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Central Projection in Excel – an Introduction to Virtual Reality

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The article presents a method of explaining the principles of virtual reality through making revolvable and sizable central projections of 3D figures in Excel.

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Keywords: 3D graphics, spreadsheets, central projection

1   Introduction

Virtual reality applies to environments that can simulate physical presence in the real as well as imaginary worlds. The technology is used in computer games, simulators, documentary and animated films, etc., and also in education software (see e.g. [1]) to develop spatial skills (see e.g. [2]). 3D computer graphics constitutes the basics of virtual reality. Traditionally, virtual reality environments are developed by using programming. However, the equations that govern the transformation from 3D to 2D can easily be evaluated and visualized in spreadsheets [3]. Creating a revolvable and sizeable projection of a 3D figure in such a simple way can be an interesting introduction to 3D graphics without any need for programming.

There are two main types of projection. The parallel projection is made by lines going through the projected points so that they are parallel to a given line, which gives the direction of the projection. The simplest version is the orthographic parallel projection when the direction of the projection is perpendicular to the projection plane [3]. Central projection is made by rays going from a point, which is the centre of the projection, to the projected points. The difference between the two types of projection is how the lines that are parallel in the reality are projected onto the projecting plane. In the parallel projection, they are parallel. That is why the projection is used by mathematicians, scientists, and engineers [4]. In the central projection, some of them seem to be intersecting in the distance, e.g., like the rails of a railroad. That is how humans see the world. The central projection was discovered in the 15th century by Italian painters. It is called perspective [5]. It gives the paintings space depth, so the viewer has the feeling of watching a real scene through a window [6], [7]. That is why central projection is used in advertisements, games, and films.

This article presents a way of learning the principles of the central projection through making revolvable and sizable projections of 3D figures in Excel™. The principles are explained and the governing equations are derived in Section 2. Excel implementations are in Sections 3 – 5. Central projection of a cube if the overlapping of the faces is not solved is in

Section 3. That is the projection of e.g. a wire model of cube (largely used in school geometry) or e.g. a children wooden cube if all faces are depicted including those that are invisible in the reality. Central projection if the overlapping of the faces is solved is in Section 4. That is the projection of e.g. the wooden cube if the visible faces are depicted only. Central projection of a house, as an example of a general convex figure, is in Section 5.

2 Central projection

Let $Oxyz$ be an orthonormal right-handed coordinate system (Figs. 1 and 2). Let ρ be a plane defined in the $Oxyz$ coordinate system so that $O \notin \rho$. Let a line be going through point O perpendicularly to plane ρ. Let O' be the intersection point of the line and plane ρ. Let the length of vector $\mathbf{OO'}$ be u. Let vector $\mathbf{OO'}$ be given by angles ϕ and θ as depicted in Figs 1 and 2. Then, $O' = (u\cos\theta\cos\phi, u\cos\theta\sin\phi, u\sin\theta)$. Let S be a point on ray OO' in distance v from point O so that $v \neq u$. Then, $S = (v\cos\theta\cos\phi, v\cos\theta\sin\phi, v\sin\theta)$. Let $\theta = 0$ and $\phi = 0$. Then, plane ρ is parallel to coordinate plane yz, and vector $\mathbf{OO'}$ merges with axis x. In this position, let the intersection line of plane ρ and coordinate plane xy be x', and the intersection line of plane ρ and coordinate plane xz be y'. Let the lines x' and y' be scaled, starting at point O', by the unit of axes x and y. Then, the orthonormal coordinate system $O'x'y'$ is defined in plane ρ. Let A be a point in space. The intersection point A' of ray SA and plane ρ is the central projection of point A onto plane ρ. Point S is the projection centre, and plane ρ is the projection plane.
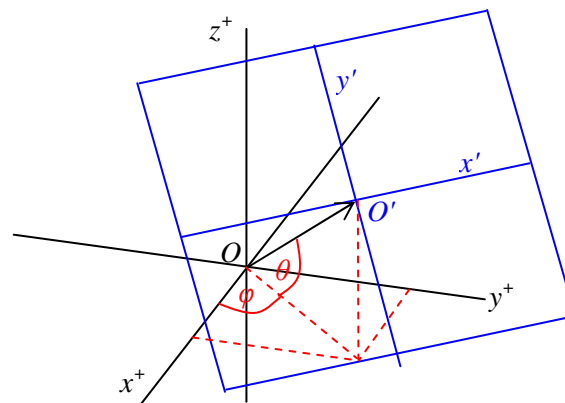

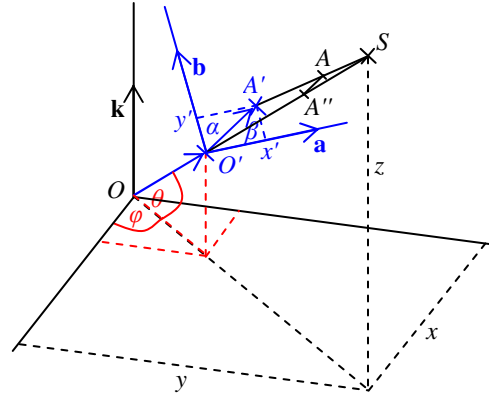Figure 1: Projection plane (blue) in the $Oxyz$ coordinate system (black)

Figure 2: Central projection $A'(x', y')$ of point $A(x, y, z)$ onto the projection plane (blue)

Let $\mathbf{k} = (0, 0, 1)$ be the unit vector of axis z. Let $\mathbf{a}$ be the unit vector of axis x', and $\mathbf{b}$ be the unit vector of axis y'. It holds that

$$\mathbf{a} = \frac{\mathbf{k} \times \mathbf{OO'}}{|\mathbf{k} \times \mathbf{OO'}|} = (-\sin\phi, \cos\phi, 0), \tag{1}$$

$$\mathbf{b} = \frac{\mathbf{OO'} \times \mathbf{a}}{|\mathbf{OO'} \times \mathbf{a}|} = (-\sin\theta\cos\phi, -\sin\theta\sin\phi, \cos\theta), \tag{2}$$

where $\times$ is the vector product and $|\ |$ is the absolute value. It holds that

$$x' = |\mathbf{O'A'}|\cos\beta = \mathbf{a} \cdot \mathbf{O'A'} = \mathbf{a} \cdot \mathbf{SA'} - \mathbf{a} \cdot \mathbf{SO'} = \mathbf{a} \cdot \mathbf{SA'} = \mathbf{a} \cdot (c \cdot \mathbf{SA}) = c \cdot (\mathbf{a} \cdot \mathbf{SA}), \ c \in R, \tag{3}$$

because $\mathbf{a} \cdot \mathbf{SO'} = 0$ as $\mathbf{a} \perp \mathbf{SO'}$. Similarly,

$$y' = |\mathbf{O'A'}|\cos\alpha = \mathbf{b} \cdot \mathbf{O'A'} = \mathbf{b} \cdot \mathbf{SA'} - \mathbf{b} \cdot \mathbf{SO'} = \mathbf{b} \cdot \mathbf{SA'} = \mathbf{b} \cdot (c \cdot \mathbf{SA}) = c \cdot (\mathbf{b} \cdot \mathbf{SA}), \ c \in R, \tag{4}$$

because $\mathbf{b} \cdot \mathbf{SO'} = 0$ as $\mathbf{b} \perp \mathbf{SO'}$. It holds that

$$\mathbf{SA} = (x - v\cos\theta\cos\phi, y - v\cos\theta\sin\phi, z - v\sin\theta). \tag{5}$$

Substituting into Eq. (3) and (4) gives

$$x' = c(-x\sin\phi + y\cos\phi), \tag{6}$$

$$y' = c(-x\sin\theta\cos\phi - y\sin\theta\sin\phi + z\cos\theta). \tag{7}$$

It holds that

$$c = \frac{|\mathbf{SA'}|}{|\mathbf{SA}|} = \frac{|\mathbf{SO'}|}{|\mathbf{SA''}|}, \tag{8}$$

where A'' is the orthogonal projection of point A onto ray $\mathbf{SO'}$. Let ω be the angle between vectors $\mathbf{SO'}$ and $\mathbf{SA}$. Then

$$|\mathbf{SA''}| = |\mathbf{SA}|\cos\omega,\qquad(9)$$

where

$$\cos\omega = \frac{\mathbf{SO'}\cdot\mathbf{SA}}{|\mathbf{SO'}||\mathbf{SA}|},\qquad(10)$$

which gives

$$c = \frac{|\mathbf{SO'}|}{|\mathbf{SA}|}\frac{|\mathbf{SO'}||\mathbf{SA}|}{\mathbf{SO'}\cdot\mathbf{SA}} = \frac{|\mathbf{SO'}|^2}{\mathbf{SO'}\cdot\mathbf{SA}} = \frac{(v-u)^2}{\mathbf{SO'}\cdot\mathbf{SA}} = \frac{v-u}{v - x\cos\phi\cos\theta - y\sin\phi\cos\theta - z\sin\theta},\qquad(11)$$

Substituting into Eq. (6) and (7) yields

$$x' = (v-u)\frac{-x\sin\phi + y\cos\phi}{v - x\cos\phi\cos\theta - y\sin\phi\cos\theta - z\sin\theta},\qquad(12)$$

$$y' = (v-u)\frac{-x\sin\theta\cos\phi - y\sin\theta\sin\phi + z\cos\theta}{v - x\cos\phi\cos\theta - y\sin\phi\cos\theta - z\sin\theta}.\qquad(13)$$

Equations (12), (13) govern the central projection of a point onto the projecting plane.

If we identify the projection plane with the computer screen, then we can project any 3D body on the screen provided we know the x, y, z coordinates of the vertices of the body. The position of the projection plane is given by angles ϕ and θ. They are equivalent to geographic longitude and latitude. Angle ϕ goes from $-180°$ to $180°$ and angle θ goes from $-90°$ to $90°$ so the body can be projected from any direction. If angle ϕ grows, then the projection plane moves to the right around the projected body; however, it seems that the figure rotates to the left. If angle θ grows, the projection plane moves upwards around the projected body but it seems that the body rotates downwards.

3 Projection of a cube if the overlapping of the faces is not solved

Let *ABCDEFGH* be a cube of edge length $a = 2$ (see cell C4 in Fig. 3). Let it be situated in the *Oxyz* coordinate system so that its centre is at the origin *O* of the system and the faces are parallel to the coordinate planes. The x, y, z coordinates of the vertices are calculated in cells C10:E17 by formulas that refer to cell C4, which makes the projection sizable. It is =$C$4/2 in cell C10. The x', y' coordinates of the vertices are calculated in cells F10:G17 by Eqs. (12) and (13). Each edge is drawn as a two-point xy line graph (12 graphs for 12 edges). The maximum and minimum of axes x' and y' are set to –3 and 3 (Fig. 3, right side), then the axes are switched off (Fig. 3, left side). If an edge becomes invisible when the cube is rotated, the user can double-click it and change to dashed.
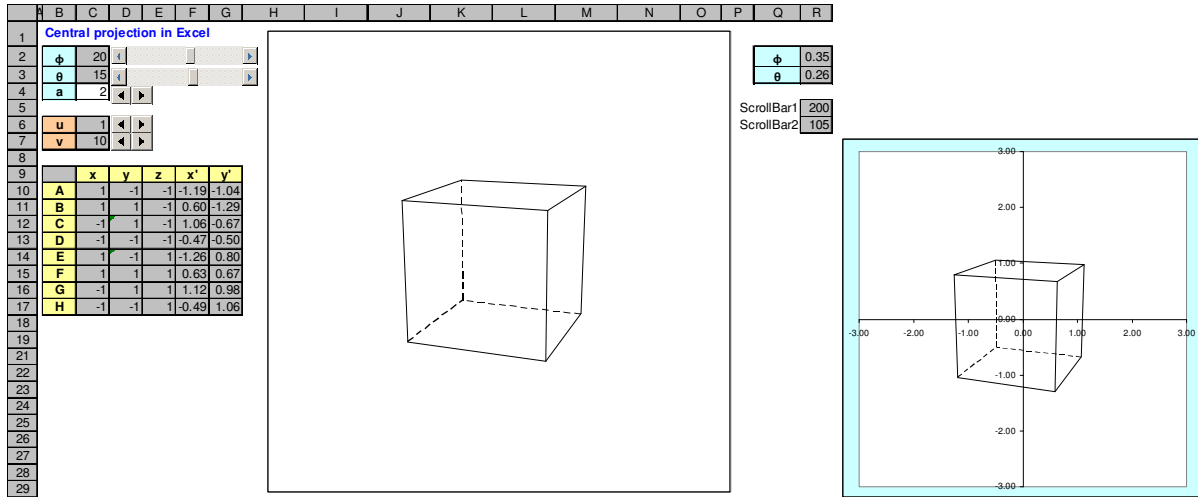
Figure 3: Cube with non-solved overlapping of the faces; the axes switched off (left) and on (right)

Angles ϕ and θ are converted into radians in cells R2 and R3. The rotation is controlled by scrollbars that change the angles by $1°$ or $5°$. They are linked to cells R5 and R6. Their property Min is set to 0. The property Max is set to 360 for the first one, and 180 for the second one. Cells C2 and C3 contain the formulas =R5-180 and =R6-90. The spinbuttons next to cells C4, C6 and C7 enable the user to change *a*, *u* and *v* by 0.1 using a short VBA code. For cell C4, it is `Range("C4") = Range("C4") – 0.1` for spinning down, and `Range("C4") = Range("C4") + 0.1` for spinning up. That allows changing *a*, *u*, and *v* also manually. Angles ϕ and θ can be changed by the scrollbars only.

4 Projection of a cube if the overlapping of the faces is solved

If a face is visible, then the angle between its normal vector oriented out of the figure and each vector going from a vertex of the face to point *S* is smaller than $90°$, so the scalar product of the vectors is positive. For example, if the bottom face *ABCD* is visible, then $\mathbf{n}\cdot\mathbf{SA} = |\mathbf{n}|\cdot|\mathbf{SA}|\cos\alpha > 0$, $\mathbf{n}\cdot\mathbf{SB} = |\mathbf{n}|\cdot|\mathbf{SB}|\cos\beta > 0$, $\mathbf{n}\cdot\mathbf{SC} = |\mathbf{n}|\cdot|\mathbf{SC}|\cos\gamma > 0$, and $\mathbf{n}\cdot\mathbf{SD} = |\mathbf{n}|\cdot|\mathbf{SD}|\cos\delta > 0$, where α, β, γ and δ are the angles between vector $\mathbf{n}$, and $\mathbf{SA}$, $\mathbf{SB}$, $\mathbf{SC}$ and $\mathbf{SD}$. However, $|\mathbf{SA}|\cos\alpha = |\mathbf{SB}|\cos\beta = |\mathbf{SC}|\cos\gamma = |\mathbf{SD}|\cos\delta$ as they all give the distance between point *S* and the bottom face. Hence, $\mathbf{n}\cdot\mathbf{SA} = \mathbf{n}\cdot\mathbf{SB} = \mathbf{n}\cdot\mathbf{SC} = \mathbf{n}\cdot\mathbf{SD}$, so it is enough to test just $\mathbf{n}\cdot\mathbf{SA}$. If it is negative, then the face is invisible and it collapses into point *O*' in the application (compare ranges C25:D29 and W34:X38 in Fig. 4). The face is drawn as a five-point xy line graph made over the vertices so that the fifth point merges with the first one (range V34:V38). The vertices are inputted anticlockwise if looking from outside the cube. The coordinates of the normal vector, which are clear from the position of the cube in the *Oxyz* coordinate system, are inputted manually in range W31:Y31. The *x*, *y*, *z* coordinates of point *A* are returned in cells W32:Y32 by the functions =VLOOKUP($V32,$B$12:$G$19,2), =VLOOKUP(…,3) and =VLOOKUP(…,4). The coordinates of point *S* are calculated in range C9:E9. The product $\mathbf{n}\cdot\mathbf{SA}$ is calculated in cell Y34. Checkbox "Overlap" is linked with cell Y4. The *x*' and *y*' coordinates of the vertices are returned in cells W34:X38 by function VLOOKUP. In W34, X34, they are =IF(AND($Y$4,Y$34<0),0,VLOOKUP(V34,$B$12:$G$19,5))

and =IF(…,6)). The same method is used for drawing the other faces. If checkbox "Overlap" is not checked, then all faces are depicted (Fig. 4, right side).
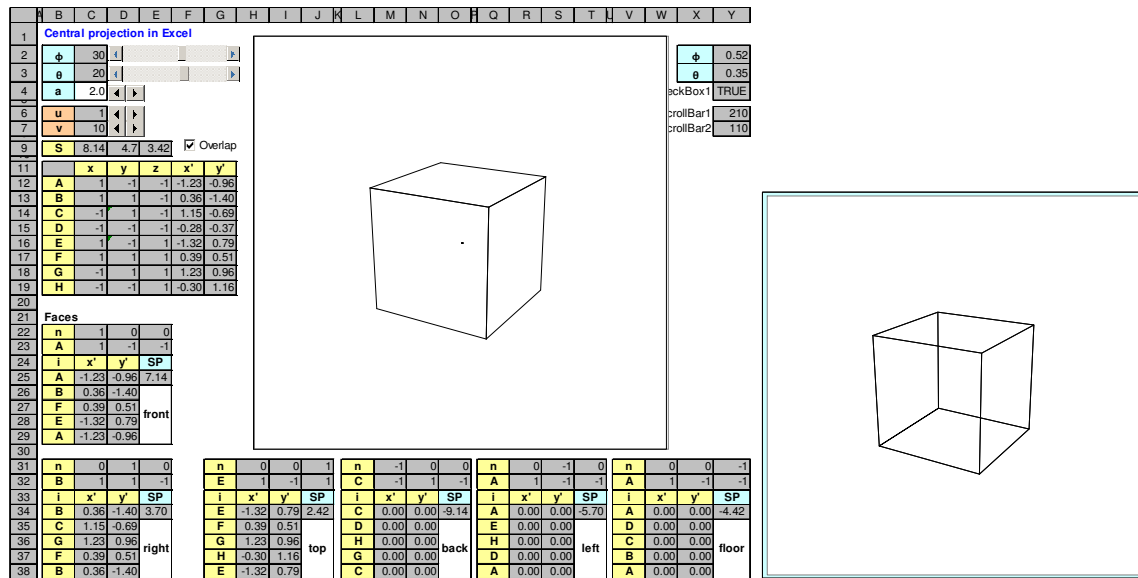
Figure 4: Cube with solved overlapping of the faces; overlapping switched on (right) and off (left)

5   Projection of an irregular convex figure

Central projection of a house is in Fig. 5. The data for the floor are in range B39:E52. The vertices go anticlockwise if looking from outside the house. The last one merges with the first one (B46:D52). Normal vector **n** is calculated in range B39:E44 as vector product of two nonparallel vectors **v1** and **v2** defined in the face. First, the first three vertices of the face are returned in range B39:E41: cell B39 contains =B46, cells C39:E39 contain =VLOOKUP(B39,$B$12:$G$37,2), =VLOOKUP(…,3) and =VLOOKUP(…,4). The formulas are copied to the next two rows. Vectors **v1** and **v2** are calculated in range C42:E43. Cells C42 and C43 contain the formulas =C40-C39 and =C41-C39, which are copied to the right. Vector **n** is calculated in range C44:E44 by the formulas =D42*E43-E42*D43, =E42*C43-C42*E43, =C42*D43-D42*C43 that give the vector product of vectors **v1** and **v2**. As the vertices go anticlockwise and **v1** goes from the first vertex to the second one while **v2** goes from the first vertex to the third one, then $\mathbf{n} = \mathbf{v1} \times \mathbf{v2}$ is oriented out of the figure. Checkbox "Overlap" is linked with cell W4. The scalar product of the normal vector and the vector going from *S* to the first vertex of the face (range B39:E39) is in cell E46. Cells C46 and D46 contain =IF(AND($W$4,E$46<0),0,VLOOKUP(B46,$B$12:$G$37,5)) and =IF(…,6)), which are copied down. The face is depicted as a xy line graph made over range C46:D52. The other faces are depicted in the same way.

Central projection in Excel

φ	-20	
θ	20	
size	1	
u	1	
v	10	

φ	-0.35
θ	0.35
TRUE	CheckBox1
160	ScrollBar1
110	ScrollBar2

$$x' = (v-u)\frac{-x\sin\phi + y\cos\phi}{v - x\cos\phi\cos\theta - y\sin\phi\cos\theta - z\cos\theta}$$

$$y' = (v-u)\frac{-x\sin\theta\cos\phi - y\sin\theta\sin\phi + z\cos\theta}{v - x\cos\phi\cos\theta - y\sin\phi\cos\theta - z\sin\theta}$$

| S | 8.83 | -3.21 | 3.42 | ☑ Overlap |

i	x	y	z	x'	y'	
1	1	1	-1	1.18	-1.05	
2	1	-1	-1	-0.59	-1.36	floor
3	-1	-1	-1	-1.06	-0.61	
4	-1	1	-1	0.47	-0.39	
5	-0.5	1.87	-1	1.25	-0.44	
6	0.5	1.87	-1	1.65	-0.76	
7	1	1	1	1.27	0.73	
8	1	-1	1	-0.64	0.53	ceiling
9	-1	-1	1	-1.13	1.01	
10	-1	1	1	0.5	1.14	
11	-0.5	1.87	1	1.33	1.11	
12	0.5	1.87	1	1.76	0.91	
13	0	-1	2	-0.94	1.76	roof
14	0	1	2	0.88	1.86	
15	-0.3	-1	-1	-0.91	-0.84	
16	0.3	-1	-1	-0.77	-1.06	door
17	0.3	-1	0.25	-0.81	0.02	
18	-0.3	-1	0.25	-0.95	0.2	
19	1	-0.5	0	-0.13	-0.38	
20	1	-0.25	0	0.11	-0.35	window 1
21	1	-0.25	0.5	0.11	0.12	
22	1	-0.5	0.5	-0.13	0.09	
23	1	0.25	0	0.56	-0.29	
24	1	0.5	0	0.79	-0.26	window 2
25	1	0.5	0.5	0.8	0.2	
26	1	0.25	0.5	0.58	0.18	

1	1	1	-1		3	-1	-1	-1		4	-1	1	-1		2	1	-1	-1		10	-1	1	1
2	1	-1	-1		2	1	-1	-1		3	-1	-1	-1		1	1	1	-1		9	-1	-1	1
3	-1	-1	-1		8	1	-1	1		9	-1	-1	1		7	1	1	1		13	0	-1	2
v1	0	-2	0		v1	2	0	0		v1	0	-2	0		v1	0	2	0		v1	0	-2	0
v2	-2	-2	0		v2	2	0	2		v2	0	-2	2		v2	0	2	2		v2	1	-2	1
n	0	0	-4		n	0	-4	0		n	-4	0	0		n	4	0	0		n	-2	0	2

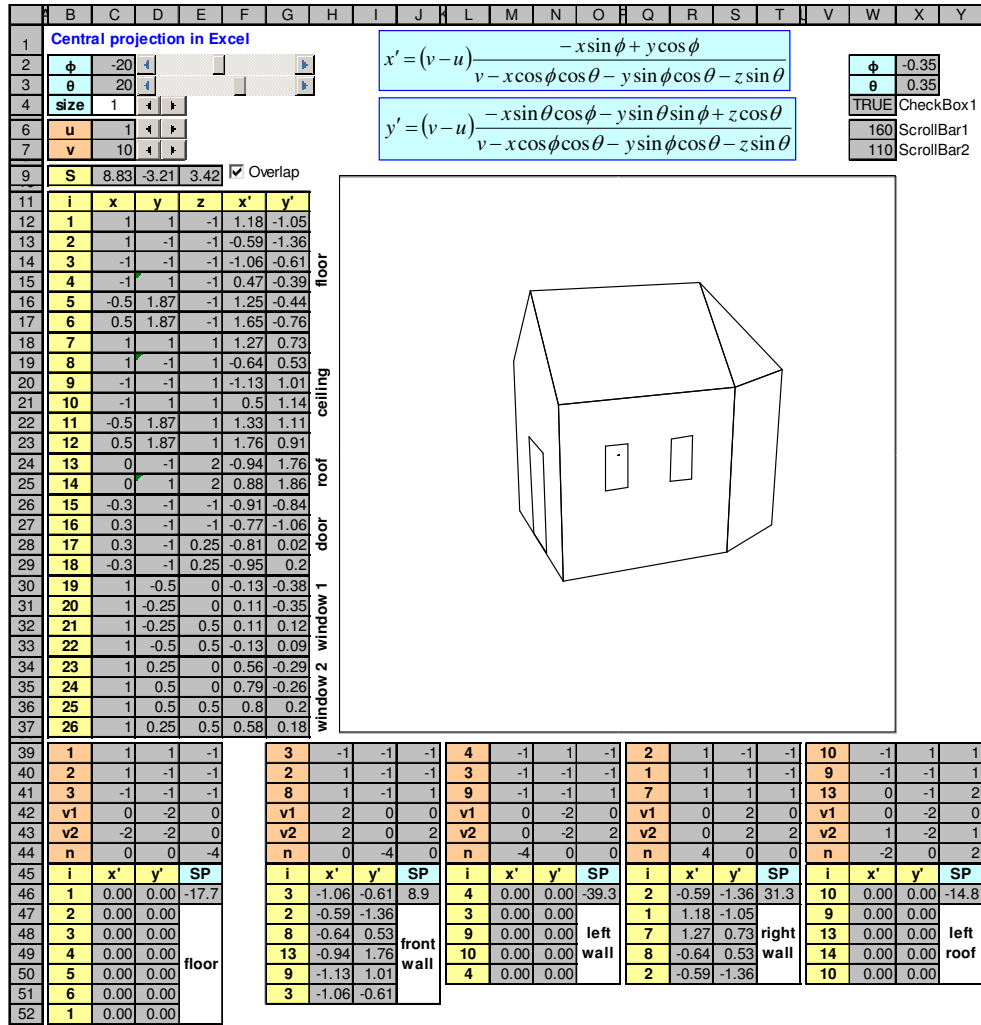| i	x'	y'	SP		i	x'	y'	SP		i	x'	y'	SP		i	x'	y'	SP		i	x'	y'	SP
1	0.00	0.00	-17.7		3	-1.06	-0.61	8.9		4	0.00	0.00	-39.3		2	-0.59	-1.36	31.3		10	0.00	0.00	-14.8
2	0.00	0.00			2	-0.59	-1.36			3	0.00	0.00			1	1.18	-1.05			9	0.00	0.00	
3	0.00	0.00			8	-0.64	0.53	front		9	0.00	0.00	left		7	1.27	0.73	right		13	0.00	0.00	left
4	0.00	0.00	floor		13	-0.94	1.76	wall		10	0.00	0.00	wall		8	-0.64	0.53	wall		14	0.00	0.00	roof
5	0.00	0.00			9	-1.13	1.01			4	0.00	0.00			2	-0.59	-1.36			10	0.00	0.00	
6	0.00	0.00			3	-1.06	-0.61																
1	0.00	0.00																				

Figure 5: House with solved overlapping; the ranges for the rest parts are not depicted

Other views of the house are in Figs. 6, 7 and 8. In Fig. 6, the overlapping is switched off. In Fig. 7, the position of the projecting plane is changed to $\phi = 50°$ and $\theta = -18°$. In Fig. 8, the centre of the projection, that is, the position of the observer, is closer to the house.
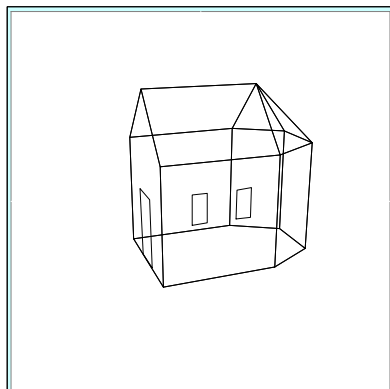
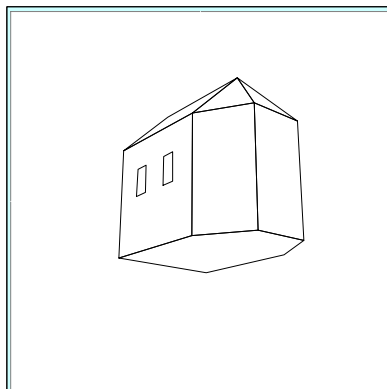Figure 6: The house from Fig. 5 if overlapping is switched off

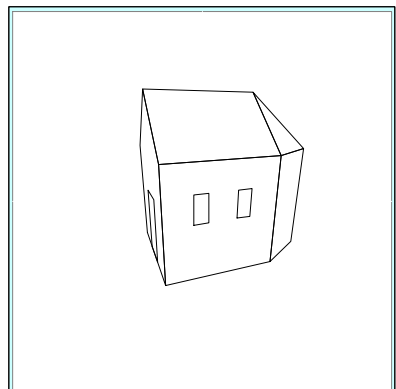Figure 7: The house from Fig. 5 if $\phi = 50°$, $\theta = -18°$

Figure 8: The house from Fig. 5 if $v = 5$

6   Conclusions

The principles of the central projection were presented in this article. The governing equations were derived. Developing revolvable and sizable projections of cube and a house in Excel™ was described if (1) all faces are depicted including those that are invisible in the reality (the wire model) and (2) the visible faces are depicted only (the real model).

Traditionally, the edges that are invisible in the reality are drawn dashed. Changing a full edge to dashed automatically when the figure is rotated and the edge becomes invisible in the reality is only possible if VBA programming is used. Also adding colours and depicting non-convex figures require programming. The algorithms and Excel implementations are in [3].

An interesting follow up to the presented application would be adjusting it so that the house would be depicted from inside.

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