	0.50	0.53	4	0.43	0.37	1.070	0.084			0.012		
23	9	1.05	0.95	0.50	0.53	5	0.31	0.26	1.207	0.052			0.016		
24	10	1.05	0.95	0.50	0.53	6	0.22	0.17	1.125	0.029			0.012		
25	11	1.05	0.96	0.50	0.52	6	0.33	0.28	1.189	0.034			0.003	0.003	
26	12	1.04	0.96	0.50	0.52	7	0.24	0.20	1.157	0.003	0.003		0.019		
27	13	1.04	0.96	0.50	0.52	7	0.35	0.29	1.168	0.014			0.008	0.008	
28	14	1.04	0.96	0.50	0.52	8	0.26	0.21	1.173	0.019			0.016		
29	15	1.04	0.96	0.50	0.52	8	0.36	0.31	1.147	0.007	0.007		0.006	0.006	15
30	16	1.04	0.96	0.50	0.52	9	0.28	0.23	1.179	0.025			0.009	0.009	16
31	17	1.04	0.96	0.50	0.52	9	0.37	0.32	1.127	0.027			0.001	0.001	17
32	18	1.04	0.97	0.50	0.52	10	0.29	0.24	1.180	0.026			0.001	0.001	18
33	19	1.04	0.97	0.50	0.52	11	0.22	0.18	1.124	0.030			0.002	0.002	19
34	20	1.03	0.97	0.50	0.52	11	0.31	0.25	1.177	0.023			0.004	0.004	20
35	21	1.03	0.97	0.50	0.52	12	0.24	0.19	1.141	0.013			0.005	0.005	21
36	22	1.03	0.97	0.50	0.52	12	0.32	0.26	1.172	0.018			0.002	0.002	22

Figure 3 An Excel Worksheet Illustrating the Computations of the Lowest Numbers of Binomial Periods for Numerical Convergence of the Binomial Model and Black-Scholes Model.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
37	23	1.03	0.97	0.50	0.52	13	0.25	0.20	1.152	0.002	0.002		0.008	0.008	23
38	24	1.03	0.97	0.50	0.52	13	0.33	0.27	1.165	0.011			0.005	0.005	24
39	25	1.03	0.97	0.50	0.52	14	0.26	0.21	1.160	0.006	0.006		0.009	0.009	25
40	26	1.03	0.97	0.50	0.52	14	0.33	0.28	1.158	0.004	0.004		0.005	0.005	26
41	27	1.03	0.97	0.50	0.52	15	0.27	0.22	1.165	0.011			0.007	0.007	27
42	28	1.03	0.97	0.50	0.51	15	0.34	0.29	1.150	0.004	0.004		0.003	0.003	28
43	29	1.03	0.97	0.50	0.51	16	0.28	0.23	1.168	0.013			0.005	0.005	29
44	30	1.03	0.97	0.50	0.51	16	0.35	0.30	1.142	0.012			0.001	0.001	30
45	31	1.03	0.97	0.50	0.51	17	0.29	0.24	1.169	0.014			0.001	0.001	31
46	32	1.03	0.97	0.50	0.51	17	0.36	0.30	1.134	0.020			0.003	0.003	32
47	33	1.03	0.97	0.50	0.51	18	0.30	0.25	1.168	0.014			0.003	0.003	33
48	34	1.03	0.97	0.50	0.51	19	0.24	0.20	1.139	0.015			0.000	0.000	34
49	35	1.03	0.97	0.50	0.51	19	0.30	0.25	1.167	0.013			0.001	0.001	35
50	36	1.03	0.98	0.50	0.51	20	0.25	0.20	1.146	0.008	0.008		0.002	0.002	36
51	37	1.02	0.98	0.50	0.51	20	0.31	0.26	1.165	0.011			0.001	0.001	37
52	38	1.02	0.98	0.50	0.51	21	0.26	0.21	1.152	0.002	0.002		0.004	0.004	38
53	39	1.02	0.98	0.50	0.51	21	0.32	0.26	1.162	0.008	0.008		0.003	0.003	39
54	40	1.02	0.98	0.50	0.51	22	0.26	0.22	1.156	0.002	0.002		0.005	0.005	40
55	41	1.02	0.98	0.50	0.51	22	0.32	0.27	1.159	0.005	0.005		0.003	0.003	41
56	42	1.02	0.98	0.50	0.51	23	0.27	0.22	1.159	0.005	0.005		0.005	0.005	42
57	43	1.02	0.98	0.50	0.51	23	0.33	0.27	1.156	0.001	0.001		0.003	0.003	43
58	44	1.02	0.98	0.50	0.51	24	0.28	0.23	1.161	0.007	0.007		0.004	0.004	44
59	45	1.02	0.98	0.50	0.51	24	0.33	0.28	1.152	0.002	0.002		0.002	0.002	45
60	46	1.02	0.98	0.50	0.51	25	0.28	0.23	1.163	0.008	0.008		0.003	0.003	46
61	47	1.02	0.98	0.50	0.51	25	0.34	0.28	1.148	0.006	0.006		0.001	0.001	47
62	48	1.02	0.98	0.50	0.51	26	0.29	0.24	1.163	0.009	0.009		0.001	0.001	48
63	49	1.02	0.98	0.50	0.51	26	0.34	0.29	1.144	0.010			0.001	0.001	49
64	50	1.02	0.98	0.50	0.51	27	0.29	0.24	1.163	0.009	0.009		0.001	0.001	50
65	51	1.02	0.98	0.50	0.51	28	0.25	0.20	1.141	0.013			0.002	0.002	51
66	52	1.02	0.98	0.50	0.51	28	0.30	0.25	1.163	0.009	0.009	52	0.002	0.002	52
67	53	1.02	0.98	0.50	0.51	29	0.25	0.21	1.145	0.009	0.009	53	0.000	0.000	53
68	54	1.02	0.98	0.50	0.51	29	0.30	0.25	1.162	0.008	0.008	54	0.000	0.000	54
69	55	1.02	0.98	0.50	0.51	30	0.26	0.21	1.149	0.005	0.005	55	0.001	0.001	55
70	56	1.02	0.98	0.50	0.51	30	0.31	0.25	1.161	0.007	0.007	56	0.001	0.001	56
71	57	1.02	0.98	0.50	0.51	31	0.26	0.22	1.152	0.002	0.002	57	0.002	0.002	57
72	58	1.02	0.98	0.50	0.51	31	0.31	0.26	1.160	0.006	0.006	58	0.002	0.002	58

Figure 3 An Excel Worksheet Illustrating the Computations of the Lowest Numbers of Binomial Periods for Numerical Convergence of the Binomial Model and Black-Scholes Model (Continued).

of them to cover.[17] This requires repeated copy-and-paste operations for such combinations to record the corresponding values in B11:B12 of the worksheet for Figure 3. Table 1 shows the summary statistics of the lowest numbers of binomial periods as required to achieve numerical convergence of the two models. The convergence criterion, just like that in Figure 3, is the absolute price difference being less than \$0.010. The summary statistics include the average, the median, the minimum, and the maximum of such numbers, for each given value of any of the four input parameters. Case (i) and case (ii) differ only in that, in the latter case, the average of two adjacent values of c is used instead for establishing $|c - C|$.

For example, in the first row, the summary statistics for $r = 1\%$ pertain to those of the $3^3 = 27$ combinations of T, σ, and S. For this specific r, although case (i) and case (ii) have the same minimum in the lowest numbers of binomial periods to achieve numerical convergence of the two models, the latter case shows much lower average, median, and maximum. The displayed values in all other rows, each of which focuses on a specific value of a given input parameter, show a similar pattern. Case (ii) even provides much lower minimum figures than case (i) does for some input parameters; examples include $\sigma = 30\%$ and 50%, as well as $S = \$50$.

Regardless of whether the average or the median is used to compare the two cases, it is clear that case (ii) represents an effective approach to reduce the number of binomial periods for numerical convergence of the two models. This is also confirmed by the numbers in the last row of Table 1, which represent the overall results for the 81 different combinations of the input parameters. For each of the two cases, the lowest number of binomial periods for numerical convergence of the two models increases with increasing T, as well as increasing σ.

For example, in case (i), for $T = 0.25$, 1, and 4, the medians of the lowest numbers of binomial periods are 52, 131, and 194, respectively. Likewise, for $\sigma = 10\%$, 30%, and 50%, the corresponding medians are 33, 131, and 209 periods. Given equation (28), such monotonic relationships are as expected. As an increase in N tends to allow the two models to converge, an increase in T or in σ will require a higher N to provide a similar set of u and d for use in equation (16).

6 Concluding Remarks

This paper has presented, from a pedagogic perspective, a binomial option pricing model, which eventually converges to the well-known Black-Scholes (1973) formula as the number of binomial periods increases. The binomial model considered is the Cox-Ross-Rubinstein (1979) version,

[17] The combination of $r = 10\%$, $T = 4$ years, $\sigma = 10\%$, and $N \leq 4$ is problematic, as it inevitably leads to unacceptable p and p' as probability measures. However, as numerical convergence of the two models requires more than 4 binomial steps for such a combination of input parameters, the exclusion of these problematic cases has no impact on the summary statistics we seek.

Table 1 The Minimum Number of Periods N for the Binomial Model in Equation (16), as Required to Achieve Numerical Convergence to the Black-Scholes Model in Equation (2). [Convergence criterion: the absolute difference in the option prices from the two models be consistently less than \$0.010, for N binomial periods or more; input parameters: for an exercise price of \$50, there is a total of $3^4 = 81$ different combinations of input parameters, with each row (except the last row) in the table pertaining to $3^3 = 27$ combinations; case (i) and case (ii): each option price from the binomial model for computing the absolute difference is based on N binomial periods [case (i)] and the average of two option prices for N and $N-1$ binomial periods [case (ii)]; notation: r = the risk-free annual interest rate, T = the expiry of the option in years, σ = the standard deviation of annual returns of the underlying stock, and S = the price of the underlying stock.

		Case (i)				Case (ii)			
		Average	Median	Min	Max	Average	Median	Min	Max
$r =$	1%	148.74	111	5	467	35.81	19	5	114
	5%	144.56	112	4	441	40.00	38	4	95
	10%	135.37	123	4	413	40.15	32	5	113
$T =$	0.25	61.33	52	4	125	17.93	9	4	49
	1	130.56	131	7	248	38.89	30	5	92
	4	236.78	194	12	467	59.15	56	8	114
$\sigma =$	10%	50.81	33	4	178	25.00	9	4	113
	30%	139.48	131	46	294	40.22	24	7	114
	50%	238.37	209	96	467	50.74	49	9	86
$S =$	\$45	144.70	131	5	377	48.59	46	6	113
	\$50	163.96	123	25	467	23.19	12	4	90
	\$55	120.00	107	4	449	44.19	47	5	114
All 81 Combinations		142.89	112	4	467	38.65	32	4	114

which can be derived by using algebraic tools alone. Though being the simplest among all available binomial models in the finance literature, the Cox-Ross-Rubinstein model still compares well against various other versions, in terms of convergence to the Black-Scholes model. It also retains the crucial insights of the Black-Scholes model. Thus, this simple binomial model is suitable for classroom coverage, as it allows students to understand the concepts of option pricing, all without the encumbrance of advanced analytical tools.

Excel plays a very important role in this pedagogic illustration. It is the interactivity of the various examples provided, in conjunction with the Excel graphical features utilized, that allows students to visualize how the results from the binomial model can eventually match those from the Black-Scholes model. The graphical part of Figure 1, which is similar to typical textbook illustrations of the Black-Scholes model, is readily suitable for classroom use. By varying the input parameters via the various scroll bars in the corresponding worksheet, students can explore interactively how well the binomial model works.

The remaining figures in this paper can be presented successively in class to illustrate different aspects of how changes in individual input parameters affect the convergence of the two models. Alternatively, students can be asked to produce the corresponding spreadsheets as exercises. As the three figures in this paper do have many common computational features, such exercises are not likely to be burdensome for students, once they are made aware of the workings of the worksheet for Figure 1. Such hands-on experience will enhance students' understanding of the binomial model and reduce the perceived mystery of option pricing.

The scope of this study has been confined to pricing of European call options on stocks that pay no dividends. Extensions to other options, such as American and European call and put options on dividend-paying stocks — where put options are options to sell the underlying stocks instead — are analytically much more challenging. It is hoped that this paper can improve the foundation for students on option pricing, so that they can be better prepared when encountering more option pricing models.

References

Black, F., and Scholes, M., (1973), "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81(3), 637-654.

Chance, D.M., (2008), "A Synthesis of Binomial Option Pricing Models for Lognormally Distributed Assets," *Journal of Applied Finance*, Spring/Summer, 38-56.

Copeland, T.E., Weston, J.F., and Shastri, K., (2005), *Financial Theory and Corporate Policy, Fourth Edition*, Pearson Addison Wesley, Boston, MA.

Cox, J.C., Ross, S.A., and Rubinstein, M., (1979), "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, 7, 229-263.

Hull, J.C., (2002), *Options*, *Futures*, and *Other Derivatives*, *Fifth Edition*, Pearson Prentice Hall, Upper Saddle River, NJ.

Hull, J., and White, A., (1988), "The Use of the Control Variate Technique in Option Pricing," *Journal of Financial and Quantitative Analysis*, 23(3), 237-251.

Leisen, D.P.J., and Reimer, M., (1996), "Binomial Models for Option Valuation — Examining and Improving Convergence," *Journal of Mathematical Finance*, 3, 319-346.

MacKenzie, D., (2006), *An Engine*, *Not a Camera*: *How Financial Models Shape Markets*, MIT Press, Cambridge, MA.

Rendleman Jr., R.J., and Bartter, B.J., (1979), "Two-State Option Pricing," *Journal of Finance*, 34(5), 1093-1110.

Sharpe, W.F., Alexander, G.J., Bailey, J.V., Fowler, D.J., and Domian, D.L., (2000), *Investments*, *Third Canadian Edition*, Prentice Hall Canada, Scarborough, ON.

Tucker, A.L., (1991), *Financial Futures*, *Options, and Swaps*, West Publishing Company, St. Paul, MN.

Appendix: Risk-Neutral Probabilities and Pure Security Prices

In a single-period setting of the state preference framework, the end of the period is characterized as having a known number of states of nature. The end-of-period payoff of each risky security is state-contingent, and there is a known payoff for each state. However, at the beginning of the period, it is unknown which state will occur.

Suppose that there are k states in total and that at least k securities are available in the market for investing at the beginning of the period. For the market to be arbitrage-free, all securities and portfolios of securities with the same set of state-contingent payoffs must have the same beginning-of-period price. For the market to be complete, we must be able to generate a complete set of k pure securities by forming portfolios of available securities in the market.

One share of pure security i will pay \$1 if state i occurs, but it will pay nothing if state i does not occur. Suppose that the beginning-of-period price of pure security i is $\$s_i$, for $i = 1, 2, \ldots, k$. A risk-free investment can be achieved by holding one share of each pure security for a total of $\$\left(\sum_{i=1}^{k} s_i\right)$, because the end-of-period pay-off is always \$1, regardless of which state occurs. Therefore, the condition of

$$\frac{1}{\sum_{i=1}^{k} s_i} = 1 + r_f \tag{A1}$$

must hold. Implicitly, we have $0 < s_i < 1$, for $i = 1, 2, \ldots, k$.

Conversely, a risky security that will pay $\$x_i$ if state i occurs, for $i = 1, 2, \ldots, k$, can be replicated by a portfolio of pure securities. The portfolio is to consist of x_i shares of pure security i, for $i = 1, 2, \ldots, k$. The equivalence of this risky security and the portfolio of pure securities ensures that the beginning-of-period price of the security be $\$\left(\sum_{i=1}^{k} s_i x_i\right)$.

Letting

$$p_i = s_i(1 + r_f), \text{ for } i = 1, 2, \ldots, k, \tag{A2}$$

we can write equation (A1) equivalently as

$$\sum\nolimits_{i=1}^{k} p_i = 1. \tag{A3}$$

Given equation (A3), the set of compounded prices of pure securities, $p_1, p_2, \ldots, p_k$, with $1 + r_f$ being a compounding factor, is mathematically equivalent to a set of probabilities. As the beginning-of-period price of a risky security that will pay $\$x_i$ if state i occurs, for $i = 1, 2, \ldots, k$, is $\$\left(\sum_{i=1}^{k} p_i x_i\right) \Big/ (1 + r_f)$, we can treat this price as the discounted value of the expected payoff, with $1/(1 + r_f)$ being the discounting factor.

In a single-period setting of the binomial option pricing model as described in Section 3 of the main text, there are only two end-of-period states; that is, $k = 2$. The risky security in question is a European call option, with end-of-period state-contingent prices being $x_1 = c_u = \max(0, uS - X)$ and $x_2 = c_d = \max(0, dS - X)$. The corresponding compounded prices of pure securities are $p_1 = p$ and $p_2 = 1 - p$. The beginning-of-period price of the option, which is $c = \left(\sum_{i=1}^{2} p_i x_i\right) \Big/ (1 + r_f)$, is equivalent to that given by equation (8). Thus, the risk-neutral probabilities p and $1 - p$ are the same as compounded prices of pure securities under the state preference framework.