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Partitions of integers, Ferrers-Young diagrams, and representational efficacy of spreadsheet modeling

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Keywords

partition of integers, manipulative, experiential learning, Ferrers-Young diagram, Pascal's triangle, quadratic sequence, spreadsheet modeling.

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**Partitions of integers, Ferrers-Young diagrams,
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Abstract

This article shows how concrete materials can be used in the context of partitions of integers to develop recursive definitions of concepts amenable to spreadsheet modeling. It demonstrates how numerical evidence made possible by the representational power of a spreadsheet when combined with other computational tools such as *Maple* allows for the conjecturing of rather sophisticated relationships among different types of integer partitions. A spreadsheet is then used to confirm these relations in special cases yet at a higher cognitive level than at the beginning of partitioning experiments. Finally, concrete materials are used again to demonstrate the correctness of computations. It is argued that whereas a spreadsheet is powerful enough to be used as a single computational tool in various mathematical contexts, its joint use with other tools, both digital and physical, is worth exploring in education.

Key words: partition of integers, manipulative, experiential learning, Ferrers-Young diagram, Pascal's triangle, difference equation, spreadsheet modeling.

1 Introduction

This article is an extended reflection on the author's paper presented at a mathematics teaching conference [1] and work with prospective elementary teachers enrolled in master's level elementary mathematics methods and content courses. One of the didactic foci of both the paper and the courses is the use of manipulative materials and visualization techniques in the development of abstract yet grade-appropriate mathematical concepts. One such concept the importance of which was first recognized at the end of the 18th century by Euler [11], concerns partition of positive integers into like summands. In today's classroom, using either physical or virtual manipulative, partitioning problems can be introduced even before the study of arithmetic, providing young learners of mathematics with understanding the multiplicity of answers in the tasks that involve counting numbers, informal learning to reason recursively in resolving quantitative queries, experience in recognizing different conditions under which counting problems can be formulated, and appreciating the innate complexity of the problems and their natural extensions already within a seemingly mundane context.

The material presented in this article, as its title suggests, goes beyond the pre-operational mathematics classroom with its only developmentally appropriate focus on hands-on explorations of quantitative situations that can be later evolved into abstract symbolic representations in other parts of vast K-12 mathematics curricula. Following the viewpoint, "What students can learn at any particular grade level depends upon what they have learned before" [8, p5], the article uses an elementary context to motivate more and more advanced discourse made possible by the use of technology. By the same token, the top to bottom approach to mathematics curricula [6] enables certain elements of higher concepts to be introduced at a lower level, using technology as appropriate. In particular, this article demonstrates how an electronic spreadsheet—a remarkable piece of digital technology used in many areas of education [7]—can complement the use of non-digital technological tools and how the latter type of a learning environment can be put to work to inform activities using the former setting. More specifically, the goal of this article is to use the context of partition of integers to demonstrate a possible learning trajectory spanning from visual and hands-on (experiential) activities to symbolic to computational ones and then back to symbolic and/or experiential discourse yet at a higher cognitive level.

The intended readership of the teaching ideas that follow includes K-12 teachers of mathematics (both prospective and practicing), mathematics educators interested in integrating hands-on activities and computers in the teaching of mathematics, and mathematicians who teach prospective teachers and undergraduate mathematics majors by connecting K-12 and university mathematics. The author's experience presenting a number of these ideas at the conference indicates that there are quite a few individuals teaching at the elementary level who are capable of not only capturing the essence of rather advanced mathematical concepts but better still, can use knowledge gathered through professional development of various kind to help their students build stronger foundations for further learning. By the same token, the material of this article can be of help to those teaching beyond the primary grades by offering means to appreciate how abstract mathematics structures originate in the physical world around us and therefore, in many cases, mathematical formalism, which often scares students away from learning, is just an extended reflection on this world.

The article begins with a few problems (tasks) the (hands-on) solution of which, to a certain extent, does not require any mathematical knowledge but common sense and counting skills. Two of these solutions are then confirmed by spreadsheet modeling for which quite involved algebraic techniques are utilized. This transition from the iconic representation to the symbolic one made possible by the use of mathematical formalism demonstrates the power of mathematical concepts in computing applications. Even if those concepts are beyond one's immediate mastery, the appreciation of the utility of conceptualization and the need for abstraction and generalization when teaching a machine to perform calculations greatly enhances his or her mathematical experience. Next, a modification of the hands-on solutions is used in developing recursive definitions of several classic concepts of combinatorial mathematics to be modeled within a spreadsheet. Observing the results of modeling leads to establishing connections among different concepts and developing abstract relations that can be verified experimentally. In that way, a hands-on experiment is situated both at the beginning and the end of spreadsheet-based activities. The concept map [14] of this approach is shown at the conclusion of the article in Figure 17.

2 Partition of integers into summands

A partition of a positive integer n is a representation of n as a sum of positive integers. There may be different conditions under which such representations are created. Consider a few tasks formulated in the context of constructing towers out of blocks—a typical standards-based mathematical practice setting recommended for a junior elementary classroom [8], [12] and that of their future teachers enrolled in mathematics methods and content courses [9], [2].

Task 1. Find all ways to build towers out of four blocks and then arrange them in different orders.

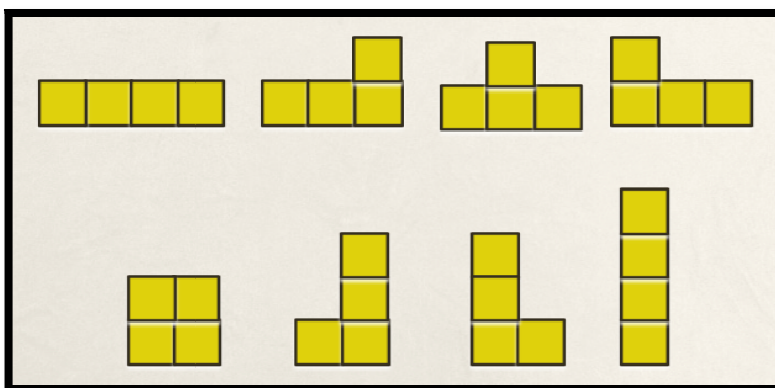


Figure 1. Eight towers built out of four blocks.

Solution. As shown in both an iconic (Figure 1) and symbolic $4 = 1 + 1 + 1 + 1 = 1 + 1 + 2 = 1 + 2 + 1 = 2 + 1 + 1 = 2 + 2 = 1 + 3 = 3 + 1 = 4$ forms, there are eight ways to build towers out of four blocks and arrange them in different orders.

Task 2. Find all ways to build towers out of four blocks and arrange them in a non-decreasing order.

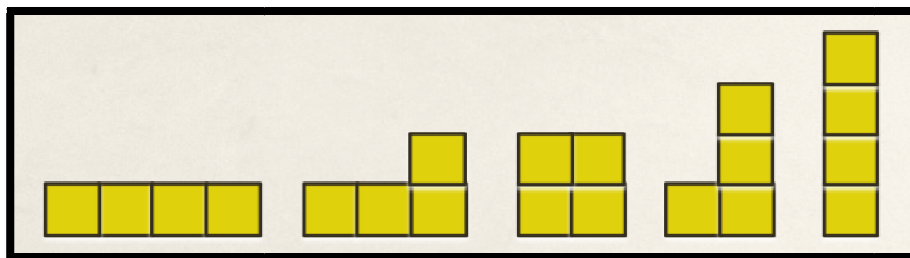


Figure 2. Five towers built out of four blocks.

Solution. As shown in both an iconic (Figure 2) and symbolic

$4 = 1 + 1 + 1 + 1 = 1 + 1 + 2 = 2 + 2 = 1 + 3 = 4$ forms, there are five ways to build towers out of four blocks and arrange them in a non-decreasing order.

Task 3. Find all ways to build three towers out of eleven blocks and arrange them in a non-decreasing order.

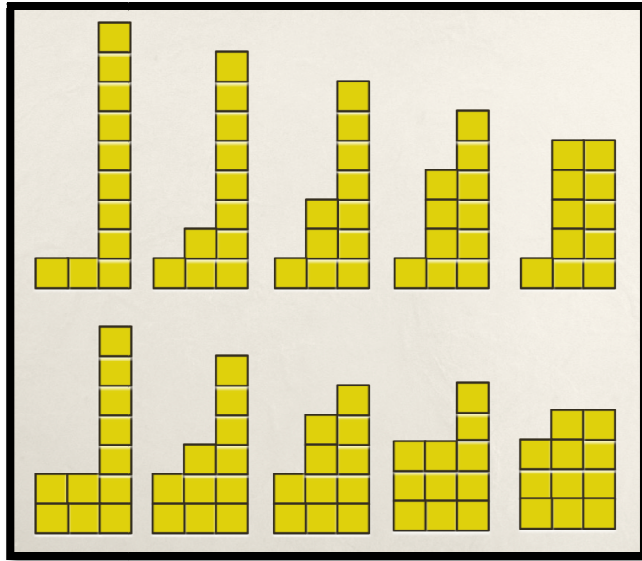


Figure 3. Partition of eleven into three parts using a system.

Solution. Figure 3 shows that three towers can be constructed out of eleven blocks in ten ways. Note that the towers are constructed through a system which is reflected in the following numerical description of Figure 3:

$$11 = 1 + 1 + 9 = 1 + 2 + 8 = 1 + 3 + 7 = 1 + 4 + 6 = 1 + 5 + 5 = 2 + 2 + 7 = 2 + 3 + 6 \\ = 2 + 4 + 5 = 3 + 3 + 5 = 3 + 4 + 4$$

where the first unit is combined with all representations of ten as a sum of two positive integers; then the number two, used first, is combined with all representations of nine as a sum of two positive integers excluding the unit; then the number three, used first, is combined with all representations of eight as a sum of two positive integers excluding integers smaller than three. These ten partitions of eleven can also be generated by using the spreadsheet shown in Figure 4 (the programming of which is discussed in section 3 below) where cell **A1** includes the partitioning number, 11; cell **A7** displays the total number of partitions, 10; and the triple of cells (**D3**, **C6**, **D6**) corresponds to one such partition, $11 = 1 + 3 + 7$. Such a (ready-made) spreadsheet can be used in a classroom as a tool for verifying the completeness of the activity with blocks because it may not be based on a systematic reasoning. Furthermore, the use of

the spreadsheet fosters “the ability to contextualize ... in order to probe into the referents for the symbols involved” [8, p6].

	A	B	C	D	E	F	G	H
1	11							
2								
3				1	2	3		
4			1	9				
5			2	8	7			
6			3	7	6	5		
7	10		4	6	5	4		
8			5	5				
9								
10								

Figure 4. Numerical solution to Task 3.

Task 4. Find all ways to build three different size towers out of twelve blocks and then line them up in the increasing order.

Solution. Figure 5 shows that three towers can be constructed in seven ways out of twelve blocks. Note that the towers are constructed through a system which is reflected in the following numerical description of Figure 5:

$$12 = 1 + 2 + 9 = 1 + 3 + 8 = 1 + 4 + 7 = 1 + 5 + 6 = 2 + 3 + 7 = 2 + 4 + 6 = 3 + 4 + 5.$$

These seven partitions of twelve can be generated by using the spreadsheet of Figure 6 where cell A1 includes the partitioning number, 12, cell A7 displays the total number of partitions, 7, and the triple of cells (D3, C6, D6) corresponds to the partition $12 = 1 + 3 + 8$.



Figure 5. Partition of twelve into three unequal parts using a system.

	A	B	C	D	E	F	G	H
1	12							
2								
3				1	2	3		
4			1					
5			2	9				
6			3	8	7			
7	7		4	7	6	5		
8			5	6				
9								
10								

Figure 6. Numerical solution to Task 4.

3 Spreadsheet programming informed by algebraic reasoning

The programming of the spreadsheet of Figure 4 is worth noting in detail to demonstrate how algebraic skills contribute to the efficacy of computing. It should be noted that while these skills may indeed be beyond a typical master's level elementary teacher candidate, they can be taught within a course such as

Using Spreadsheets in Teaching School Mathematics [4] developed by the author for students enrolled in different teacher education programs at SUNY Potsdam with a special interest in both technology and mathematics. However, all teacher candidates enrolled in elementary mathematics methods and content courses that the author teaches on a regular basis are expected to master operating a spreadsheet like the one shown in Figure 4, introduced to them as an environment to use, and learn how to interpret information it generates in terms of the corresponding hands-on tasks.

On the other hand, one of the recommendations of the Conference Board of the Mathematical Sciences for the mathematical preparation of prospective high school teachers includes the need to “examine the crucial role of algebra in use of computer tools like spreadsheets” [9, p41]. Such a recommendation shifts the focus of mathematical activities from using a spreadsheet for solving algebraic problems to utilizing algebraic ideas for improving the tool’s computational efficiency. Thus, the material of this section demonstrates how a spreadsheet can be used as a medium for revisiting algebra through computing applications in a capstone course for the teachers. More specifically, algebra will be used below as a means for developing a spreadsheet capable to provide a visual representation of all partitions of an integer into three like parts without regard to their order. Towards this end, it is important to distinguish between using a spreadsheet that generates data for an efficient representation of a concept and data for developing a new concept. Algebraically speaking, one has to display all solutions to the Diophantine equation

$$x + y + z = n \quad (1)$$

where $x \leq y \leq z$. That is why not displaying the values of x and y in row 3 and column C, respectively, that are not solutions to equation (1) is important from the point of view of efficiency of computing. Furthermore, this efficiency enhances mathematical visualization (the values of z are displayed within the range D4:H10), a pedagogical construct introduced in education in the early 1990s [17]. Toward this end,¹ given n , one has to find the largest value of y first. The largest y requires $x = 1$ (the smallest value) and $z = y_{\max}$ (as $z \geq y$) whence $2y_{\max} = n - 1$. Two cases need to be considered: n is odd and n is even. The former

¹ More information on using inequalities as tools in spreadsheet modeling can be found elsewhere [5].

case yields $y_{\max} = \text{int}(\frac{n}{2})$; the latter case yields $y_{\max} = \text{int}(\frac{n-1}{2})$ where $\text{int}(x)$ is the largest integer not greater than x . Indeed, for both $n = 11$ (as in Figure 4) and $n = 12$ one has $y_{\max} = 5$ ($11 = 1 + 5 + 5$, $12 = 1 + 5 + 6$) and $\text{int}(\frac{11}{2}) = \text{int}(\frac{12-1}{2}) = 5$. The largest value of x requires $y = z = x_{\max}$ whence $3x_{\max} = n$ or $x_{\max} = \text{int}(\frac{n}{3})$. Indeed, $11 = 3 + 4 + 4$, where $3 = \text{int}(\frac{11}{3})$ and $4 = \text{int}(\frac{12}{3})$. The above reasoning explains the formulas:

cell **A1** is given the name **n**; ranges to the right of cell **D3** and down cell **C4** are given the names **x** and **y**, respectively; (C4): =1, (C5): =IF(MOD(n,2)=0,IF(C4<INT((n-1)/2),1+C4," "), IF(C4<INT(n/2),1+C4," ")) which is replicated down column C. (D3): = 1; (E3): =IF(D3<INT(n/3),1+D3," ") which is replicated across row 3. Finally, (D4): =IF(OR(x=" ",y=" "), " ",IF(AND(y>=x, n-x-y>=y),n-x-y," ")) which is replicated across rows and down columns.

Note that the spreadsheet displays all ten partitions in a way they were presented above in Figure 3 as well as in the form of the chain of equalities: in column D one can see the combination of all representations of ten as a sum of two integers combined with the unit; in column E—all representations of nine as a sum of two integers combined with the two, and, finally, in column F—all representations of eight as a sum of two integers combined with the three. This suggests the method of solving a three-dimensional problem by reducing it to several two-dimensional problems. So, although this reduction was not specifically included in the above spreadsheet formulas, it manifested itself in the final representation.

The spreadsheet of Figure 6 is programmed similar to that of Figure 4, yet there are some differences due to the strict inequalities $x < y < z$. The largest x requires the smallest possible y and z , implying that the numbers x_{\max} , y_{\min} , and z_{\min} are as close to each other as possible. In the case of three consecutive counting numbers, their sums, known as trapezoidal numbers,² are always divisible by three. The smallest such sum is $6 = 1 + 2 + 3$ where the numbers 1 and 6 are related through the equality $1 = \text{int}(\frac{6}{3}) - 1$. Replacing 6 either by 7 or by 8 does not change x_{\max} .

² For using a spreadsheet in modeling trapezoidal numbers see [3].

Therefore, given n , one can define $x_{\max} = \text{int}(\frac{n}{3}) - 1$. The largest y requires $x = 1$ (the smallest value) and $z = y_{\max} + 1$ (as $z > y$) whence $2y_{\max} = n - 2$. This time, regardless whether n is odd or even $y_{\max} = \text{int}(\frac{n-2}{2})$. For example, when $n = 12$ one has $12 = 1 + 5 + 6$, i.e., $y_{\max} = \text{int}(\frac{10}{2}) = 5$; when $n = 11$ one has either $11 = 1 + 4 + 6$ or $11 = 2 + 4 + 5$, i.e., $y_{\max} = \text{int}(\frac{9}{2}) = 4$. This explains the following change in the three main spreadsheet formulas used in programming the spreadsheet of Figure 6 (in comparison with Figure 4):

(E3): =IF(D3<INT(n/3-1),1+D3," "); (C5): =IF(C4<INT((n-2)/2),1+C4," "); and

(D4): =IF(OR(x=" ",y=" ")," ",IF(AND(y>x, n-x-y>y),n-x-y," ")).

4 Notation and the next steps

Below, the following notation will be used: $P(n)$ and $p(n)$ —the number of partitions introduced in the context of Task 1 and Task 2, respectively, where the number 4 is replaced by n ; $P(n, m)$ and $Q(n, m)$ —the number of partitions introduced in the context of Task 3 and Task 4, respectively, where the number 4 is replaced by n and the number 2 is replaced by m . We will refer to $P(n)$ and $p(n)$ as partition of n with and without regard to the order of parts, respectively; $P(n, m)$ as partition of n into m parts and $Q(n, m)$ as partition of n into m non-equal parts.

It will be shown how one can use concrete materials to develop mathematical definitions for the partitions $P(n, m)$ and $Q(n, m)$ to enable their numerical modeling for different n and m within a spreadsheet. In turn, a spreadsheet environment can then be used to develop grade-appropriate activities for “mathematically proficient students” [8] and their future teachers that enable the discovery of interesting relationships among different kinds of partitions mentioned above and other mathematical concepts, including Pascal’s triangle and the binomial coefficients, triangular numbers, difference equations and quadratic sequences as their solutions, and systems of linear equations.

5 Ferrers-Young diagrams

A representation of a positive integer as a sum of like numbers through an array of dots in which each addend is represented by a row or a column of dots is called Ferrers diagram [15], named after a 19th century British mathematician Norman M. Ferrers. To reflect possible activities which, in the context of Tasks 1–4, are appropriate for young children and their future teachers alike, the diagrams will be represented below using squared shapes (often, in a classroom practice, called square tiles). Such diagrams (i.e., when squares are used instead of dots) are also called Young diagrams [16], named after a 20th century British mathematician Alfred Young. So, Figures 1, 2, 3 and 5 may be called Ferrers-Young diagrams representing, respectively, partition of four into all parts, partition of four into ordered parts, partition of eleven into three parts, and partition of twelve into three non-equal parts. Whereas the use of the spreadsheets pictured in Figures 4 and 6 is limited to describing Ferrers-Young diagrams with three parts, the corresponding diagrams can be helpful in developing recursive definitions of partitions $P(n,m)$ and $Q(n,m)$ for all (appropriate) values of m .

6 Recursive definition of $P(n, m)$ informed by a hands-on experiment

Figure 7 shows how eleven can be partitioned into three parts through the following (recursive) definition. First partition ten blocks into two parts and then make three parts by adding a single block (shown at the far-left); then partition eight blocks into three parts and increase each part by one block. That is, $P(11,3) = P(10,2) + P(8,3)$ where $10 = 11 - 1$, $2 = 3 - 1$, and $8 = 11 - 3$.

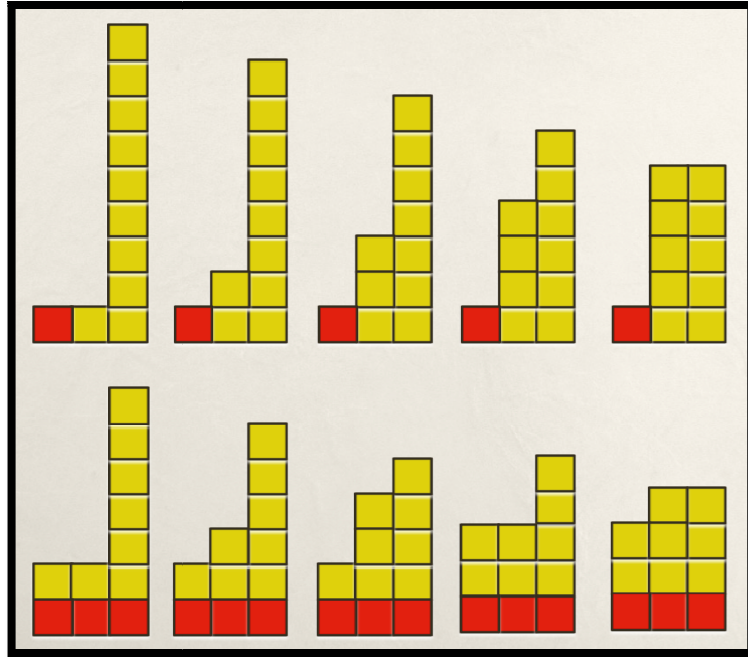


Figure 7. Representing equation (2) in the case $n = 11$ and $m = 3$.

Likewise, in general, all partitions of n into m parts, $P(n, m)$, can be put in two groups. The first group includes all partitions with the unit present; the second group includes all other partitions. Then the number of partitions in the first group is equal to $P(n - 1, m - 1)$ because after partitioning $n - 1$ into $m - 1$ parts it is possible to add one to each of the partitions, thus creating all the partitions in the first group. Likewise, the number of partitions in the second group is equal to $P(n - m, m)$ because after partitioning $n - m$ into m parts, it is possible to increase each part by one thereby creating all the partitions in the second group. From here, the following recursive definition of the number of partitions of n into m parts follows

$$P(n, m) = P(n - 1, m - 1) + P(n - m, m) \quad (2)$$

In order to numerically model equation (2) within a spreadsheet, the following (obvious) initial conditions for $P(n, m)$ should be taken into consideration:

$$P(n, n) = P(n, 1) = 1 \text{ and } P(1, m) = 0 \text{ for } m > 1 \quad (3)$$

The spreadsheet shown in Figure 8 represents a triangular array of the numbers $P(n, m)$. In particular, $P(11, 3) = 10$, the result having been confirmed through the

Ferrers-Young diagrams of Figures 3 and 7 as well as through the spreadsheet of Figure 4. Note that

$$\sum_{m=1}^n P(n,m) = p(n) \tag{4}$$

Therefore, the values of $p(n)$ can be found by adding all integers within each of the columns of the spreadsheet in Figure 8 (row 25). These sums—1, 2, 3, 5, 7, 11, 15, 22—will be discovered below within a spreadsheet designed to the number of partitions of integers into unequal parts.

Furthermore, the diagram of Figure 7 makes it possible to be interpreted as the partition of eleven into parts with three being the largest part. In general, the number of partitions of n into m parts is equal to the number of partitions of n with the largest part m . In particular, eleven can be partitioned into parts with the largest part three in ten ways.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	
1																								
2		m\ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	
3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
4	2		1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10		
5	3			1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37		
6	4				1	1	2	3	5	6	9	11	15	18	23	27	34	39	47	54	64	72		
7	5					1	1	2	3	5	7	10	13	18	23	30	37	47	57	70	84	101		
8	6						1	1	2	3	5	7	11	14	20	26	35	44	58	71	90	110		
9	7							1	1	2	3	5	7	11	15	21	28	38	49	65	82	105		
10	8								1	1	2	3	5	7	11	15	22	29	40	52	70	89		
11	9									1	1	2	3	5	7	11	15	22	30	41	54	73		
12	10										1	1	2	3	5	7	11	15	22	30	42	55		
13	11											1	1	2	3	5	7	11	15	22	30	42		
14	12												1	1	2	3	5	7	11	15	22	30		
15	13													1	1	2	3	5	7	11	15	22		
16	14														1	1	2	3	5	7	11	15		
17	15															1	1	2	3	5	7	11		
18	16																1	1	2	3	5	7		
19	17																	1	1	2	3	5		
20	18																			1	1	2	3	
21	19																				1	1	2	
22	20																					1	1	
23	21																						1	
24																								
25	p(n)	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627	792		

Figure 8. An array of the numbers $P(n, m)$ with the numbers $p(n)$ in the bottom row.

Unlike the spreadsheets of Figures 4 and 6, the spreadsheet of Figure 8 does not display all possible partitions for an individual number but rather, it displays the

total number of partitions. That is, there is a difference between solving equation (1) and finding its number of solutions.

Here are the programming details for the spreadsheet of Figure 8. To reflect initial conditions (3), the range (C3:W3) is filled with units and the range (C4:C23) is filled with (hidden) zeros; (D4): =IF(D\$2-\$B4<0,0,IF(\$B4-D\$2=0,1, C3+INDEX(\$C4:C4,1, D\$2-\$B4))) which is replicated across rows and down columns; (C25): =SUM(C3:C23)—replicated across row 25.

7 Connecting different concepts as a problem-solving method

One can note that the array of the numbers $P(n, m)$ resembles Pascal’s triangle—a triangular array filled with the binomial coefficients so that any element of the triangle is the sum of elements immediately to its left in the same row and in the row above it. It is interesting to explore if Pascal’s triangle can be connected to partitioning problem. With this in mind, one can use a spreadsheet to generate Pascal’s triangle shown in Figure 9 (programming details are omitted because of their simplicity). Compare the numbers in the ranges (F3:F6) of the spreadsheets of Figures 8 and 9. In Figure 8, $p(4) = 1 + 2 + 1 + 1 = 5$; in Figure 9, $C_3^0 + C_3^1 + C_3^2 + C_3^3 = 1 + 3 + 3 + 1 = 8$. How can one interpret the entries of Pascal’s triangle in term of partitions? For example, how can one partition four into two summands in order to generalize Task 1 to any positive integer n ?

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
1																							
2		m/n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3		0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4		1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
5		2	0	0	1	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153	171	190
6		3	0	0	0	1	4	10	20	35	56	84	120	165	220	286	361	455	560	680	816	969	1140
7		4	0	0	0	0	1	5	15	35	70	126	210	330	495	715	1001	1365	1820	2380	3060	3876	4845
8		5	0	0	0	0	0	1	6	21	56	126	252	462	792	1287	2002	3003	4368	6188	8568	11628	15504
9		6	0	0	0	0	0	0	1	7	28	84	210	462	924	1716	3003	5005	8008	12376	18584	27132	38760
10		7	0	0	0	0	0	0	0	1	8	36	120	330	792	1716	3432	6435	11440	19448	31824	50388	77520
11		8	0	0	0	0	0	0	0	0	1	9	45	165	495	1287	3003	6435	12870	24310	43758	75582	125970
12		9	0	0	0	0	0	0	0	0	0	1	10	55	220	715	2002	5005	11440	24310	48620	92378	167960
13		10	0	0	0	0	0	0	0	0	0	0	1	11	66	286	1001	3003	8008	19448	43758	92378	184756
14		11	0	0	0	0	0	0	0	0	0	0	0	1	12	78	364	1365	4368	12376	31824	75582	167960
15		12	0	0	0	0	0	0	0	0	0	0	0	0	1	13	91	455	1820	6188	18564	50388	125970
16		13	0	0	0	0	0	0	0	0	0	0	0	0	0	1	14	105	560	2380	8568	27132	77520
17		14	0	0	0	0	0	0	0	0	0	0	0	0	0	1	15	120	680	3060	11628	38760	
18		15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	16	146	816	3876	15504	
19		16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	17	165	560	4845	
20		17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	18	171	1340	
21		18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	19	190	
22		19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	20	
23		20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	
24																							
25		2^n	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768	65536	131072	262144	524288	1048576

Figure 9. Pascal’s triangle with the powers of two at the bottom (row 25).

To answer this question, consider the following diagram

$$\begin{aligned} \{1 \ 1 \ 1\} + \{1\} &\rightarrow 4 = 3 + 1 \\ \{1 \ 1\} + \{1 \ 1\} &\rightarrow 4 = 2 + 2 \\ \{1\} + \{1 \ 1 \ 1\} &\rightarrow 4 = 1 + 3 \end{aligned}$$

in the right-hand side of which the partitioning of four into two summands is carried out by placing the plus sign between two groups of units. The question to be answered is: How many ways a single plus sign can be put in three spaces that separate four units? The answer is C_3^1 , that is, the number of ways to choose one object out of three objects. We have one plus sign because of two summands. We have three spaces because four units require three spaces to be separated. When four is partitioned into three summands, we have three spaces where two plus signs have to be placed and this can be done in C_3^2 ways. In general, a number n can be partitioned into m summands (counting partitions into the same but differently ordered parts) in C_{n-1}^{m-1} ways. Therefore, with regard to the order of summands, C_3^0 is the number of ways to partition four in one summand, C_3^1 – to partition four in two summands, C_3^2 – to partition four in three summands, and C_3^3 – to partition four in four summands. Now one can see that the situation introduced in Task 1 could be resolved by connecting the array of numbers $P(n, m)$ to Pascal’s triangle. It follows from the well-known identity (e.g., [10])

$$\sum_{m=0}^{n-1} C_n^m = 2^{n-1}$$

and the fact that the largest number of plus signs that can be put between n units is equal to $n - 1$, that n can be partitioned in all differently ordered positive integer summands in 2^{n-1} ways. In other words, the equality $P(n) = 2^{n-1}$ hold true. An alternative way of arriving at the same result by using two-color counters and tree diagrams is discussed elsewhere [2].

8 Recursive definition of $Q(n, m)$ informed by a hands-on experiment

Figure 10 shows how twelve can be partitioned into three unequal (distinct) parts through the following (recursive) definition. First partition nine blocks into two parts (four ways) and then make three parts by placing each of the four partitions

on the top of three blocks (shown at the top of Figure 10); then partition nine blocks into three parts (three ways) and increase each part by three blocks (shown at the bottom of Figure 10). That is, $Q(12,3) = Q(9,2) + Q(9,3)$ where $9 = 12 - 3$ and $2 = 3 - 1$.

In general, similar to the case of $P(n, m)$, all partitions of n into m distinct parts, $Q(n, m)$, can be put in two groups. The first group includes all partitions with the unit present; the second group includes all other partitions. Then the number of partitions in the first group is equal to $Q(n - m, m - 1)$ because after partitioning $n - m$ into $m - 1$ parts it is possible to add a unit to each of the partitions and increase each part of any partition by one thus creating all the partitions in the first group. Likewise, the number of partitions in the second group is equal to $Q(n - m, m)$ because after partitioning $n - m$ into m parts, it is possible to increase each part by one thereby creating all the partitions in the second group. This explains the following recursive definition of the number of partitions of n into m parts:

$$Q(n,m) = Q(n - m, m - 1) + Q(n - m, m) \tag{4}$$

In order to numerically model equation (4), the following properties of $Q(n, m)$ should be taken into consideration to serve as initial conditions

$$Q(n, 1) = 1 \text{ and } Q(1, m) = 0 \text{ for } m > 1. \tag{5}$$

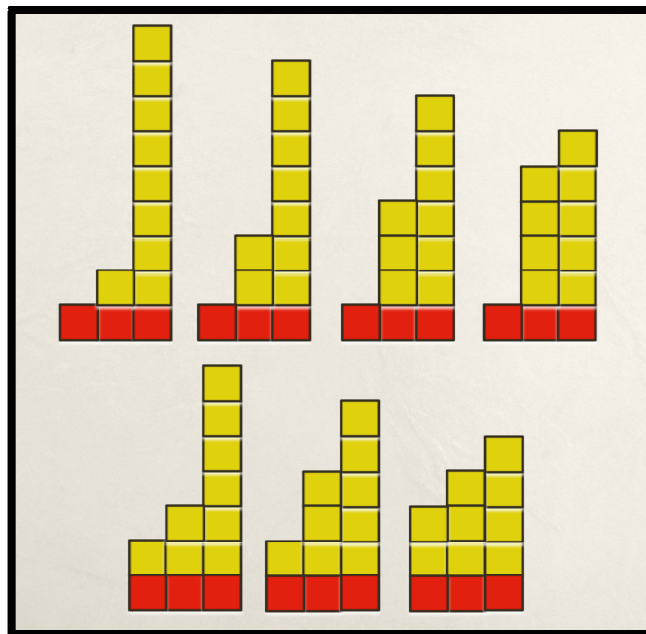


Figure 10. Representing equation (4) in the case $n = 12$ and $m = 3$.

m\n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	1																					
2		1																				
3			1																			
4				1																		
5					1																	
6						1																
7							1															

Figure 11(a). A spreadsheet-generated array of the numbers $Q(n, m)$: $1 \leq n \leq 22$.

m\n	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44
23	1																					
24		11																				
25			33																			
26				39																		
27					18																	
28						2																
29							1															
30								1														
31									1													

Figure 11(b). A spreadsheet-generated array of the numbers $Q(n, m)$: $23 \leq n \leq 44$.

m\n	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66
45	1																					
46		22																				
47			147																			
48				441																		
49					674																	
50						532																
51							201															
52								29														
53									1													
54										1												
55											1											

Figure 11(c). A spreadsheet-generated array of the numbers $Q(n, m)$: $45 \leq n \leq 66$.

Here are the details of spreadsheets programming for Figures 11(a–c). To reflect initial conditions (5), row 3 is filled with units; column C from cell C4 is filled with zeros (which are not displayed); (D4): =IF(OR(\$B4-D\$2>=0,\$B3-C\$2>=0),0,IF(AND(\$B4-D\$2=0,\$B3-C\$2=0),1, INDEX(\$C3:C3,1,C\$2-\$B3)+INDEX(\$C4:C4,1,D\$2-\$B4))).

8 Connection to triangular numbers

Recall that the sum of the first n counting numbers is called a triangular number of rank n , a concept introduced already at the elementary level [13, p14], for which the notation t_n is typically used. Numerical data generated by the spreadsheets introduced above makes it possible to be used as a tool in making mathematical connections between different concepts. For example, the array of the spreadsheet pictured in Figure 11(a) has a quasi-triangular form with each step in the ladder being one longer than the previous step. How can this phenomenon be explained? Note that the smallest integer that can be represented as the sum of two different positive numbers is 3, thus $Q(n, 2) = 0$ for $n < 3$; the smallest integer that can be represented as the sum of three different positive numbers is 6, thus $Q(n, 3) = 0$ for $n < 6$; the smallest integer that can be represented as the sum of four different positive numbers is 10, thus $Q(n, 4) = 0$ for $n < 10$. What is special about the numbers 3, 6, and 10? These are triangular numbers of rank 2, 3, and 4, respectively. Generalizing from this observation yields the relation

$$Q(n, m) = 0 \text{ for all } n < t_m$$

where t_m is the triangular number of rank m .

10 Quadratic sequences as solutions to difference equations

Difference equations as modeling tools to be studied both by high school students and their future teachers are included in the number of standards and recommendations for teaching mathematics across the board [9], [12]. For some sequences (e.g., natural numbers), the form of the corresponding difference equation immediately follows from the rule according to which a sequence develops. Other sequences don't reveal their nature without an effort on the part of the learner. Numerical evidence provided by a spreadsheet can motivate one to "see trends in data by noticing change in the form of differences" [12, p305]. From such an observation, a difference equation may result.

For example, the data presented in the spreadsheets of Figures 11 (a–c) allows one to recognize the following repeated equalities

$$Q(9, 3) = Q(13, 4) = Q(18, 5) = Q(24, 6) = Q(31, 7) = Q(39, 8) = \dots = 3 \quad (6)$$

Can this observation be generalized? To answer this question, consider the sequence $x_n = \{9, 13, 18, 24, 31, 39, \dots\}$ representing the numbers each of which can be partitioned in three, four, five, and so on unequal integer parts in three ways. One can check to see that the second difference of this number sequence,

$\Delta_2 x_n = x_{n+2} - 2x_{n+1} + x_n$, is equal to one. From this observation, one can develop a

closed formula for the sequence x_n by solving the difference equation

$$x_{n+2} - 2x_{n+1} + x_n = 1 \quad (7)$$

subject to the initial conditions

$$x_1 = 9, x_2 = 13 \quad (8)$$

To this end, x_n can be found as the sum

$$x_n = x_n^h + x_n^p \quad (9)$$

where x_n^h and x_n^p are, respectively, the general solution to the homogeneous difference equation

$$x_{n+2} - 2x_{n+1} + x_n = 0 \quad (10)$$

and a partial solution to a non-homogeneous equation (7). Because the characteristic equation, $\lambda^2 - 2\lambda + 1 = 0$, for equation (10) has double root, $\lambda = 1$, the general solution to equation (10) has the form $x_n^h = C_1 + C_2 n$. Furthermore, one

can check to see that $x_n^p = \frac{n(n+1)}{2}$ satisfies equation (7). Therefore,

$$x_n = C_1 + C_2n + \frac{n(n+1)}{2} \quad (11)$$

In order to find C_1 and C_2 , initial conditions (8) should be used as follows:

$9 = C_1 + C_2 + 1$ and $13 = C_1 + 2C_2 + 3$ whence $C_1 + C_2 = 8$ and $C_1 + 2C_2 = 10$. Solving the last two equations yields $C_1 = 6, C_2 = 2$. From formula (11) it follows that

$$x_n = 6 + 2n + \frac{n(n+1)}{2} = \frac{n^2 + 5n + 12}{2}.$$

Thus, the following closed formula has been found

$$x_n = \frac{n^2 + 5n + 12}{2} \quad (12)$$

Noting that $\frac{n^2 + 5n + 12}{2} = \frac{(n+2)(n+3)}{2} + 3$, makes it possible to generalize equalities (6) to the form

$$Q\left(\frac{(n+2)(n+3)}{2} + 3, n+2\right) = 3 \quad (13)$$

11 Using technology in developing formula (12)

Alternatively, formula (12) can be developed by defining $x_n = an^2 + bn + c$ and, using initial conditions (8), solving the following system of three linear equations

$$a + b + c = 9, 4a + 2b + c = 13, 9a + 3b + c = 18 \quad (14)$$

Toward this end, one can use a spreadsheet function **MINVERSE** to find the values of the unknowns a, b , and c satisfying equations (14). In the spreadsheet of

Figure 12 the matrix of coefficients of system (14) is entered in the range (A1: C3); the range (A5:C7) computes the inverse matrix; and the right-hand sides of the equations of system (14) are entered in the range (E1:E3).

	A	B	C	D	E	F
1	1	1	1		9	
2	4	2	1		13	
3	9	3	1		18	
4						
5	0.5	-1	0.5			
6	-2.5	4	-1.5			
7	3	-3	1			
8						
9	0.5	2.5	6			

Figure 12. Solving a system of three linear equations within a spreadsheet.

It should be noted that using a spreadsheet for solving systems of linear equations can only be justified in the absence of other (mathematical) software. For example, in the context of *Maple*, one can use the operator *fsolve* in a rather straightforward way (Figure 13).

```
fsolve({a + b + c = 9, 4 a + 2 b + c = 13, 9 a + 3 b + c = 18})
[a = 0.5000000000, b = 2.500000000, c = 6.]
```

Figure 13. Using *Maple* in solving systems of linear equations.

Finally, difference equation (7) subject to initial conditions (8) can also be solved in the context of *Maple* using the operator *rsolve* as shown in Figure 14.

```
rsolve({x(n+2) - 2*x(n+1) + x(n) - 1, x(1)=9, x(2)=13}, x(n))
n + 5 + (n + 1) (1/2 n + 1)
expand(%)
5/2 n + 6 + 1/2 n^2
```

Figure 14. Using *Maple* in solving a difference equation.

It should be noted that there is no pedagogically justified reason for using a spreadsheet instead of *Maple* in developing quadratic sequences other than for demonstrating the former tool's various features, thereby preserving this paper's technological consistency. At the same time, the use of a spreadsheet in modeling Pascal's triangle as well as the numbers $P(n, m)$ and $Q(n, m)$ has clear pedagogic advantage over other computational tools for the outcome of modeling is not the numbers per se, but rather their array-like, visually powerful representations enabling subsequent search for patterns, conjecturing, and generalizing/symbolizing from numerical evidence. The didactic value of a spreadsheet as a learning tool is both in its mathematically rich tool kit of computing devices and didactically unparalleled representational efficacy of mathematical concepts. The next section will further demonstrate the validity of the last characterization of the software. Yet the joint use of a spreadsheet with other educational tools, both digital and tactile is worth exploring as appropriate. This, in turn, points to the usefulness of using in mathematics and other areas of education the so-called integrated spreadsheets.

12 A spreadsheet-motivated generalization connecting Tasks 2 and 4

The modeling data generated by the spreadsheets of Figure 8 and Figures 11(a–c) allow one to observe that

$$Q(14, 4) = Q(19, 5) = Q(25, 6) = \dots = 5$$

$$Q(20, 5) = Q(26, 6) = Q(33, 7) = \dots = 7$$

$$Q(27, 6) = Q(34, 7) = Q(42, 8) = \dots = 11$$

and recognize that $p(4) = 5$, $p(5) = 7$, $p(6) = 11$. As this observation is consistent with equalities (6), one can first rewrite identity (13) in the form

$$Q\left(\frac{(n+2)(n+3)}{2} + 3, n+2\right) = p(3), \quad n \in Z \tag{15}$$

and then use techniques described in the previous section to develop new identities

$$Q\left(\frac{(n+3)(n+4)}{2} + 4, n+3\right) = p(4), \quad n \in Z \tag{16}$$

$$Q\left(\frac{(n+4)(n+5)}{2} + 5, n+4\right) = p(5), \quad n \in Z \tag{17}$$

$$Q\left(\frac{(n+5)(n+6)}{2} + 6, n+5\right) = p(7), \quad n \in Z \tag{18}$$

where Z is the set of positive integers.

Finally, identities (15)–(18) can be (inductively) generalized to the form

$$Q\left(\frac{(n+k-1)(n+k)}{2} + k, n+k-1\right) = p(k), \quad n, k \in Z \tag{19}$$

Identity (19), motivated by the use of a spreadsheet, can be formulated in the following form:

The number of partitions of k into integer parts without regard to order is equal to the number of partitions of the number $\frac{(n+k-1)(n+k)}{2} + k$ into $n+k-1$ unequal parts arranged from the least to the greatest for any positive integer value of n .

In particular, substituting $k = 3$ and $n = 1, 2, 3$ in identity (19) yields

$Q\left(\frac{3 \cdot 4}{2} + 3, 3\right) = Q\left(\frac{4 \cdot 5}{2} + 3, 4\right) = Q\left(\frac{5 \cdot 6}{2} + 3, 5\right) = 3$ and $p(3) = 3$. As shown in Figure 15, the number of partitions of the number 3 into three parts without regard to order is equal to the number of partitions of the number 9 into three unequal parts. The latter partition situation is reflected in the spreadsheet of Figure 16 showing the following three partitions: $9 = 1 + 2 + 6 = 1 + 3 + 5 = 2 + 3 + 4$. Similarly, the Ferrers-

Young diagrams for $Q(13, 4)$ and $Q(18, 5)$ and their spreadsheet equivalents can be constructed to experimentally confirm the above abstract statement.

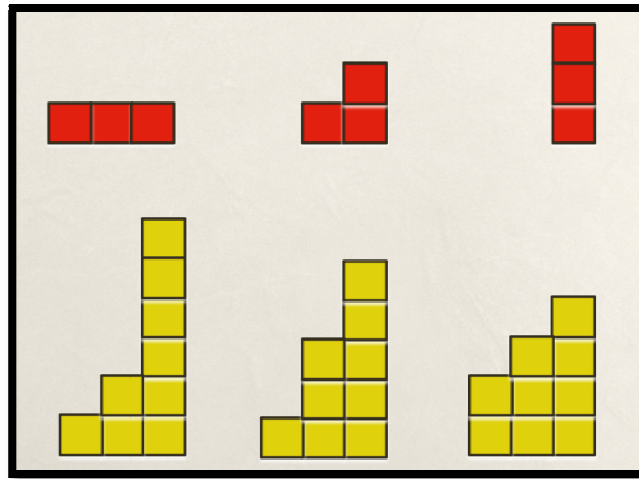


Figure 15. Confirming theory through constructing Ferrers-Young diagrams.

	A	B	C	D	E	F	G
1	9						
2							
3				1	2		
4			1				
5			2	6			
6			3	5	4		
7	3						
8							

Figure 16. Three partitions of nine into three unequal parts.

13 Conclusion

This article demonstrated how an experiential approach to mathematics that incorporates concrete materials as means of creating diagrammatic representations of problematic situations can be put to work to develop rather sophisticated mathematical models associated with the theory of partitions. In a spreadsheet environment, these models were described in terms of the spreadsheet function INDEX enabling their numerical representation that was powerful enough to enable one's search for patterns and regularities among numbers with different combinatorial meaning. Furthermore, numerical evidence so obtained allowed for making mathematical connections through which abstract and apparently not self-evident symbolic relationships were discovered. In turn, special cases of these relationships were verified experimentally by using concrete materials and a spreadsheet yet at a higher cognitive level. In that way, a spreadsheet mediated between the original set of experiments designed to develop a theory and the follow-up set of experiments used to confirm the theory. Figure 17 represents a concept map of this approach. Although the author's current teaching responsibilities do not include work for a mathematics department, it appears that this approach to teaching topics in discrete mathematics would be useful in the undergraduate preparation of mathematicians who could use the material of this article as a source for research projects.

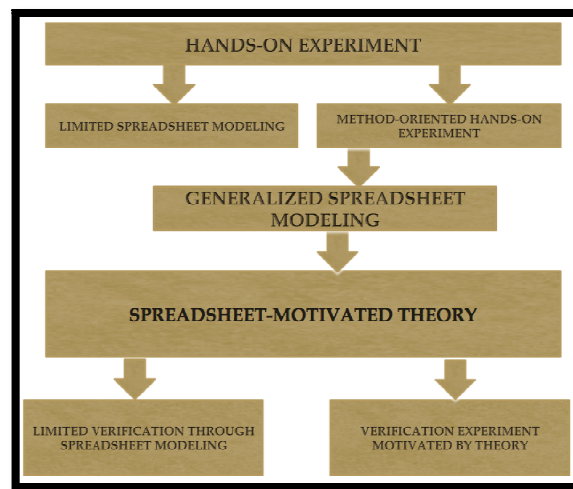


Figure 17. From experiment to theory to experiment: a concept map.

Finally, the article suggested that spreadsheets can be effectively integrated with other technological tools, just like technology, in general, has been integrated with paper-and-pencil mathematics for the last four decades. The representational efficacy of a spreadsheet as a mathematical modeling tool is not

debated but rather, it is suggested that one can be augment the software by (easy to learn and use) auxiliary computational tools. By the same token, mathematical models amenable to spreadsheet modeling may not only emerge from learners' experimentation with concrete materials but these very materials can subsequently be employed for the tactile verification of spreadsheet-motivated mathematical propositions.

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