Spreadsheets in Education (eJSiE)

Volume 5 | Issue 3 Article 4

November 2012

Geometric Brownian Motion, Option Pricing, and Simulation: Some Spreadsheet-Based Exercises in Financial Modeling

Kevin D. Brewer
McMaster University, kbrewer@mcmaster.ca

Yi Feng
Ryerson University, yi.feng@ryerson.ca

Clarence C. Y. Kwan
McMaster University, kwanc@mcmaster.ca

Geometric Brownian Motion, Option Pricing, and Simulation: Some Spreadsheet-Based Exercises in Financial Modeling

Abstract

This paper presents some Excel-based simulation exercises that are suitable for use in financial modeling courses. Such exercises are based on a stochastic process of stock price movements, called geometric Brownian motion, that underlies the derivation of the Black-Scholes option pricing model. Guidance is provided in assigning appropriate values of the drift parameter in the stochastic process for such exercises. Some further simulation exercises are also suggested. As the analytical underpinning of the materials involved is provided, this paper is expected to be of interest also to instructors and students of investment courses.

Keywords

Stochastic process, geometric Brownian motion, Black-Scholes model, put-call parity, simulation

Cover Page Footnote

Kwan is the corresponding author. We wish to thank P.C. Miu for helpful discussions.

Geometric Brownian Motion, Option Pricing, and Simulation: Some Spreadsheet-Based Exercises in Financial Modeling

1 Introduction

In spreadsheet-based financial modeling courses, students learn how to apply spreadsheet tools, such as those in Microsoft Excel^{TM}, to various financial settings, for which analytical models are available. Settings requiring only basic present-value concepts tend to include annuity, amortization, capital budgeting, and bond valuation. More advanced financial settings cover also topics in risk management, portfolio analysis, alternative risk measures, pricing of various derivative assets, and simulation. Given the objectives of financial modeling courses, sophisticated analytical materials, if covered, are typically confined to end results or recipes. So are such materials in standard textbooks of financial modeling.[1]

Relevant Excel tools for financial modeling courses include Data Validation, Lookup Tables, Data Tables, and various Form Controls and ActiveX Controls from the Developer tab, in addition to those numerical and graphical tools that are familiar to general users. Students also have opportunities to use Excel's various financial and statistical functions pertaining to the financial topics covered. Further, students are introduced to Macros and coding in Visual Basic for Applications (VBA). Individual or team projects requiring the use of various Excel tools, as well as VBA programming, are often a key component of the course materials, for enhancing students' learning experience and for strengthening their technical skills.

In financial modeling courses at both undergraduate and M.B.A. levels currently taught by one of the co-authors of this paper, students are often assigned projects requiring the use of the Black-Scholes (1973) option pricing model.[2] Although the derivation of the model is well beyond the scope of the standard finance curriculum, its implementation in Excel is straightforward. As the well-known Black-Scholes formula contains expressions of the cumulative standardized normal distribution, the Excel function NORMSDIST can be applied directly for computational purposes. Further, in view of an exact relationship between the corresponding call and put prices, known as put-call parity, Excel-based computations of put prices under the assumptions of the Black-Scholes

[1] See, for example, Benninga (2008), Holden (2011), and Sengupta (2004)

[2] The Black-Scholes model is for pricing a European call option on a stock that pays no dividend. The option holder has the right, not the obligation, to buy from the option writer on the expiry date a share of the underlying stock at a predetermined price, called exercise price or strike price. A put option is the opposite; it gives the option holder the right to sell the underlying stock to the option writer instead. An American option differs from the corresponding European version in that it can also be exercised any time prior to the expiry date. For analytical convenience, the options considered in this paper are confined to European options on stocks that pay no dividends.

model are equally straightforward.

Drawing on an idea pertaining to the above-mentioned student projects, this paper presents some simulation exercises based on a stochastic process, known as geometric Brownian motion, for characterizing stock price movements. Such a stochastic process is relevant, as it underlies the original derivation of the Black-Scholes model. While the model is consistent with various stochastic processes, such as those identified in Lo and Wang (1995), geometric Brownian motion is the simplest among them. Its simplicity has made it particularly suitable for pedagogic purposes.

Based on available information on the day of a call or put option investment, which includes the corresponding stock and option prices at the time, of interest to the option investor is how these prices can potentially vary over the remaining life of the option. When used to characterize the underlying stock price movements, geometric Brownian motion will allow the potential time paths to be simulated. Such simulation exercises are intended to enhance students' understanding of not only what a stochastic process is all about, but also how stock price movements over time affect the corresponding option price movements and thus the profitability of an option investment. As a result, students will have a deeper understanding of the risk-return characteristics of an option investment, from the perspectives of the buyer and the writer of the option.[3]

To facilitate the Excel illustrations later in this paper, Section 2 first describes, without the encumbrance of analytical details, how values of the two crucial parameters in geometric Brownian motion, as required for simulation exercises, can be inferred from available information at the time of an option investment. Stated in annual terms, the two parameters — which are generally known as drift and volatility — represent the expected instantaneous return and the standard deviation of returns of the underlying stock, respectively. The issue as to why the drift parameter, though absent from the Black-Scholes formula, is relevant in simulation exercises is also discussed in Section 2. Such descriptions and discussions are intended to place the analytical materials of Section 3 in a proper context.

Section 3 has seven subsections. The first three subsections are for describing essential materials on geometric Brownian motion, the Black-Scholes formula, and put-call parity. To avoid digressions, some derivations pertaining to geometric Brownian motion are provided separately in Appendix A. The next three subsections cover, in analytical detail, the various ideas in Section 2.[4] To the best of our knowledge, Subsections 3.5 and 3.6 are original contributions. The final

[3]It is up to individual instructors of financial modeling courses to decide whether actual or artificial data are used, for the day of an option investment, in simulation exercises. The use of actual data will make such exercises more practically relevant to students.

[4]For pedagogic purposes, we derive in this paper most of the analytical materials involved, although the same materials can also be found elsewhere. In such cases, appropriate references are provided.

subsection shows how various analytical materials can be utilized to simulate daily stock and option price movements.[5]

Some simulation exercises are illustrated in Section 4. It has two subsections. The first subsection provides a simple illustration that requires only Excel's general numerical and graphical tools, in order to show the ideas involved. For computational and graphical convenience in performing simulation exercises, VBA coding is utilized in the second subsection. Section 5 suggests further simulation exercises, with materials presented in three subsections. Some of these suggested exercises are also suitable for use in investment courses that cover stochastic processes of stock price movements. Finally, Section 6 provides some concluding remarks.

2 The Two Parameters in Geometric Brownian Motion

Of the two parameters in geometric Brownian motion, only the volatility parameter is present in the Black-Scholes formula. The absence of the drift parameter is not surprising, as the derivation of the model is based on the idea of arbitrage-free pricing. The derivation requires that risk-free hedged portfolios, based on the call option and the underlying stock, be formed continuously. Accordingly, the predictability of stock returns in terms of the stock's expected return over the remaining life of the option, which can be inferred from the drift parameter, becomes irrelevant.[6]

For the purpose of simulation exercises involving geometric Brownian motion, values of both the drift parameter and the volatility parameter are required. The latter parameter can be inferred directly from the corresponding stock and option prices on the day of the option investment or estimated empirically from historical time series of stock returns. If the corresponding stock and option prices are deemed available on the day of the option investment, the use of an implied volatility for simulation exercises will ensure internal consistency of the data involved. To use Excel's Solver or Goal Seek to deduce the volatility parameter from stock and option prices is straightforward.

Estimation of the volatility parameter based on time series data, which is associated with the empirical aspect of option pricing, is beyond the scope of typical financial modeling courses.[7] Thus, for simulation exercises, students are seldom required to estimate the volatility parameter

[5] Readers who are primarily interested in the Excel-based simulation exercises can skip the derivations in Section 3 and go directly to Section 4. Any specific equations in Section 3 as required for such exercises are indicated there.

[6] Lo and Wang (1995) have shown that the predictability of stock returns, if also considered, can improve the Black-Scholes option pricing results. The task, which requires the use of stochastic processes that are more sophisticated than geometric Brown motion, is beyond the pedagogic scope of this paper.

[7] See, for example, Hull (2009, Chapter 13) for an estimation method that can easily be implemented in Excel. See also Campbell, Lo, and MacKinlay (1997, Chapter 9) and Yang and Zhang (2000), as well as the references there, for some other estimation methods.

themselves; rather, they can attempt different values for it as user input, with guidance from empirical results elsewhere. For internal consistency of the input data, however, the volatility parameter and the option price on the day of an option investment cannot both be from user input. One of them must be computed instead.

In contrast, the drift parameter is more difficult to deduce. Guidance from textbooks is inadequate as to what values of the drift parameter ought to be used in simulation exercises. Textbook examples either use the risk-free interest rate as the drift parameter or simply assign an arbitrary value for it. Justification for the former case implicitly requires the assumption that investors are risk neutral.[8]

The use of a higher (lower) value of the drift parameter will tend to result, on average, higher (lower) simulated stock prices over the remaining life of each option that the stock underlies. With option prices depending on the underlying stock prices, the expected profitability of an option investment inevitably depends on the value of the drift parameter used. There are two opposite sides in an option investment, the buyer side and the writer side. As a rational individual never willingly invests for an expected loss, the drift parameter must be perceived to be higher for the buyer of a call option than for the writer. For an investment in a put option, the opposite is true. Given different perspectives from the two sides of an option investment, a relevant question now is what values of the drift parameter are appropriate for use in simulation exercises.

To seek an answer, let us start with a scenario where an option is not exercised on the expiry date. In this scenario, the writer of the option will achieve the maximum profit, and the buyer will incur the maximum loss. In the case of a call option, this scenario requires the price of the underlying stock on the expiry date to be no greater than the exercise price. For this scenario not to occur, the stock price must be higher. In the case of a put option, the situations are reversed. Thus, for either a call option or a put option, if we equate the exercise price and the stock price that is expected for the expiry date, we can deduce a threshold value of the drift parameter. We can do so because, as shown in Section 3, there is an analytical expression to relate the expected stock price and the drift parameter. In simulation exercises involving a call option, if the assigned value of the drift parameter is above (below) this threshold value, the option will be more (less) likely to be exercised. In the case of a put option, it is the opposite.

However, exercising an option does not necessarily result in a profit for the buyer and a loss for the writer. Thus, a different threshold value of the drift parameter is required to separate

[8] See, for example, Hull (2009, Chapter 13), where the drift parameter is set equal to the risk-free interest rate. See also Hull (2009, Chapter 12) and Sengupta (2004, Chapters 11 and 22), where arbitrary values are used instead.

the option investment outcomes into profitable and nonprofitable cases. The idea as mentioned earlier that a rational individual never willingly invests for an expected loss is applicable here. For an option investment that precludes neither the buyer nor the writer from expecting a profit, we simply set a threshold value of the drift parameter to correspond to expected profits being zero for both parties.

Specifically, in the case of a call option, this requires that the expected stock price at the expiry date of the option be equal to the total investment by the buyer, which includes both the exercise price and the original purchase price of the option as measured at the expiry date. As the value of the drift parameter thus determined provides a zero expected profit for each side of the option investment, the use of a higher (lower) value will give the buyer (writer) side an advantage. Analogously, in the case of a put option, this requires that the exercise price be equal to the expected total investment by the buyer, which includes both the expected stock price at the expiry date and the original purchase price of the option as measured at the expiry date.[9]

Given put-call parity, we are able to establish how various threshold values of the drift parameter compare to each other and to the risk-free interest rate. As shown in Section 3, the threshold value of the drift parameter based on zero expected profits for both the buyer and the writer of a call option is always above the risk-free interest rate. In the case of an investment in a put option, the opposite is true. Accordingly, the range of values between these two threshold values always encloses the risk-free interest rate. An implication is that, if the risk-free interest rate is used as the drift parameter, as suggested in textbooks, simulation results of the profitability of an option investment will tend to favour the writer, on average, regardless of whether it is a call option or a put option.

The same range of values also encloses the threshold value of the drift parameter for expecting an option to be exercised. However, whether such a threshold value is above or below the risk-free interest rate depends on the sign of the difference between the corresponding call and put prices. The analytical details of all these properties are provided in Section 3.

As long as the two sides of an option investment have different expectations about the investment outcomes, subjectivity in the choice of values of the drift parameter for simulation exercises is inevitable. The threshold values considered in this study are intended to guide students in setting reasonable — rather than totally arbitrary — values of the drift parameter. For the Excel-based

[9] For analytical convenience, an option investment is evaluated under the assumption that the investment is over the remaining life of the option considered. In practice, however, the option can always be traded prior to its expiry date, if desired. To establish a threshold value of the drift parameter in such a setting not only is a formidable task, but also will cause digressions from the original pedagogic objectives of this paper.

exercises in Section 4, different choices for values of the drift parameter are available from a menu. Specifically, the menu items include not only the threshold values as identified above and the risk-free interest rate, but also user input.

3 Analytical Materials for Simulation Exercises

This section, which consists of seven subsections, covers the analytical materials that are essential for the simulation exercises in Section 4. As the materials on geometric Brownian motion are generally unfamiliar to business students, they are covered in some detail in the first subsection, with derivations also provided in Appendix A. In contrast, the Black-Scholes formula, which is part of the standard finance curriculum even at the introductory level, does not require a detailed description; it is briefly stated in the second subsection. The connection of corresponding call and put option prices is also provided in this section via the concept of put-call parity in the third subsection. As the corresponding materials, though analytically simple, are usually not covered in introductory finance textbooks, they are provided there. The next three subsections restate and expand, from an analytical perspective, the various ideas in Section 2. The final subsection describes analytically how the time paths of daily stock and option prices can be simulated.

3.1 Geometric Brownian Motion

A key assumption of the Black-Scholes option pricing model is that the instantaneous stock price movements can be characterized by

$$dS = \mu S dt + \sigma S dz. \tag{1}$$

Here, S is the stock price, μ and σ are constants, t is time, and z follows a stochastic process called Wiener process, under which $dz = \epsilon\sqrt{dt}$, where ϵ is a random draw from the standardized normal distribution. Equation (1) is commonly known as geometric Brownian motion, with μ and σ called the drift parameter and the volatility parameter, respectively.

As shown in Appendix A, equation (1) implies

$$d \ln S = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz. \tag{2}$$

The stochastic process as characterized by equation (2) indicates that $\ln S$ is normally distributed. Equivalently, S is lognormally distributed. With S_0 and S_T denoted as the stock prices at time 0 and time T, respectively, equation (2) leads to

$$S_T = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma\epsilon\sqrt{T}\right]. \tag{3}$$

Further, the expected value and the variance of S_T are

$$E(S_T) = S_0 \exp(\mu T) \tag{4}$$

and

$$Var(S_T) = S_0^2 \left[\exp(2\mu T)\right] \left[\exp(\sigma^2 T) - 1\right] = \left[E(S_T)\right]^2 \left[\exp(\sigma^2 T) - 1\right], \tag{5}$$

respectively. Equation (5) can be written as

$$\sqrt{Var(S_T)} = E(S_T)\sqrt{\exp(\sigma^2 T) - 1} \tag{6}$$

for graphical convenience.

Given equation (4), we have

$$\frac{\partial E(S_T)}{\partial \mu} = T\, E(S_T) \tag{7}$$

and

$$\frac{\partial E(S_T)}{\partial T} = \mu\, E(S_T). \tag{8}$$

Accordingly, $E(S_T)$ increases with μ for any given $T > 0$. It increases with T if $\mu > 0$, decreases instead if $\mu < 0$, and remains unchanged if $\mu = 0$.

Likewise, given equation (5), we have

$$\frac{\partial Var(S_T)}{\partial \mu} = 2T\, Var(S_T), \tag{9}$$

$$\frac{\partial Var(S_T)}{\partial \sigma} = 2\sigma T\ \left[E(S_T)\right]^2 \exp(\sigma^2 T), \tag{10}$$

and

$$\frac{\partial Var(S_T)}{\partial T} = S_0^2 \left\{(2\mu + \sigma^2)\exp\left[(2\mu + \sigma^2)T\right] - 2\mu \exp(2\mu T)\right\}. \tag{11}$$

Thus, $Var(S_T)$ increases with μ and σ for any given $T > 0$. As $\exp\left[(2\mu + \sigma^2)T\right] > \exp(2\mu T) > 0$ for $T > 0$, $Var(S_T)$ increases with T if $2\mu + \sigma^2 \geq 0$.[10] However, if μ is negative and low enough to make $2\mu + \sigma^2$ negative, the graph of $Var(S_T)$ versus T will not always be upward sloping; if $2\mu + \sigma^2 < 0$, the graph will be downward sloping once T exceeds a threshold value. Specifically, with the threshold value of T being

$$T^* = \frac{1}{\sigma^2} \ln\left(\frac{2\mu}{2\mu + \sigma^2}\right), \tag{12}$$

$T \lesseqgtr T^*$ corresponds to $\partial Var(S_T)/\partial T \gtreqless 0$.[11]

[10] As $\sigma^2 > 0$, we always have $2\mu + \sigma^2 > 2\mu$ regardless of the sign of μ. If $\mu \geq 0$, it follows from $\exp\left[(2\mu + \sigma^2)T\right] > \exp(2\mu T) > 0$ that $(2\mu + \sigma^2)\exp\left[(2\mu + \sigma^2)T\right] > 2\mu \exp(2\mu T)$ and thus $\partial Var(S_T)/\partial T > 0$. If $\mu < 0$ instead, as $-2\mu \exp(2\mu T) > 0$, a positive sign of $\partial Var(S_T)/\partial T$ is assured if we also have $2\mu + \sigma^2 \geq 0$.

[11] According to equation (11), if $2\mu + \sigma^2 < 0$, to achieve $\partial Var(S_T)/\partial T < 0$ requires $\exp(\sigma^2 T) > 2\mu/(2\mu + \sigma^2)$. It follows from $2\mu < 2\mu + \sigma^2 < 0$ that $2\mu/(2\mu + \sigma^2) > 1$. As $\exp(\sigma^2 T)$ increases monotonically with T, there is a threshold value of T beyond which $\exp(\sigma^2 T) > 2\mu/(2\mu + \sigma^2)$. This idea leads to equation (12).

3.2 The Black-Scholes Formula

In addition to the various symbols already defined, let r be the continuously compounded annual risk-free interest rate, C be the value of a European call option on a stock that pays no dividend, X be the exercise price of the option, and $N(\cdot)$ be the cumulative standardized normal distribution. Further, let T now denote the proportion of a year before the option expires. The Black-Scholes formula is

$$C = S\ N(d_1) - X\ \exp(-rT)\, N(d_2), \tag{13}$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S}{X}\right) + rT\right] + \frac{1}{2}\sigma\sqrt{T} \tag{14}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}. \tag{15}$$

3.3 Put-Call Parity

For European options on stocks that pay no dividends, there is a relationship between put and call prices — P and C, respectively — for the same exercise price X and the same maturity T on an underlying stock. This is what put-call parity is all about. To establish such a relationship, let us construct a portfolio by holding the underlying stock and the put option and writing the call option, for a net investment of $S + P - C$.

On the expiry date, which is time T, the underlying stock price is S_T. The following represents three potential scenarios:

	Value of Stock	+ Value of Put	− Value of Call	=	Value of Portfolio
If $S_T < X$	S_T	$+ (X - S_T)$	$- 0$	$=$	X
If $S_T = X$	S_T	$+ 0$	$- 0$	$=$	$S_T = X$
If $S_T > X$	S_T	$+ 0$	$- (S_T - X)$	$=$	X

The idea is that, if $S_T < X$, we exercise the put option in the portfolio, but the holder of the call option does not exercise it. If $S_T = X$, neither option is exercised. If $S_T > X$ instead, we do not exercise the put option, but the holder of the call option exercises it. Regardless of which scenario prevails, the investment outcome is still X. That is, the net investment of $S + P - C$ is risk-free. Thus, with r being a continuously compounded annual risk-free interest rate, we must have

$$(S + P - C)\exp(rT) = X \tag{16}$$

or, equivalently,

$$C - P = S - X\exp(-rT). \tag{17}$$

Equation (17) allows the put price to be determined once the corresponding call price is known.[12]

Upon substituting the expression of C in equation (13), we can write

$$\begin{aligned} P &= -S[1-N(d_1)] + X\ \exp(-rT)[1-N(d_2)] \\ &= -S\ N(-d_1) + X\ \exp(-rT)\ N(-d_2). \end{aligned} \tag{18}$$

With $N(\cdot)$ being a cumulative distribution, $1-N(\cdot)$ is the corresponding complementary cumulative distribution. From an algebraic perspective, the expressions in equations (13) and (18) are similar to each other. Thus, for the purpose of simulation exercises, to change from a setting involving a call option to a setting involving the corresponding put option requires only minor algebraic adjustments.

3.4 Implied Volatility

Regardless of whether a simulation exercise requires the use of equation (13) or equation (18), we consider the available information for an option investor to be confined to the corresponding stock and option prices on the day of the option investment, the exercise price and the expiry date of the option, and the risk-free interest rate. The day of the option investment is also the day for price simulation over the remaining life of the option. The investor in question can be the buyer or the writer of the option.

However, values of the drift parameter and the volatility parameter are not considered to be part of the available information. Given the presence of the volatility parameter in the Black-Scholes formula, to deduce its value objectively from the available information is straightforward. Thus, a brief description here is adequate.[13] In contrast, to find appropriate values of the drift parameter for simulation exercises is not as simple. The analytical detail is provided in the next two subsections.

To determine values of σ for simulation exercises, let us label the corresponding stock and option prices on the day of the option investment as S_0 and C_0, or as S_0 and P_0, depending on whether a call option or a put option is involved. As C_0 and P_0 are each a function of S_0, X, T, r, and σ according to equations (13) and (18), there must be a specific value of σ, when combined with the given values of S_0, X, T, and r, that allows C_0 and P_0 to be deduced. Such a value of σ is commonly known as the implied volatility.

It has been well established in the finance literature that the price of an option increases with the volatility of its underlying stock. That is, for a given set of parameters for option pricing,

[12] The derivation of equation (17) draws on Copeland, Weston, and Shastri (2005, Chapter 7).

[13] See, for example, Chance and Brooks (2008, Chapter 5) for an Excel illustration.

there is a monotonic relationship between the option price and the volatility parameter.[14] Thus, there will not be any concern of whether there are some other values of the volatility parameters that also correspond to the same option price. In view of this option property, Excel's Solver or Goal Seek can be used to infer σ numerically from values of S_0, X, T, r, and C_0 or P_0, depending on whether equation (13) or equation (18) is applicable to the simulation exercise involved. By using either numerical tool, we can search for the value of σ that allows the given price and the computed price of the option to be matched.

Implicit in the above numerical search is that the available options are properly priced. For this to be true, the user input of S_0, X, T, r, and C_0 or P_0 in each simulation exercise must not be inconsistent with each other. Specifically, the no-arbitrage conditions of $0 \leq C_0 \leq S_0$, $C_0 \geq S_0 - X\exp(-rT)$, $0 \leq P_0 \leq X\exp(-rT)$, and $P_0 \geq X\exp(-rT) - S_0$ must be satisfied. Otherwise, the implied value of σ will not be meaningful; neither will the simulation results.

The potential inconsistency of user input can be avoided in simulation exercises if the input is to consist of S_0, X, T, r, and σ instead. Given equations (13) and (18), each computed option price for the day of an option investment will always satisfy the no-arbitrage conditions as noted above. In this revised setting, we assume that an option with a computed price based on the values of S_0, X, T, r, and σ is available for investing and that the simulation exercise to follow is for such an option.

3.5 Threshold Values of the Drift Parameter

As explained in Section 2, to set appropriate values of the drift parameter for simulation exercises, some of its threshold values are relevant. One of such values is from matching the expected stock price and the exercise price; that is,

$$E(S_T) = X. \tag{19}$$

Let us label the corresponding drift parameter as μ_x. Given equation (4), we can deduce μ_x from

$$S_0 \exp(\mu_x T) = X. \tag{20}$$

or, equivalently,

$$\mu_x = \frac{1}{T} \ln\left(\frac{X}{S_0}\right). \tag{21}$$

For an investment in a call option, also relevant is the threshold value of the drift parameter that allows the expected stock price to match the total investment by the buyer, as measured at the expiry date of the option. Let us label the drift parameter here as μ_c. At time 0, the buyer pays

[14] See, for example, Hull (2009, Chapter 9).

C_0 for the option. If the option is to be exercised at the expiry date, which is time T, the buyer pays the exercise price X in addition. Thus, the total amount of the investment, as measured at the expiry date, is $C_0 \exp(rT) + X$. The investment is profitable for the buyer only if S_T is greater than $C_0 \exp(rT) + X$.

Thus, we can deduce the value of μ_c that benefits neither the buyer nor the writer of the call option by specifying that

$$E\left(S_T\right) = C_0 \exp(rT) + X. \tag{22}$$

Given equation (4), we can write

$$S_0 \exp(\mu_c T) = C_0 \exp(rT) + X. \tag{23}$$

It follows that

$$\mu_c = \frac{1}{T} \ln \left[\frac{C_0 \exp(rT) + X}{S_0} \right]. \tag{24}$$

For an investment in a put option, the corresponding threshold value of the drift parameter, labeled as μ_p, can be determined in an analogous manner. Specifically, at time 0, an investor who buys the put option pays P_0 for it. At time T, the expiry date of the option, if the option is to be exercised, the investor acquires the underlying stock for S_T and sells it for X.

The initial investment of P_0, as evaluated at time T, is $P_0 \exp(rT)$. At time 0, if the option investment is expected to benefit neither the buyer nor the writer, the following equality must hold:

$$P_0 \exp(rT) + E(S_T) = X. \tag{25}$$

Given equation (4) for an explicit expression of $E(S_T)$, we can write

$$S_0 \exp(\mu_p T) = X - P_0 \exp(rT), \tag{26}$$

which leads to

$$\mu_p = \frac{1}{T} \ln \left[\frac{X - P_0 \exp(rT)}{S_0} \right]. \tag{27}$$

3.6 A Comparison of Threshold Values of the Drift Parameter

In view of the analytical expressions in equations (21), (24), and (27), we can establish the following properties:

$$\begin{array}{ll} (i) & \mu_p < \mu_x < \mu_c; \\ (ii) & \mu_p < r < \mu_c; \\ (iii) & \mu_x \gtreqless r \ \text{ for } C_0 \lesseqgtr P_0. \end{array} \tag{28}$$

These properties can provide some guidance in setting appropriate values of the drift parameter for simulation exercises. Thus, after deriving each of these properties, we also explore the corresponding implications.

As $\ln a$ (where a is any positive real variable) increases with a, property (i) can be deduced directly by comparing the expressions in equations (21), (24), and (27). On the day of a call option investment, if the buyer of the option considers the drift parameter μ in the stochastic process of the underlying stock price movements to be above μ_c, a profit is expected. The expected profit increases with μ that is above μ_c. For the writer of the same option, it is the opposite; an expected profit requires the value of the drift parameter to be below μ_c. As μ decreases from μ_c to μ_x, the writer's expected profit increases. The maximum expected profit requires $\mu \leq \mu_x$.

The corresponding implications pertaining to a put option investment are analogous. Specifically, for the buyer, the drift parameter μ must be below μ_p for an expected profit, and the profit increases as μ decreases from this threshold value. For the writer, an expected profit requires $\mu > \mu_p$, and the maximum expected profit requires $\mu \geq \mu_x$. A further implication of property (i) is that the use of μ_x for the drift parameter will favour the writer for expecting a profit from an option investment, regardless of whether the investment is in a call option or a put option.

To derive properties (ii) and (iii), we rely on put-call parity repeatedly. When applied to day 0, put-call parity that equation (17) represents can be stated as

$$C_0 \exp(rT) + X = (S_0 + P_0) \exp(rT). \tag{29}$$

Thus, equation (24) is equivalent to

$$\begin{aligned} \mu_c &= \frac{1}{T} \ln \left[\frac{(S_0 + P_0) \exp(rT)}{S_0} \right] \\ &= r + \frac{1}{T} \ln \left(1 + \frac{P_0}{S_0} \right). \end{aligned} \tag{30}$$

With P_0 being strictly positive, we must have

$$\frac{1}{T} \ln \left(1 + \frac{P_0}{S_0} \right) > 0 \tag{31}$$

and, accordingly,

$$\mu_c > r. \tag{32}$$

Likewise, with equation (29) written equivalently as

$$X - P_0 \exp(rT) = (S_0 - C_0) \exp(rT), \tag{33}$$

equation (27) becomes

$$\begin{aligned}\mu_p &= \frac{1}{T}\ln\left[\frac{(S_0 - C_0)\exp(rT)}{S_0}\right] \\ &= r + \frac{1}{T}\ln\left(1 - \frac{C_0}{S_0}\right). \end{aligned} \tag{34}$$

As C_0 is strictly positive, we must have

$$\frac{1}{T}\ln\left(1 - \frac{C_0}{S_0}\right) < 0. \tag{35}$$

It follows that

$$\mu_p < r. \tag{36}$$

Thus, property (ii) is confirmed.

The implications of property (ii) are essentially the same as those of property (i). The only exception is that the risk-free interest rate does not correspond to a threshold value for the maximum expected profit to the writer in an option investment. If the value of the drift parameter is set to equal the risk-free interest rate, as recommended in textbooks, it will favour the writer for expecting a profit. This implication of property (ii) pertains to both call and put options.

To derive property (iii), let us state put-call parity equivalently as

$$X = (S_0 - C_0 + P_0)\exp(rT). \tag{37}$$

Combining equations (20) and (37) leads to

$$S_0 \exp(\mu_x T) = (S_0 - C_0 + P_0)\exp(rT), \tag{38}$$

which can be written as

$$\exp(\mu_x T - rT) = \frac{S_0 - C_0 + P_0}{S_0} \tag{39}$$

and then

$$\mu_x - r = \frac{1}{T}\ln\left(1 - \frac{C_0 - P_0}{S_0}\right). \tag{40}$$

Accordingly, the sign of $\mu_x - r$ is opposite to that of $C_0 - P_0$, with $\mu_x - r = 0$ for $C_0 - P_0 = 0$. Thus, property (iii) is confirmed.

With equation (37) being equivalent to

$$C_0 - P_0 = S_0 - X\exp(-rT), \tag{41}$$

we can state property (iii) alternatively as $\mu_x \gtreqless r$ for $S_0 \lesseqgtr X\exp(-rT)$. That is, the sign of $\mu_x - r$ is opposite to that of the difference between S_0 and the present value of X, which is

$X \exp(-rT)$. Thus, for the purpose of simulating stock and option price movements to evaluate an option investment on day 0, if S_0 is greater (less) than the present value of X, using r instead of μ_x for the drift parameter will produce, on average, simulation results that are relatively more favourable, from the perspective of the writer of a put (call) option.

3.7 Simulated Daily Stock and Option Price Movements

Although geometric Brownian motion is a stochastic process in continuous time, its implementation in simulation exercises requires that it be approximated in a discrete time setting. We assume for now that a day as a proportion of a year is short enough for such an approximation to work well. The issue as to whether there is any need for using a shorter time interval and, if so, how the Excel-based simulation exercises as described in the next section can be revised accordingly will be addressed in Section 5.

To simulate the time paths of daily stock and option prices, from the day of an option investment to the expiry date of an option, we need an explicit expression of the stock price on each day in terms of the stock price a day earlier. Such an expression is a recursive version of equation (3). Specifically, if we use t and $t + \triangle t$, instead of 0 and $T > 0$, to indicate two successive points in time, equation (3) can be written as

$$S_{t+\triangle t} = S_t \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)\triangle t + \sigma\epsilon\sqrt{\triangle t}\right]. \tag{42}$$

Now, let n be the number of days in a year. Here, the number can be based on calendar days or trading days; however, the latter is more common in practice. The time interval $\triangle t$ between two adjacent days is the proportion $1/n$ of a year. For notational convenience, let S_t and S_{t+1} be the stock prices on two adjacent days, for $t = 0, 1, 2, \ldots$, until the expiry date of the option that the stock underlies. Provided that μ and σ are stated in annual terms, we can write equation (42) as

$$S_{t+1} = S_t \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)\frac{1}{n} + \frac{\sigma}{\sqrt{n}}\epsilon\right]. \tag{43}$$

For a given initial price S_0 and given constant values of μ and σ, equation (43) will allow $S_1, S_2, S_3, \ldots$ to be generated. The idea is to use equation (43) recursively, starting from day 0; for each day, we generate a new random draw of ϵ from the standardized normal distribution for the equation to simulate the stock price of the next day. These simulated daily stock prices, in turn, will allow the corresponding call and put option prices, $C_1, C_2, C_3, \ldots$ and $P_1, P_2, P_3, \ldots$, based on equations (13) and (18), to be computed successively until the expiry date of each option.

Given the stochastic nature of price movements as characterized by geometric Brownian motion, each set of simulated time paths of stock and option prices will inevitably differ from any other set as generated repeatedly in simulation runs. From a statistical perspective, we are interested in knowing what simulated prices can be expected and how widely dispersed are such prices. Equations (4) and (6) can be used directly to compute the expected stock price and the standard deviation of stock prices, respectively, on each day until the expiry of the option that the stock underlies. With t being a day label, we simply substitute T on the right hand side of each of the two equations with t/n, for $t = 1, 2, \dots,$ until the expiry date of the option; that is,

$$E(S_t) = S_0 \exp\left(\frac{\mu t}{n}\right) \tag{44}$$

and

$$\sqrt{Var(S_t)} = E(S_t)\sqrt{\exp\left(\frac{\sigma^2 t}{n}\right) - 1}. \tag{45}$$

Suppose that, for some given values of S_0, μ, and σ, we have the results of a set of simulation runs. On each day t, the sample average of the simulated stock prices and the sample standard deviation of such prices can easily be computed with Excel. The signs as established for the various expressions in equations (7)-(11) are useful for expecting how the sample average and the sample standard deviation vary with t and with one or both of μ and σ. Further, the threshold value of T as established in equation (12), with $T = t/n$, is useful for determining whether a downward sloping segment in the time path of the sample standard deviations of simulated prices can exist before the expiry date of an option that the stock underlies.

4 Some Excel-Based Simulation Exercises

Having covered various analytical materials, we now turn our attention to some Excel-based simulation exercises. The illustrations below, based on four Excel files accompanying this paper, are presented in two subsections. The first subsection is to illustrate, given available information on the day of an option investment, how a set of time paths of corresponding stock and option prices can be simulated. The task requires primarily Excel's general computational and graphical tools. The second subsection illustrates how multiple simulation runs can be generated, with the simulation results organized and displayed. For computational and graphical convenience, VBA programming in Excel is utilized.

4.1 Individual Simulation Runs

Figure 1 shows part of an Excel worksheet that illustrates a simulation exercise pertaining to a call option investment. The shaded cells are for user input. They include B2:B5 for r, S_0, X, and C_0, B15 for σ, F2:F3 for T, and F23 for μ. The value of T, as shown in F5 (=F3/F2), is deduced from the number of days in a year (in F2) and the number of days to expiry (in F3). Such numbers can be based on calendar days or trading days. Besides the obvious requirements that none of the input data, with the exception of μ, be negative, the basic option conditions of $0 \leq C_0 \leq S_0$ and $C_0 \geq S_0 - X\exp(-rT)$ must also be satisfied to ensure self-consistency of the input data. Given that r and X, as well as T on the day of an option investment, are strictly positive and finite for simulation runs, these conditions can be restated as $0 < C_0 < S_0$ and $C_0 > S_0 - X\exp(-rT)$, with the equality signs omitted.

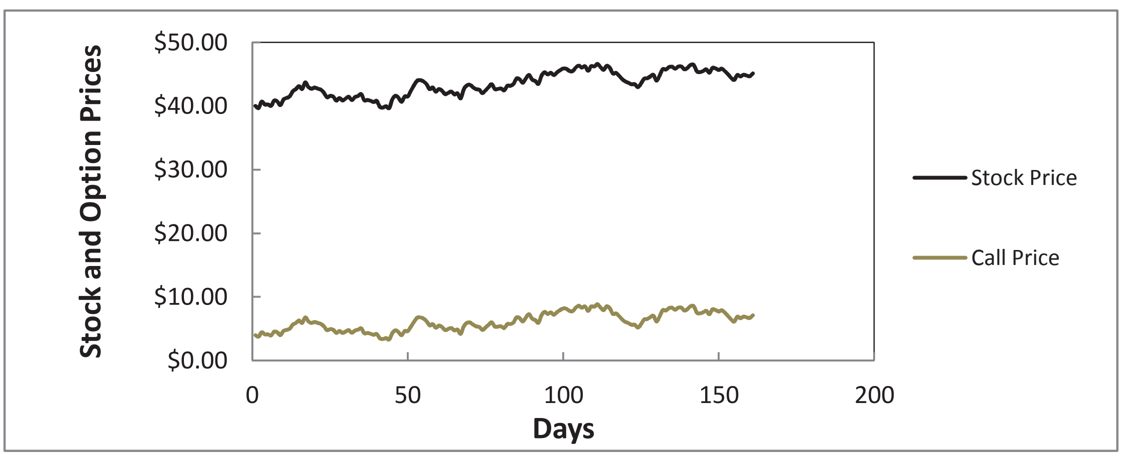
Further guidance for user input is also available. Specifically, we show in B7 $\max[0, S_0 - X\exp(-rT)]$, the lower bound of C_0. For the input data to be self-consistent, C_0 must exceed the computed value in B7. Consistency check of the input data is performed in B8 via the formula =IF(OR(call>=IPr,call<=IPr-exPr*EXP(-Rf*Time/dC),call<=0),"Fail","Pass"). Here, "Rf, IPr, exPr, call, dC," and "Time" are the names assigned to B2, B3, B4, B5, F2, and F3, respectively. If B5 is the last cell among B2:B5 and F2:F3 for data entry, data validation can be performed there as well. This can be achieved via Data Validation under Data on the menu bar. In the Settings tab there, the validation criteria are set for Allow|Custom with the following Formula:

```
=AND(CELL("contents")<IPr,CELL("contents")
   >IPr-exPr*EXP(-Rf*Time/dC),CELL("contents")>0)
```

Here, CELL("contents") represents user input for B5. A nice feature of Data Validation is that any violation of the three conditions — $C_0 < S_0$, $C_0 > S_0 - X\exp(-rT)$, and $C_0 > 0$ — will cause an error message to appear and that the simulation exercise will not proceed until the input data are properly corrected. In the example as shown in Figure 1, as $r = 3\%$, $S_0 = \$40$, $X = \$38$, $T = 160/250 = 0.64$, $C_0 = \$4$, and $\max[0, S_0 - X\exp(-rT)] = \$2.72$, none of the basic option conditions are violated.

To compute the implied volatility, we first enter an arbitrary initial value of σ to B10, named "StdDev" for computational convenience. Based on this initial value of σ and the given values of S_0, X, r, and T, the corresponding C_0 is computed, by using equations (13)-(15), and stored in B13. The difference between (1) the given C_0 in B5 and (2) the computed C_0 in B13 is computed in F8. We then use Solver to search for the value of σ in "StdDev" that makes the difference equal

	A	B	C	D	E	F
1	Simulation of Stock and Call Option Prices					
2	Risk-free Int Rate	3.00%			Number of Days in a Year	250
3	Initial Stock Price	$40.00			Number of Days to Expiry	160
4	Exercise Price	$38.00			(enter up to 500)	
5	Call Price, User Input (1)	$4.00			Expiry, in Years	0.64
6						
7	Call Price, Lower Bound	$2.72			Run Solver for Implied Volatility	
8	Consistency Check	Pass			Difference, (1) - (2)	$0.00
9						
10	Implied Volatility	19.46%			Choice of Volatility	Indicator
11	d1	0.53059			Implied Volatility	1
12	d2	0.37489			User Input	
13	Call Price, Computed (2)	$4.00				
14					Choice of Drift	Indicator
15	Volatility, User Input	25.00%			Expected Price = Exercise Price	2
16	d1	0.45247			Expected Profit = 0	
17	d2	0.25247			Risk-free Interest Rate	
18	Call Price, Computed	$4.63			User Input	
19						Drift
20	Volatility for Simulation	19.46%			Exp. Price = Exercise Price	-8.01%
21	Call Price for Simulation	$4.00			Expected Profit = 0	7.91%
22	Drift for Simulation	7.91%			Risk-free Interest Rate	3.00%
23					User Input	5.50%
24	Stock Price at Expiry	$45.13				
25	Profit for Buyer	$3.05				
26–37						



	A	B	C	D	E	F
38	d1	d2	Random	Day	Stock Price	Call Price
39	0.53059	0.37489		0	$40.00	$4.00
40	0.47808	0.32287	0.2461	1	$39.67	$3.76
41	0.64037	0.48564	0.97817	2	$40.68	$4.47
42	0.56749	0.41325	0.173	3	$40.22	$4.13

Figure 1 An Excel Example Illustrating the Time Paths of Corresponding Stock and Call Option Prices from a Simulation Run

to 0. The search result is the implied volatility.[15] Alternatively, if the value of σ for use in the simulation exercise is based on user input in B15 instead, the corresponding C_0 is stored in B18.

The selections of the volatility parameter and the drift parameter are from choosing items in two List Boxes. Each box is produced via Insert|Controls under the Developer tab on the menu bar. For the volatility parameter, if the choice is the implied volatility, the indicator in F11 for the corresponding list box will show a 1, and the values of σ and C_0 for the simulation exercise will be from B10 and B13, respectively. Otherwise, with the indicator in F11 showing a 2, the corresponding values will be from B15 and B18 instead. The values of σ and C_0 in B20:B21, for use in the simulation exercise, will depend on which of the two choices is selected. Chosen for Figure 1 is the implied volatility; that is, user input of the call price in B5 is used.

For the drift parameter, the indicator for the corresponding list box will show in F15 1, 2, 3, or 4 instead. The four choices of the drift parameter are listed in F20:F23 as indicated, with F23 being from user input. The value of μ to be used for the simulation exercise is indicated in B22. Chosen for Figure 1 is μ_c, the case of zero expected profit for either the buyer or the writer of the option; it corresponds to a 2 for the indicator in F15.

To simulate the time paths of stock and option prices based on values of r, S_0, X, σ, C_0, and μ in B2:B4 and B20:B22, the corresponding values of d_1, d_2, S_0, and C_0 are computed again and displayed in A39:B39 and E39:F39, with D39 indicating day 0, the day of the option investment. The simulation results for day 1 are displayed in A40:F40. Specifically, by using the Excel function RAND, C40 generates a random draw from a uniform distribution in the range of 0 to 1. This random number, when interpreted as a cumulative probability, allows the Excel function NORMSINV to generate the corresponding value of ϵ, a random variable from the standardized normal distribution, in equation (43). For the simulated stock price S_1 in E40 according to equation (43), as $t = 1$, S_{t-1} is given by S_0 in E39, and n is given by F2, under the cell name "dC." The corresponding simulated call price C_1 in F40 is based on equations (13)-(15), with the special case of day 1 being the expiry data of the option also accommodated; in such a case, we simply let $C_1 = \max(0, S_1 - X)$.

As the day label in D40 is based on D39+1, A40:F40 when copied to A40:F540 allows us to accommodate the expiry of the option for as many as approximately 500 days since day 0. However, as shown in the representative cell formulas of the Excel worksheet in Figure 1, the use of some

[15] The numerical search can also be performed with Goal Seek. See, for example, the Macroption website <http://www.macroption.com/implied-volatility-excel/> for an illustration that uses Goal Seek to search for the implied volatility. A disadvantage with Goal Seek, however, is that the information for its dialog box has to be re-entered for each new search. In contrast, Solver retains all previously entered information in the worksheet until it is cleared or overwritten.

appropriate IF statements will ensure that the simulated prices always end on the expiry date of the option. To generate a different set of random numbers for a new simulation run, all that is required is to trigger a recalculation, by changing anything or by pressing F9.

For the example in Figure 1, the option expires in 160 days. Although the numerical results from a simulation run are displayed until row 199 of the Excel worksheet, the part below row 42 is omitted from Figure 1; however, graphical results for the simulated stock and option prices in D39:F199 are still provided.[16] With the Axis Options of the graph set to be "Auto," instead of "Fixed," we are able to accommodate various inputs for B3 and F3.

The simulated stock price at the expiry date of the option, which is displayed in E199 for day 160, is duplicated in B24. This is achieved by using the Excel function OFFSET; specifically, noting that the value in F3 under the cell name "Time" is 160, we use the formula =OFFSET(E39,Time,0) in B24 to display the value of E199, which is 160 rows below E39 in the same column. The corresponding profit for the buyer of the option, with a negative value indicating a loss, is shown in B25. The computation is based on $\max(0, S_t - X) - C_0 \exp(rT)$, where $S_t = S_{160}$ is the simulated stock price in B24 and, as of day 0, $T = 160/250$ is the proportion of a year before the option expires.

Figure 2 shows part of an Excel worksheet for simulating stock and put price movements with a different set of input data, which include $r = 3\%$, $S_0 = \$40$, $X = \$39.50$, $T = 125/250 = 0.5$, and $P_0 = \$2$. Like Figure 1, only the part containing A1:F42 is shown here. The worksheets accompanying the two figures are similar. Indeed, to change from equation (13) to equation (18) requires only minor algebraic adjustments. Further, in view of put-call parity, so does changing option properties from those of a call option to those of a put option, for the purpose of simulation exercises.

However, there are some notable differences between the two figures. Specifically, to ensure self-consistency in user input, the condition of $P_0 < X \exp(-rT)$ is shown explicitly in B6 of Figure 2. In contrast, the corresponding condition of $C_0 < S_0$ for a call option is so obvious that a reminder in B6 for Figure 1 is unnecessary. The cell formula in B8 for consistency check also differs in the two figures; in Figure 2, it is =IF(OR(put>=exPr*EXP(-Rf*Time/dC),put<=exPr*EXP(-Rf*Time/dC)-IPr,put<=0),"Fail","Pass") instead, where "put" is the name assigned to P_0 in B5. In case that B5 is the last cell among B2:B5 and F2:F3 for data entry, the corresponding formula

[16] See A24:F199 of the Excel worksheet accompanying Figure 1 for the results of a simulation run based on the current set of input data. Notice that the number of rows as required to display the simulation results depends on the number of days before the option expires. In general, if the option expires in t days as indicated in F3, the last row for the display will be row $39 + t$. See also G1:O42 in the same Excel worksheet for a complete list of cell names, as well as some representative cell formulas.

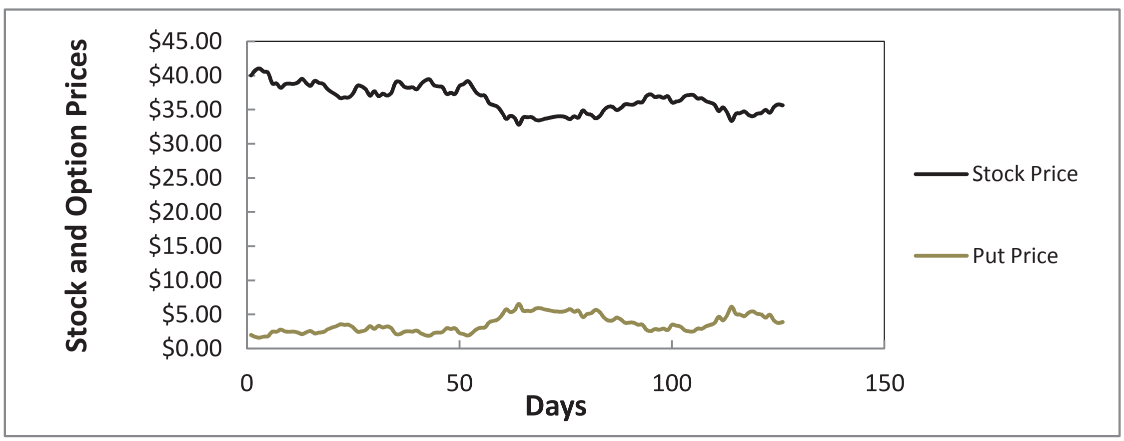
	A	B	C	D	E	F
1	Simulation of Stock and Put Option Prices					
2	Risk-free Int Rate	3.00%			Number of Days in a Year	250
3	Initial Stock Price	$40.00			Number of Days to Expiry	125
4	Exercise Price	$39.50			(enter up to 500)	
5	Put Price, User Input (1)	$2.00			Expiry, in Years	0.5
6	Put Price, Upper Bound	$38.91				
7	Put Price, Lower Bound	$0.00			Run Solver for Implied Volatility	
8	Consistency Check	Pass			Difference, (1) - (2)	$0.00
9						
10	Implied Volatility	22.55%			Choice of Volatility	Indicator
11	d1	0.25271			Implied Volatility	1
12	d2	0.09329			User Input	
13	Put Price, Computed (2)	$2.00				
14					Choice of Drift	Indicator
15	Volatility, User Input	25.00%			Expected Price = Exercise Price	2
16	d1	0.24440			Expected Profit = 0	
17	d2	0.06762			Risk-free Interest Rate	
18	Put Price, Computed	$2.27			User Input	
19						Drift
20	Volatility for Simulation	22.55%			Exp. Price = Exercise Price	-2.52%
21	Put Price for Simulation	$2.00			Expected Profit = 0	-13.07%
22	Drift for Simulation	-13.07%			Risk-free Interest Rate	3.00%
23					User Input	2.00%
24	Stock Price at Expiry	$35.62				
25	Profit for Buyer	$1.85				
26–37						
38	d1	d2	Random	Day	Stock Price	Put Price
39	0.25271	0.09329		0	$40.00	$2.00
40	0.37008	0.21130	0.9123	1	$40.75	$1.71
41	0.41378	0.25564	0.70102	2	$41.04	$1.60
42	0.34221	0.18472	0.22667	3	$40.58	$1.76

Figure 2 An Excel Example Illustrating the Time Paths of Corresponding Stock and Put Option Prices from a Simulation Run

for data validation of the cell is as follows instead:

=AND(CELL("contents")<exPr*EXP(-Rf*Time/dC),CELL("contents")
>exPr*EXP(-Rf*Time/dC)-IPr,CELL("contents")>0)

The additional cell names here are the same as those in Figure 1. The cell formula for F21 in Figure 2 — the case of a zero expected profit for either the buyer or the writer of a put option — is based on equation (27), rather than equation (24) for the corresponding cell in Figure 1. Further, the simulated profit for the buyer in B25 is based on $\max(0, X - S_t) - P_0 \exp(rT)$ instead.[17]

4.2 Multiple Simulation Runs

As the Excel files underlying Figures 1 and 2 are intended to be for illustrating one simulation run at a time, they are inconvenient for statistical purposes. To explore statistical properties of the simulation results, many repeated simulation runs are required. The Excel files, on which Figures 3 and 4 are based, utilize VBA to generate and organize multiple sets of simulated time paths of stock and option prices. The VBA code in each file can be accessed via Visual Basic under the Developer tab. Figure 3, which uses the same input data from Figure 1, is considered first.

The block A1:E23 in Figure 3 is essentially the same as A1:F23 in Figure 1. The only notable difference is the presence of a button at D6:D7 in Figure 3; pressing it will activate the part of the VBA code for finding the implied volatility with Excel's Goal Seek. Instead of responding to a dialog box each time Goal Seek is used, a three-line subroutine (or, simply, sub) in VBA — with Range("D8").GoalSeek Goal:=0, ChangingCell:=Range("B10") to search for the value in B10 that makes D8 equal to 0 — will allow the implied volatility in B10 to be determined.

The shaded cell G2 is for user input of the number of simulation runs. In Figure 3, the number is set at $2,000$. Pressing the button at F3:G4 will allow the simulation to commence. As in Figure 1, the simulation exercise is based on the use of the implied volatility, with the value of the drift parameter being μ_c, corresponding to a zero expected profit for the buyer and the writer.

Some selected simulation results for the expiry date of the option are summarized in G9:H16. Specifically, G9 shows the expected stock price, $E(S_t) = \$42.08$, based on equation (44) for $S_0 = \$40$ ("IPr" in B3), $\mu = 7.91\%$ (in E21), $n = 250$ days (in D2), and $t = 160$ days (in D3). The mean, the median, the maximum, and the minimum of the $2,000$ simulated values of S_t are displayed in G10:G13. The profit for the buyer, as measured on the expiry date of the option, is the difference of (1) the maximum of 0 and $S_t - X$ and (2) $C_0 \exp(rt/n)$, with a negative value indicating a loss. The mean, the median, the maximum, and the minimum of the $2,000$ simulated profits are

[17] See the worksheet accompanying Figure 2 for all cell names and representative cell formulas in G1:O42, as well as the results from a simulation run in A24:F164.

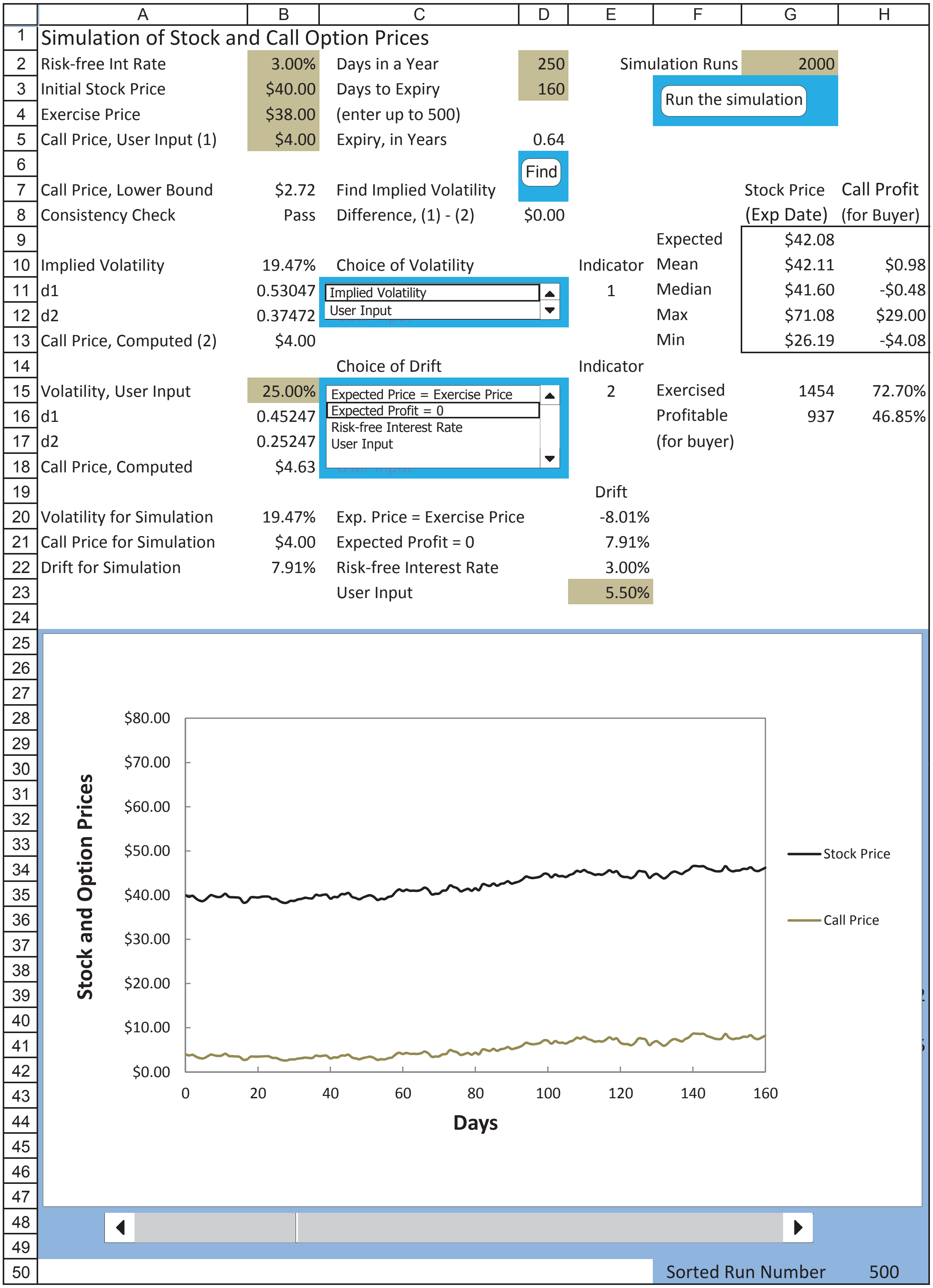


Figure 3 An Excel Example Illustrating the Time Paths of Corresponding Stock and Call Option Prices and a Summary of Various Simulation Results

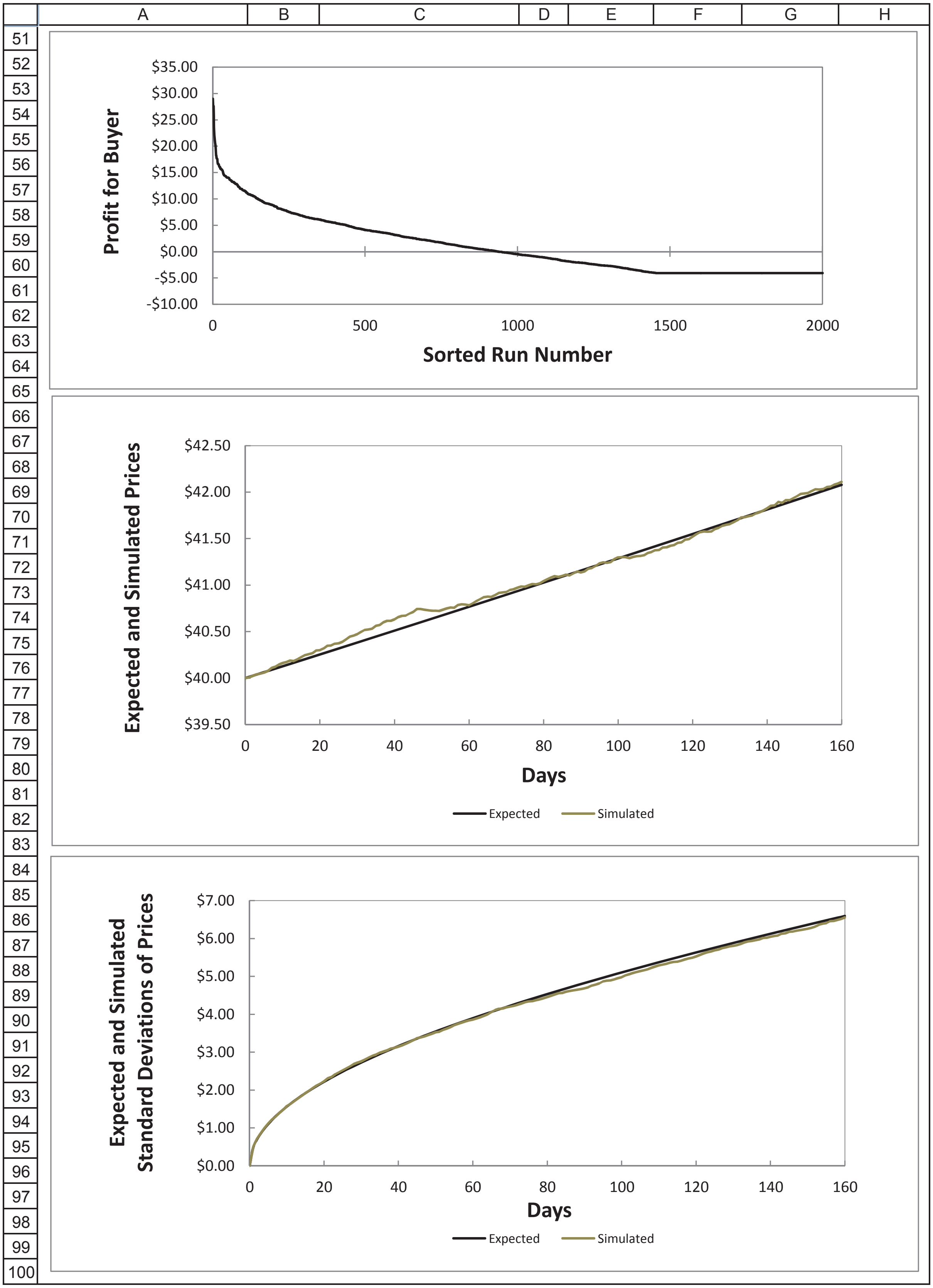


Figure 3 An Excel Example Illustrating the Time Paths of Corresponding Stock and Call Option Prices and a Summary of Various Simulation Results (Continued)

displayed in H10:H13. Out of the 2,000 simulation runs, the number of cases where the option is exercised and the number of profitable cases are shown in G15:G16. The corresponding numbers in percentage terms are provided in H15:H16 as well.

The simulated daily stock and option prices over the remaining life of the option, as of day 0, are stored in two separate worksheets. For graphical convenience, the time paths of these prices are sorted in a descending order of the simulated stock prices on the expiry date of the option, with each row of a worksheet storing a simulated time path.[18] A scroll bar is provided in the vicinity of A48:G49, with the selected value from it stored in H50, named "RunDisplayed" for programming and graphical convenience. In Figure 3, with the selected value being 500, the corresponding stock and option prices — the 500-th pair of sorted prices — over the 160 days since the option investment on day 0 are shown graphically.[19] Given the way the prices are sorted, the lower the number selected for H50, the greater will be the upward price movements over time in the graphical display.

The second graph in Figure 3 shows the simulated profit for the buyer versus the sorted run number. This graph complements the numerical results in G9:H16. The horizontal part of the graph captures all cases where $S_t \leq X$ on the expiry date of the option; in such cases, the option will not be exercised. This graph illustrates that, from the perspective of the buyer, while the downside risk of an option investment is always limited, its upside potential is great.

The third graph in Figure 3 compares the expected price and the average of the simulated prices of the underlying stock over the remaining life of the call option. The fourth graph compares instead the corresponding expected and simulated standard deviations of such prices. The expected values for each of the 160 days are based on equations (44) and (45). These values are compared against the corresponding average and standard deviation based on the 2,000 simulated stock prices each day.

It is the nature of simulation that repeating a set of simulation runs seldom yields the same set of results. However, if the number of runs is adequately high, the differences in the results among various sets of runs will not be substantial. According to the graphical results in Figure 3, as

[18] To make the Excel files accompanying this paper accessible to more readers, each is saved as a 1997-2003 version. The maximum number of columns in a worksheet in such a version, which is only 256, does limit what can be entered to D3, the cell for the number of days until expiry. Readers who use Excel 2007 or 2010 can extend the column limit to 16,384 by saving the corresponding file as a macro-enabled version.

[19] For VBA programming convenience, the simulated stock and option prices are sorted separately. To avoid any mismatch of the sorted prices, the sorting of option prices is based on the difference between the stock price on the expiry date and the exercise price, rather than the greater between zero and such a difference. The exercise price being a constant, both stock and option prices can be viewed as being sorted according to the stock price on the expiry date. As the unsorted stock and option prices are stored in the same order as they are generated, the sorting in Excel will not alter their relative positions in case of a tie.

well as those based on different sets of input data and 500 or more simulation runs, the average of simulated stock prices and standard deviations of such prices are seldom far from the corresponding expected values. Further, the signs of the various expressions in equations (7)-(11) are found to hold. With values of μ being positive, the corresponding time paths of standard deviations of simulated stock prices are upward sloping, as expected.

Figure 4, which shows the simulation results for an investment in a put option, is based on the same set of input data as in Figure 2. The part of Figure 4 that duplicates the corresponding part in Figure 2 requires no further descriptions. The Excel files corresponding to Figures 3 and 4 are so similar that only a brief mention of the latter is adequate.

Like Figure 3, Figure 4 shows a set of four graphs. The first graph displays the simulated time paths of stock and put prices for sorted run number $1,500$, out of a total of $2,000$ runs. Given the way simulated prices are sorted, with the highest stock price on the expiry date of the put option treated as that from run number 1, the time path of simulated stock prices in run number $1,500$ shows a small downward trend. Accordingly, the trend in the put prices is slightly upward. The second graph — a graph of the profit for the buyer versus the sorted run number — has a shape that is opposite to that in Figure 3. That is, the horizontal part of the second graph in Figure 4 corresponds to low sorted run numbers. As the value of the drift parameter for the simulation runs is negative, both expected and average simulated stock prices as shown in the third graph in Figure 4 are downward sloping. Provided that there are 500 or more simulation runs, the expected and average simulated stock prices tend to match quite well. So do the expected and simulated standard deviations of stock prices in the fourth graph.

Notice that, in the fourth graph, $\sqrt{Var(S_t)}$ always increases with t between day 0 and day 125, the expiry date of the option, although the value of the drift parameter used in the simulation runs is negative, with $2\mu + \sigma^2 = -21.06\%$. Thus, provided that μ and σ remain constant over time, $\sqrt{Var(S_t)}$ will eventually decrease with increasing t beyond a certain date in the future. According to equation (12), the threshold value of T for $\partial\sqrt{Var(S_t)}\Big/\partial T$ to become negative is $T^* = 4.254$ years or about $1,064$ trading days.[20]

5 Suggestions for Further Simulation Exercises

The simulation exercises as described in Section 4 can be refined and extended in various ways. In this section, we offers some suggestions that are suitable for use in student projects in financial

[20] The analytical materials in Subsection 3.1, which connect $E(S_T)$ and $Var(S_T)$ to their underlying parameters, can also be examined numerically via the same simulation exercises as illustrated in Figures 3 and 4. These exercises will enhance students' understanding of such analytical materials.

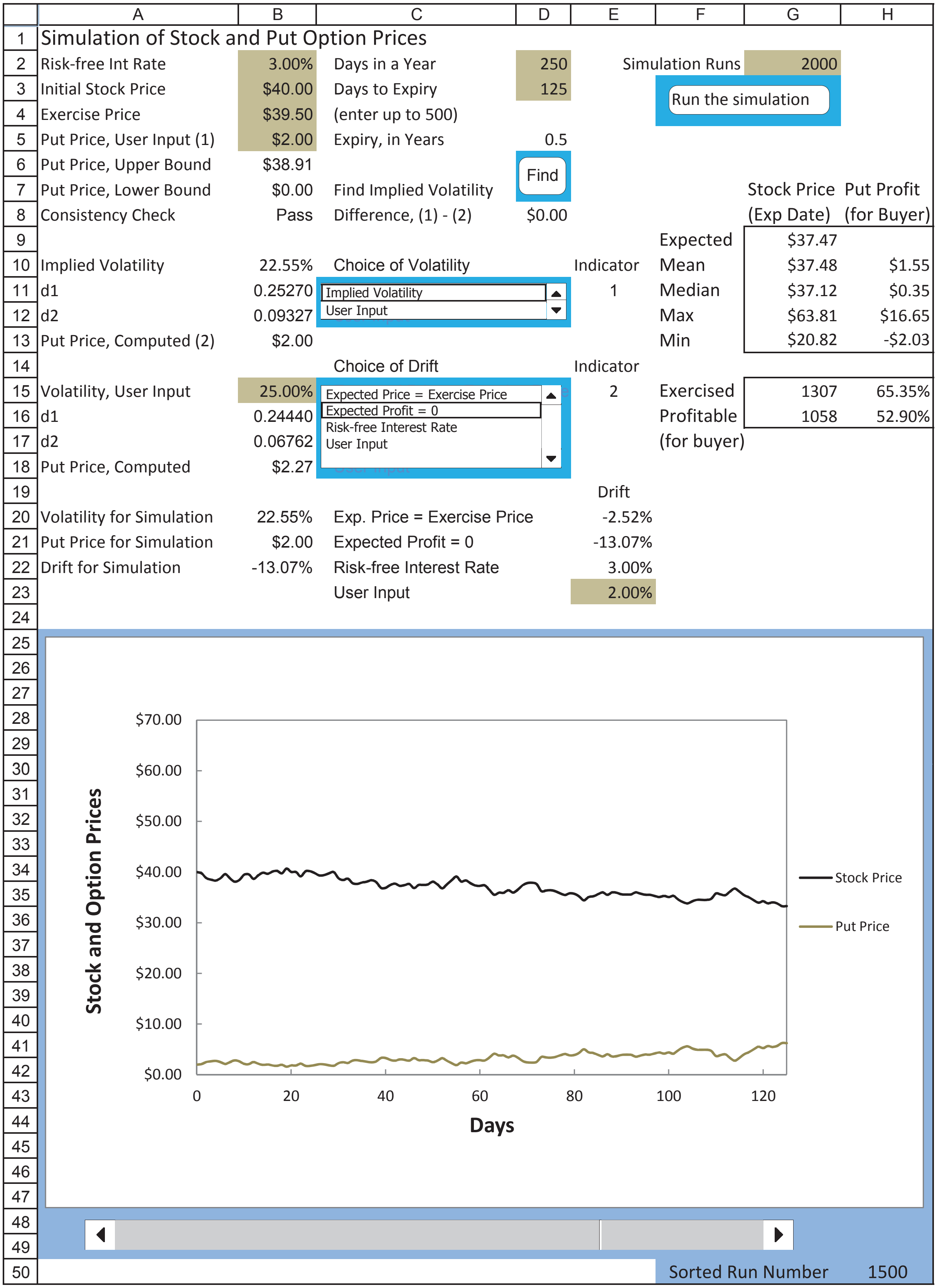


Figure 4 An Excel Example Illustrating the Time Paths of Corresponding Stock and Put Option Prices and a Summary of Various Simulation Results

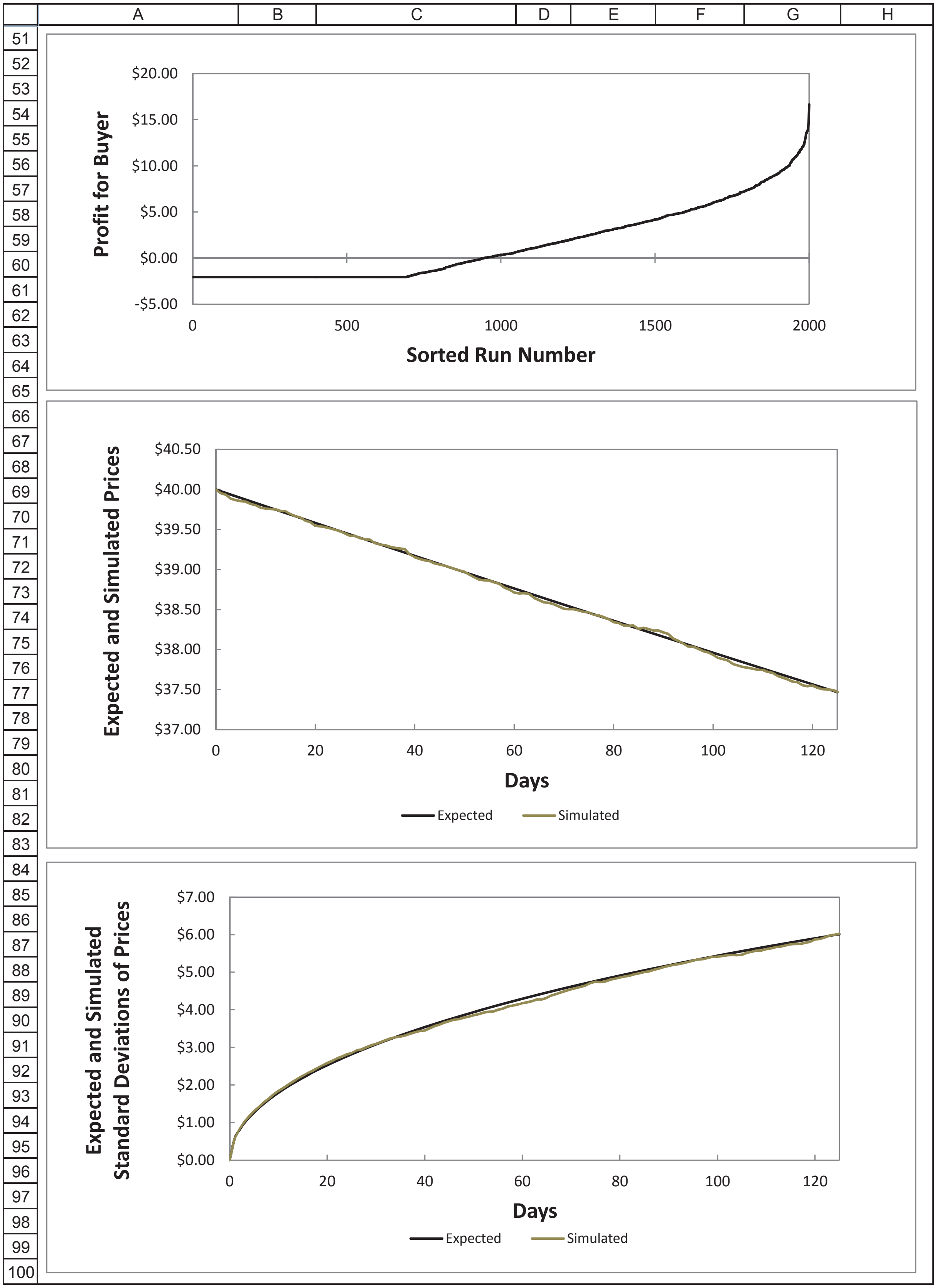


Figure 4 An Excel Example Illustrating the Time Paths of Corresponding Stock and Put Option Prices and a Summary of Various Simulation Results (Continued)

modeling courses, as well as investment courses that cover the Black-Scholes option pricing model in some detail. Although students in the latter courses may not be familiar with VBA coding, they can still use the same Excel files accompanying the four figures of this paper to explore various relevant investment issues.

5.1 An Implication of Geometric Brownian Motion: Lognormally Distributed Stock Prices

One of the implications of geometric Brownian motion for characterizing the stochastic process of stock price movements is that stock prices are lognormally distributed. That is, the logarithmic stock prices are normally distributed. The exercise as suggested below is based on such an implication. The idea of the exercise is to examine the distributions of the simulated stock prices for various combinations of the values of the two parameters in geometric Brownian motion, as observed on some selected days between day 0 and the expiry date of the option that the stock underlies.

In the simulation exercises as illustrated in Figures 3 and 4, each set of simulated stock prices is stored in a worksheet for graphical convenience. The simulated time path of stock prices in the first graph of each figure is for the sorted run number in H50 as specified by the user via a scroll bar. The same idea can also be used to select all simulated stock prices for any specified day.

In the Excel file accompanying Figure 3, the $2,000$ sets of simulated stock prices over 160 days are stored in a block of cells covering $2,000$ rows and 160 columns of a worksheet. Just like retrieving the prices in the 500-th row, as specified by the sorted run number 500 in H50, for generating the first graph of Figure 3, we can also retrieve the prices in any specified column of the same block for an exercise to examine the price distribution for the corresponding day. Guided by the idea of the VBA code for Figure 3, students in financial modeling courses are expected to be able to perform such a task. For students in investment courses who are unfamiliar with VBA coding, the simulated stock prices on any given day can still be retrieved manually from the simulation results. As Excel can graph distributions in the form of histograms, to examine how simulated stock prices are distributed is not an onerous task. Further, as normality tests are well established, statistical statements pertaining to the goodness of fit can also be made.

To illustrate, suppose that the simulated stock prices for a given day have been retrieved and transformed logarithmically. The range of values of the transformed prices can be used to produce a number of bins to place them, so that a histogram can be drawn. The histogram feature in Excel is accessible via Data Analysis under the Data tab. Alternatively, the Excel function

FREQUENCY, which is more versatile for the same task, can be used instead. The histogram as produced either way, when displayed along with a normal distribution (based on the mean and the standard deviation of the transformed prices), will reveal how well the transformed prices follow a normal distribution.

A well-known statistical test for the goodness of fit, which can easily be implemented in Excel as a normality test, is a chi-square test. The test statistic is

$$\chi^2 = \sum_{i=1}^{k} \frac{(o_i - e_i)^2}{e_i}, \tag{46}$$

where k is the number of bins, o_i is the observed frequency in bin i, and e_i is the expected frequency in bin i according to a specific distribution. The test statistic follows approximately a chi-square distribution with $k - h - 1$ degrees of freedom, where h is the number of parameters characterizing the distribution in question. In the case of a normal distribution, which is characterized by its mean and standard deviation, as $h = 2$, we have $k - 3$ degrees of freedom for the chi-square test. The Excel function CHIDIST can be used directly to provide the p-value of the test statistic.[21]

5.2 Simulation in Discrete Time Based on a Stochastic Process in Continuous Time

As a stochastic process, geometric Brownian motion is intended to be for a continuous time setting. When this stochastic process is applied to simulation exercises involving daily stock and option prices, as is done in this paper, an implicit assumption is that it can work well for such a discrete time setting. From a pedagogic perspective, therefore, a relevant issue is whether a day is short enough to approximate an instantaneous time interval well. If so, also of pedagogic relevance is whether a longer time interval, such as a week, a month, or a quarter of a year, can still be short enough for the corresponding simulation results to be meaningful. Otherwise, of interest is how short a time interval, in terms of a proportion of a day, is required for a discrete time approximation of a continuous time stochastic process to work adequately.

In the Excel files accompanying Figures 3 and 4, the two cells in D2:D3 are for user input of the number of days in a year and the number of days for an option to expire. However, they need not be measured in days. If we divide a year into 52, 12, or 4 equal time intervals instead and enter the corresponding number to D2, the expiry of the option in D3 (which must be an integer) will be measured in weeks, months, or quarters. Alternatively, if we divide a year into progressively greater numbers of equal time intervals, such as multiples of 250 (365) and replace the numbers in

[21] See, for example, Anderson, Sweeney, and Williams (2008, Chapter 12) for a description of the chi-square test and Black and Eldredge (2001, Chapter 11) for an Excel illustration.

D2:D3 with such multiples, each time interval can be interpreted as a proportion of a trading day (calendar day).

For example, instead of entering n and t to D2:D3 of either Excel worksheet that utilizes VBA, such as 250 and 160 in Figure 3, we can enter mn and mt there, with $m = 1, 2, 3, \ldots,$ for successive sets of simulation runs. If $m = 2$, each interval represents one half of a day; if $m = 3$, it is one-third of a day instead. A similar idea applies to higher values of m. Regardless of m, $T = mt/(mn) = t/n$, which is the proportion of a year before the option expires as of day 0, remains the same.

With a year divided into different sets of equal time intervals, ranging from a quarter of a year to progressively shorter intervals, we can use a common set of input data to compare the corresponding simulation results. For each set of input data, T is based on a common ratio of the integers in D3 and D2. As these simulation runs are based on a common set of input data and the same choice of the values of the two parameters in geometric Brownian motion, improvements in the simulation results for shortening the time intervals involved, if any, will be noticeable.

In the case of an investment in a call (put) option, if the value of the drift parameter for simulation runs is based on μ_c (μ_p), the benchmark for assessing the adequacy of the corresponding simulation runs is 50% for H16 in Figure 3 (Figure 4). Likewise, if the value of the drift parameter is based on μ_x, the benchmark is 50% for H15 instead. For more in-depth examination, distributions of the simulated stock prices on some selected dates, including the expiry date of the option, can be utilized as well. As indicated in the previous subsection, geometric Brownian motion implies that stock prices are lognormally distributed. It will be a good exercise for students to explore, with different sets of input data, how the simulated stock prices can match the theoretical distributions, as a year is divided into progressively shorter equal time intervals.

It is true that dividing a year into smaller time intervals always provides better approximations of a continuous time setting with a discrete time setting. However, any increases in the numbers in D2:D3, when combined with a high number of simulation runs, will inevitably increase both the computer time to perform the task and the storage space for the individual simulation results. The exercises as suggested above will help in establishing a practical trade-off between computational efficiency and adequacy of the simulation runs involved.

5.3 Combinations of Call and Put Options

There are various other option-based simulation exercises that are also suitable for pedagogic purposes. The idea is that an option investment need not be confined to a call option or a put option alone. Investment strategies involving combinations of different options can also be assessed via simulation exercises. For example, a combination of a call option and a put option on a stock, with the same expiry date and the same exercise price, is called a straddle. A large increase or decrease in the underlying stock price is beneficial to the buyer of a straddle. In contrast, it is a stable stock price that ensures the writer's profit.

In simulation exercises involving a straddle, the threshold value μ_x of the drift parameter can still be deduced from equating the expected stock price $E(S_T)$ and the exercise price X. That is, equation (21) still holds. However, instead of using the same threshold values, μ_c and μ_p, as established in equations (24) and (27), respectively, we must revise them to account for an increase in the investment cost for the buyer on day 0.

Specifically, for a revised μ_c, we start with

$$E(S_T) = (C_0 + P_0)\exp(rT) + X, \tag{47}$$

where $E(S_T)$ is given by equation (4). This is the situation where exercising the call option by the buyer of the straddle provides no expected profit for either side of the option investment. Instead of equation (24), we now have

$$\mu_c = \frac{1}{T}\ln\left[\frac{(C_0 + P_0)\exp(rT) + X}{S_0}\right]. \tag{48}$$

The threshold value here is higher than that in equation (24) because of an increased investment cost for the buyer on day 0.

Likewise, for a revised μ_p, we start with

$$(C_0 + P_0)\exp(rT) + E(S_T) = X, \tag{49}$$

which corresponds to the situation where no profit is expected for either side if the put option is exercised. Equation (27) now becomes

$$\mu_p = \frac{1}{T}\ln\left[\frac{X - (C_0 + P_0)\exp(rT)}{S_0}\right]. \tag{50}$$

Accounting for an increased investment cost to the buyer, the threshold value here is lower than that in equation (27).

The above approach for deducing threshold values of the drift parameter, which is based on a zero expected profit, will also work well for many other option strategies. For example, a strap differs from a straddle in that it is a portfolio of two call options and a put option. A strip is the opposite; it is a portfolio of a call option and two put options. Option strategies such as a butterfly and its variants involve portfolios of call options with different exercise prices. In contrast, a strangle and its variants require the use of put and call options with different exercise prices.

The presence of more than one exercise price for one type of options will inevitably increase the number of threshold values of the drift parameter. However, from a computational standpoint, the change from simulating prices of an option to that of a portfolio of options is a simple extension. Simulation exercises involving various option strategies will enhance students' understanding of geometric Brown motion as a stochastic process and simulation in Excel as a practical numerical tool.

To implement a simulation exercise based on any of the option strategies above, students in investment courses who are unfamiliar with VBA coding can still follow the same idea in the Excel files accompanying Figures 1 and 2. Although the simulation can only be performed with one run at a time, the graphical and numerical results from repeated simulation runs will still be able to provide useful information to assess the strategy involved. For students in financial modeling courses where VBA coding is part of the course elements, to revise the Excel files accompanying Figures 3 and 4, for assessing the profitability of any of the above-mentioned option strategies via a simulation exercise, is a manageable task.

6 Concluding Remarks

This paper has presented some simulation exercises that are suitable for use not only in Excel-based financial modeling courses, but also in some investment courses. Such exercises are intended to help students gain some hands-on experience in working with geometric Brownian motion and the well-known Black-Scholes formula, as well as the concept of put-call parity. Being the simplest stochastic process that underlies the derivation of the Black-Scholes model for option pricing, geometric Brownian motion is important from a pedagogic perspective.

The approach of this paper differs from that in typical financial modeling courses because of its emphasis on the analytical underpinning of the materials involved. It is true that, given the technical nature of such courses, a recipe-based approach still works very well for great many finance topics. However, a recipe-based approach, when applied to simulation exercises for assessing the profitability of an option investment, does not always work well. A problem is the lack of adequate textbook guidance in assigning appropriate values of the drift parameter for use in simulation runs.

In spite of the fact that the choice of the value of the drift parameter does depend on the subjective view of the investor involved, the choice should not be entirely arbitrary and unguided. As explained in this paper, neither should it simply be set equal to the risk-free interest rate, which tends to favour the writer when simulating the profitability of an option investment. Using the idea that a rational individual never willingly chooses to invest for an expected loss, as well as various other ideas, this paper has provided some guidance in setting appropriate values of the drift parameter for simulation runs.

To understand a stochastic process of price movements properly will require students to have knowledge in stochastic calculus and advanced statistics. As such topics are beyond the scope of the standard finance curriculum, such requirements have given us challenges as instructors of finance courses at various academic levels. It is hope that the Excel-based simulation exercises in this pedagogic paper, as supported by the underlying analytical materials and further Excel-based exercises, can help students reduce the conceptual burden in learning stochastic processes.

References

Anderson, D.R, Sweeney, D.J., and Williams, T.A., (2008), *Statistics for Business and Economics, Tenth Edition*, Thomson South-Western, Mason, Ohio.

Benninga, S., (2008), *Financial Modeling, Third Edition*, MIT Press, Cambridge, Massachusetts.

Black, F., and Scholes, M., (1973), "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81(3), 637-654.

Black, K., and Eldredge, D.L., (2001), *Business and Economic Statistics Using Microsoft Excel*, South-Western College Publishing, Cincinnati, Ohio.

Campbell, J.Y., Lo, A.W., and MacKinlay, A.C., (1997), *The Econonometrics of Financial Markets*, Princeton University Press, Princeton, New Jersey.

Chance, D.M., and Brooks, R., (2008), *An Introduction to Derivatives and Risk Management, Seventh Edition*, Thomson South-Western, Mason, Ohio.

Copeland, T.E., Weston, J.F., and Shastri, K., (2005), *Financial Theory and Corporate Policy, Fourth Edition*, Pearson Addison Wesley, Boston, Massachusetts.

Holden, C.W., (2011), *Excel Modeling in Corporate Finance, Fourth Edition,* Pearson Prentice Hall, Upper Saddle River, New Jersey.

Hull, J.C., (2009), *Options, Futures, and other Derivatives, Seventh Edition*, Pearson Prentice Hall, Upper Saddle River, New Jersey.

Lo, A.W., and Wang, J., (1995), "Implementing Option Pricing Models When Asset Returns Are Predictable," *Journal of Finance*, 50(1), 87-129.

Sengupta, C., (2004), *Financial Modeling Using Excel and VBA*, John Wiley & Sons, Inc., Hoboken, New Jersey.

Yang, D., and Zhang, Q., (2000), "Drift-Independent Volatility Estimation Based on High, Low, Open, and Close Prices," *Journal of Business*, 73(3), 477-491.

Appendix A: Derivations of Various Analytical Expressions Based on Geometric Brownian Motion

Equation (1) in the main text can be viewed as the description of an Itō process,

$$dS = a(S,t)dt + b(S,t)dz, \tag{A1}$$

where $a(S,t)$ and $b(S,t)$ are functions of S and t. Specifically, these two functions are $a(S,t) = \mu S$ and $b(S,t) = \sigma S$. According to Itō's lemma, we have, for any twice differentiable function $G(S,t)$ of S and t,

$$dG = \left(\frac{\partial G}{\partial S} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} b^2 \right) dt + \frac{\partial G}{\partial S} b dz. \tag{A2}$$

Here, for notational simplicity, the arguments of the functions $a(S,t)$ and $b(S,t)$ are not displayed. An informal derivation of Itō's lemma, which can be found in options textbooks, is as follows:[22]

The derivation starts with Taylor's expansion of $G(S,t)$, which gives

$$\triangle G = \frac{\partial G}{\partial S} \triangle S + \frac{\partial G}{\partial t} \triangle t + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} (\triangle S)^2 + \frac{\partial^2 G}{\partial S \partial t} \triangle S \triangle t + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} (\triangle t)^2 + \cdots , \tag{A3}$$

where $\triangle G$, $\triangle S$, and $\triangle t$ represent incremental changes of the corresponding variables. The idea here is to express $\triangle S$ in terms of $\triangle t$ and, as $\triangle t$ approaches zero, to ignore all terms of higher order in $\triangle t$. To implement such an idea, recall that z follows a Wiener process, under which $dz = \epsilon \sqrt{dt}$. Then, it follows from equation (A1) that

$$\triangle S = a \triangle t + b \epsilon \sqrt{\triangle t} \tag{A4}$$

and

$$(\triangle S)^2 = b^2 \epsilon^2 \triangle t + 2ab\epsilon (\triangle t)^{3/2} + a^2 (\triangle t)^2. \tag{A5}$$

[22] See, for example, Hull (2009, Chapter 12, Appendix).

The second and third terms on the right hand side of equation (A5), as compared to the first term there, are of higher order in $\triangle t$. Thus, their magnitudes attenuate at faster rates, as $\triangle t$ approaches zero.

We can deduce from $E(\epsilon) = 0$ and $Var(\epsilon) = E\{[\epsilon - E(\epsilon)]^2\} = 1$ that $E(\epsilon^2) = 1$. Accordingly, the expected value of $b^2\epsilon^2\triangle t$ is $b^2\triangle t$, and its variance is of order $(\triangle t)^2$. The term $2ab\epsilon(\triangle t)^{3/2}$ has a zero mean and a variance of $4a^2b^2(\triangle t)^3$, which is trivially small as $\triangle t$ approaches zero. Then, as $\triangle t$ approaches zero, $(\triangle S)^2$ approaches $b^2\triangle t$, which is non-stochastic; that is, we can write $(dS)^2 = b^2 dt$. As all terms beyond the first three terms in Taylor's expansion of $\triangle G$ in equation (A3) are of higher order in $\triangle t$, letting t approach zero will directly lead to Itō's lemma, as shown in equation (A2).

Now, let $G = \ln S$. As $\partial G/\partial S = 1/S$, $\partial^2 G/\partial S^2 = -1/S^2$, and $\partial G/\partial t = 0$, equation (A2) becomes

$$dG = \left[\left(\frac{1}{S}\right) a + 0 + \frac{1}{2}\left(-\frac{1}{S^2}\right)\left(b^2\right)\right] dt + \left(\frac{1}{S}\right) b dz = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz. \tag{A6}$$

We can write equation (A6) more explicitly as equation (2) in the main text.

Let us attach a time subscript to each variable in equation (2) and consider time 0 and time $T > 0$. From

$$\int_{t=0}^{T} d\ln S_t = \int_{t=0}^{T}\left(\mu - \frac{\sigma^2}{2}\right) dt + \int_{t=0}^{T} \sigma dz_t, \tag{A7}$$

we have

$$\ln S_T - \ln S_0 = \left(\mu - \frac{\sigma^2}{2}\right) T + \sigma(z_T - z_0). \tag{A8}$$

With z following a Wiener process, we can write

$$z_T - z_0 = \epsilon\sqrt{T}, \tag{A9}$$

Further, as $\ln S_T - \ln S_0 = \ln(S_T/S_0)$, equation (A8) becomes equation (3) in the main text.

Given the standardized normal distribution of ϵ, we have

$$\frac{1}{\sqrt{2\pi}}\int_{\epsilon=-\infty}^{\infty} \exp\left(-\frac{\epsilon^2}{2}\right) d\epsilon = 1 \tag{A10}$$

and

$$\begin{aligned} E\left[\exp\left(\sigma\epsilon\sqrt{T}\right)\right] &= \frac{1}{\sqrt{2\pi}}\int_{\epsilon=-\infty}^{\infty} \exp\left(\sigma\epsilon\sqrt{T}\right)\exp\left(-\frac{\epsilon^2}{2}\right) d\epsilon \\ &= \exp\left(\frac{\sigma^2 T}{2}\right)\cdot\frac{1}{\sqrt{2\pi}}\int_{\epsilon=-\infty}^{\infty} \exp\left[-\frac{\left(\epsilon - \sigma\sqrt{T}\right)^2}{2}\right] d\epsilon. \end{aligned} \tag{A11}$$

The entire term that is multiplied to $\exp\left(\sigma^2 T/2\right)$ in equation (A11) is 1, as it represents the area under a normal distribution curve with mean being $\sigma\sqrt{T}$ and variance being 1. With

$$E\left[\exp\left(\sigma\epsilon\sqrt{T}\right)\right] = \exp\left(\frac{\sigma^2 T}{2}\right) \tag{A12}$$

and

$$E(S_T) = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) T\right] E\left[\exp\left(\sigma\epsilon\sqrt{T}\right)\right], \tag{A13}$$

equation (4) in the main text follows directly.

The derivation of $Var(S_T)$ is analogous. Likewise, we start with

$$\begin{aligned} E\left[\exp\left(2\sigma\epsilon\sqrt{T}\right)\right] &= \frac{1}{\sqrt{2\pi}} \int_{\epsilon=-\infty}^{\infty} \exp\left(2\sigma\epsilon\sqrt{T}\right) \exp\left(-\frac{\epsilon^2}{2}\right) d\epsilon \\ &= \exp\left(2\sigma^2 T\right) \cdot \frac{1}{\sqrt{2\pi}} \int_{\epsilon=-\infty}^{\infty} \exp\left[-\frac{\left(\epsilon - 2\sigma\sqrt{T}\right)^2}{2}\right] d\epsilon, \end{aligned} \tag{A14}$$

which leads to

$$E\left[\exp\left(2\sigma\epsilon\sqrt{T}\right)\right] = \exp\left(2\sigma^2 T\right). \tag{A15}$$

As

$$Var(S_T) = E(S_T^2) - [E(S_T)]^2, \tag{A16}$$

$$E(S_T^2) = S_0^2 \exp\left[\left(2\mu - \sigma^2\right) T\right] E\left[\exp\left(2\sigma\epsilon\sqrt{T}\right)\right] = S_0^2 \exp\left[\left(2\mu + \sigma^2\right) T\right], \tag{A17}$$

and

$$[E(S_T)]^2 = S_0^2 \exp(2\mu T), \tag{A18}$$

equation (5) in the main text follows directly.

Equations (4) and (5) can also be deduced directly via the definition of a lognormal distribution and its basic properties. Specifically, if a random variable Y is normally distributed with expected value μ^* and variance $(\sigma^*)^2$, the distribution of $\exp(Y)$ is lognormal by definition. The expected value and the variance of $\exp(Y)$ are $\exp\left[\mu^* + (\sigma^*)^2/2\right]$ and $\exp\left[2\mu^* + (\sigma^*)^2\right]\left\{\exp\left[(\sigma^*)^2\right] - 1\right\}$, respectively. With $\ln S_T$ being normally distributed according to equations (A8) and (A9), the expected value and the variance of the distribution are $\mu^* = \ln S_0 + (\mu - \sigma^2/2)T$ and $(\sigma^*)^2 = \sigma^2 T$, respectively. Letting $Y = \ln S_T$, we have $\exp(Y) = S_T$. Once μ^* and σ^* are expressed in terms of μ, σ, and T, equations (4) and (5) will follow directly.[23]

[23] These two equations can also be found on the Wikipedia website <http://en.wikipedia.org/wiki/Geometric_Brownian_motion>.