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# Pascal Pyramids: a mathematical exploration using spreadsheets

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# Pascal Pyramids:

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## Abstract

The features of Pascal's Triangle are generalised to 3-variable and 4-variable expressions resulting in the formation of pyramids of coefficients. The steps required to create the coefficients are also given.

## 1. Introduction

The name Pascal is most readily associated with a triangular array of numbers that give the coefficients of $x^{n-r}\,y^r$ in the expansion of $(x + y)^n$. Each number in row $n$ of the triangle is the sum of the two numbers above it in the previous row. The process of creating one row from the one above it can be generalised to numbers in the configuration of a pyramid and in this paper we explore the possibilities and limitations of the number pyramids that can be made in this way. The exploration was carried out with the use of a spreadsheet which made it possible to have hypotheses about what was happening as the pyramids grow in height and to use the data generated by the spreadsheet to check if a hypothesis was correct.

## 2. Creating a Pascal Pyramid

The first four expansions of the polynomials $(x + y + z)^n$ for $n$ = 0, 1 2 and 3 lead to terms in $x$, $y$ and $z$ that can be laid out in the shape of a triangle, as shown in Figure 1.
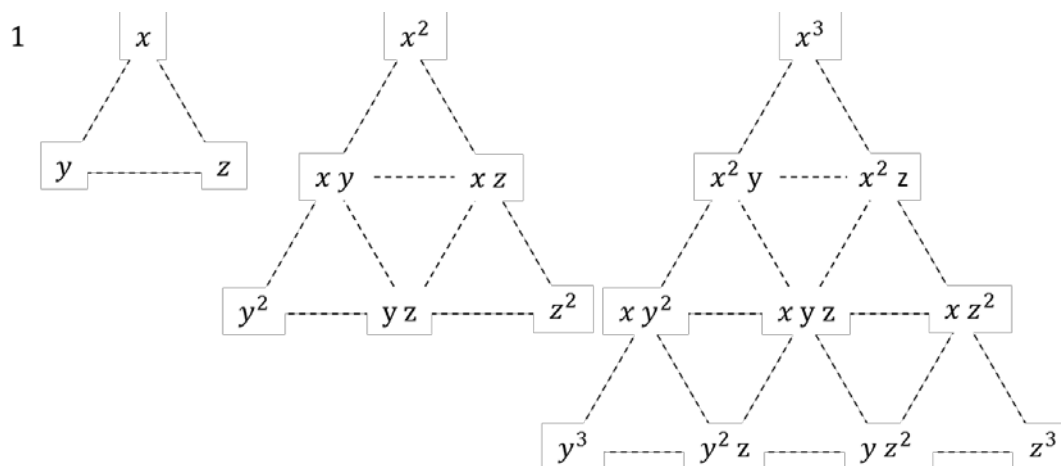


Figure 1: Terms in the expansions of $(x + y + z)^n$

In the next section we show how a spreadsheet can be used to find the corresponding coefficients of these terms, but first, note how Figure 2 shows that, when these triangles are stacked one on the other, they form a pyramid.
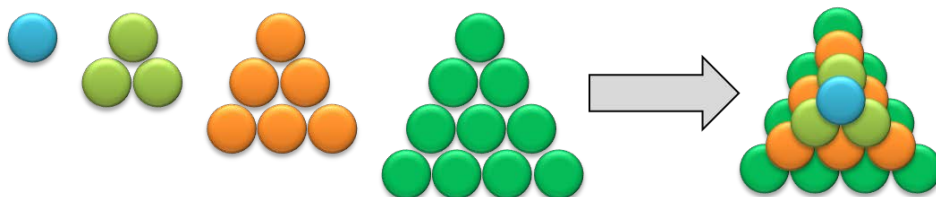


Figure 2: Making layers of terms to form a pyramid

The pyramids have been named Pascal Pyramids because they have an analogous meaning to the numbers in Pascal's Triangle, namely they are the coefficients of the terms in the expansion of $(x + y + z)^n$. Also, each layer of the pyramid is formed by combining three numbers in the layer above (rather than combining two numbers as is the case in Pascal's Triangle).

### 3.   Putting numbers in a Pascal Pyramid

We start from the top layer and use spreadsheet formulae to generate the layer below. Figure 3 shows the first three layers that the spreadsheet generates. The following steps can be used to generate the pyramid.

1.  Begin by selecting the whole worksheet and narrowing the column width. Also set the alignment to middle as this makes the patterns easier to see.

2.  Choose **File: Options: Advanced** and in the **Display Options** for this worksheet, uncheck this box:

3.  ☐ Show a zero in cells that have zero val

4.  Set L3 to 1 and label it as Layer 0 using a text box, rather than typing in a cell.

5.  Set (K6) = SUM(K3, J1, L1) and then drag K6 over the range K6:M5. This will make Layer 1 of the pyramid which should also be labelled with a text box.
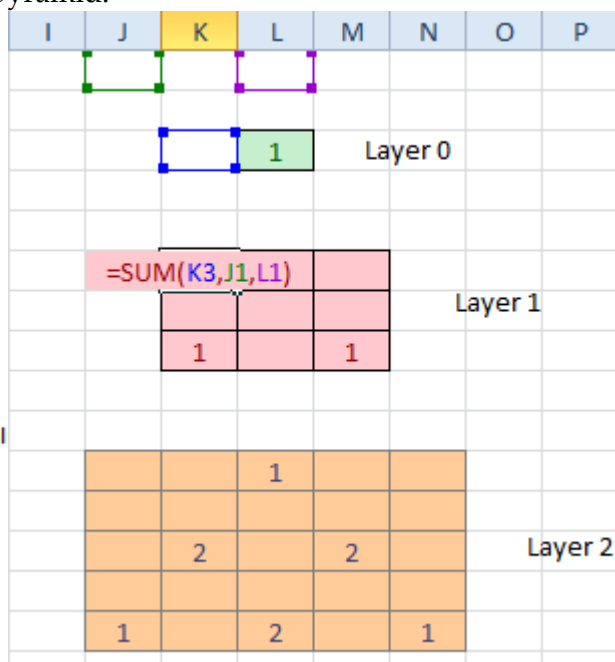


Figure 3: The first three layers of a Pascal Pyramid

The text box is used to ensure that the formula that creates the layers does not pick up the labels and we leave two rows between each layer because the formula which

spans three rows must not pick up numbers from the layer two above it. So to create Layer 2, we start in cell J11.

6. Set (J11) = SUM(J6, I4, K4) and drag this formula over the range J11:N15 to create Layer 2.

7. To create Layer 3, enter the formula (I18)=SUM(I11,J9,H9) and drag over the range I18:O24. Continue to create more layers in the same way until you have, say, Level 7.

## 4. Giving meaning to the numbers in each layer

Layer 3 of the spreadsheet is used in Figure 4 to show how the coefficients generated by the spreadsheet relate to the terms of the expansion of $(x + y + z)^3$. For subsequent layers the same process applies
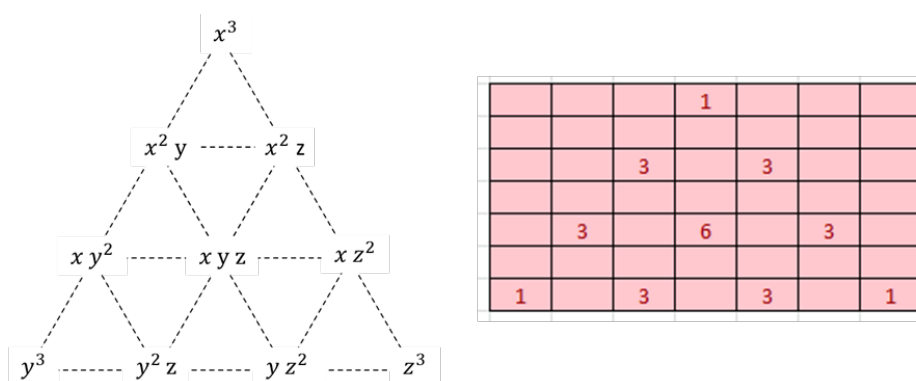


$$(x + y + z)^3 = x^3 + 3x^2y + 3x^2z + 3xy^2 + 6xyz + 3xz^2 + y^3 + 3y^2z + 3yz^2 + z^3$$

Figure 4: Matching the terms and coefficients of Layer 3 of the Pascal Pyramid.

### 4.1. Features of the Pyramid layers

There are features of the Pascal Pyramid that give a check that the coefficients have been correctly generated. Rather than explain what the features are, we have posed questions below whose answers would show a deep understanding of how the Pascal Pyramid is constructed.

1. Check the entries in each side of the triangle. How do the numbers along the sides relate to the rows of a Pascal's Triangle?

2. In Column A and in the same row as the top of each triangle, enter a formula such as

3. (A11) =SUM(O11:S15)

4. This example gives the sum of the numbers in Layer 2. You should find the sum of the numbers in the other triangles and explain the pattern in the totals.

5. All the numbers in any row (except the first and last) of a layer have a common factor. The common factor is the value of the first number in that row. Why is this number a factor of every other number in the row?

### 5. Moving to the next level

The Pascal Pyramid for $(x + y + z)^n$ is formed from triangular sections and hence has the shape of a triangular-based pyramid. Given that the expansion of $(w + x + y + z)^n$ is an expression that involves four variables, we might expect that the corresponding Pascal Pyramid would be a square-based pyramid. In the steps given below we show how a square-based pyramid can be formed of square layers.

1. Open a new worksheet and, as before, narrow the columns and centre the alignment. Also set the worksheet not to show zero values.

2. Layer 0 and Layer 1 can be created by hand with two rows of the worksheet separating each layer.

3. The formula in J13 is

4. (J13) = SUM(K8, I8, I6, K6)

5. Drag this over the range J13:N17 and label this as Layer 2.

6. Make a few more layers in this way so that the patterns that the spreadsheet creates become clear.
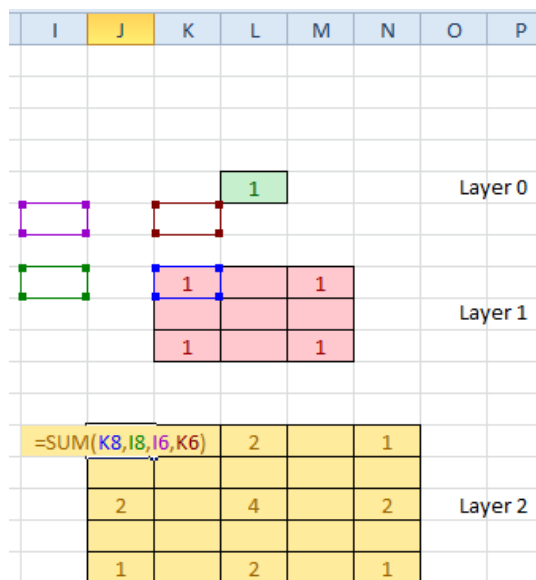


Figure 5: Creating a square-based pyramid

The layers that are created in this way have some, but not all of the properties which the layers of the triangular-based Pascal Pyramid need to have. For example:

1. In A8 enter the formula:

2. (A8) = SUM(K8:M10)

3. The value in A8 is the first of a sequence formed by the totals of the numbers in successive layers. What is the property of this sequence and why is the property true?

4. The sides of the square layers are to be found in Pascal's Triangle. Can you explain why this is so?

5. Each row of a layer (apart from the first and last) has a common factor which is the number in the first position.

The problem is that when we consider the expansion of $(w + x + y + z)^2$, we find that it has 10 terms:

$$w^2, wx, wy, wz$$
$$x^2, xy, xz$$

$$y^2, yz$$
$$z^2$$

However, the square layer that was created for Layer 2 has only 9 coefficients. Even though many of the required properties of a layer are present in the spreadsheet, the key property that each term can be associated with a coefficient is missing. The discrepancy is even larger for Layer 3 where the spreadsheet generates only 16 coefficients whereas the expansion of $(w + x + y + z)^3$ has 20 terms.

## 6. An alternative type of Pascal Pyramid

Given that there are four variables to accommodate in the expansion of $(w + x + y + z)^n$, we might also consider the possibility positioning them in a tetrahedral configuration. Figure 6 show the tetrahedron for $(w + x + y + z)^2$.
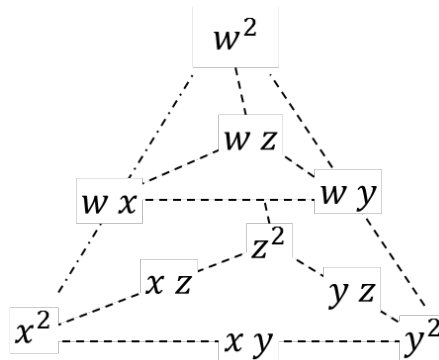


Figure 6: Terms of $(w + x + y + z)^2$ arranged to form a tetrahedron

The horizontal layers are:

$w^2 = w^2 \times 1$ at the top as Layer 0

$wx + wy + wz = w \times (x + y + z)$ as Layer 1 and

$$x^2 + xy + y^2 + yz + z^2 + xz = w^0 \times (x^2 + xy + y^2 + yz + z^2 + xz)$$
as Layer 3

These expressions have two parts. First, there is a $w^n$ term for $n = 2, 1 \ and \ 0$ multiplied by an expression in $x, y \ and \ z$ which is the same as a layer of the Pascal Pyramid for $(x + y + z)^n$. This arrangement suggests how the corresponding Pascal Pyramid can be formed.

1. Right click the tab of the worksheet that you made for the Pascal Pyramid for 3 variables and choose the Move or Copy option. Check the Create a Copy box and move the copy to Before Sheet 3.

We will make a Pascal Pyramid for $(w + x + y + z)^3$

To do this we copy the values in the first side of the pyramid into cells where we can use them as multipliers.
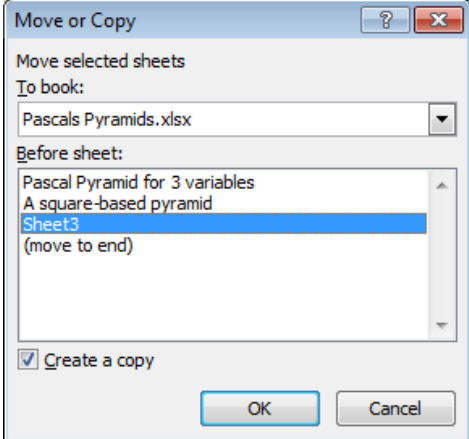
Figure 7: Making a copy of a worksheet

2. Set (C3) = 1, (C6) = 3, (C11) = 3 and (C17) = 1.

3. To create Layer 0 of the new pyramid,
   set (AA3) = $C$3*Q3.
   Set (Z6) = $C$6*P6 and drag this formula over Z6:AB8.
   Set ((Y11) = $C$11*O11 and drag this formula over Y11:AC15 and finally set (X18) =$C$18*N18 and drag this formula over X18:AD24.

4. To check that your work is correct, sum all three layers of the pyramid that you have made, Do they add to the correct total?

The above process creates the layers of the Pascal Pyramid for $(w + x + y + z)^3$. Figure 8 shows how the coefficients that have been generated relate to the terms on the expansion of $(w + x + y + z)^3$.
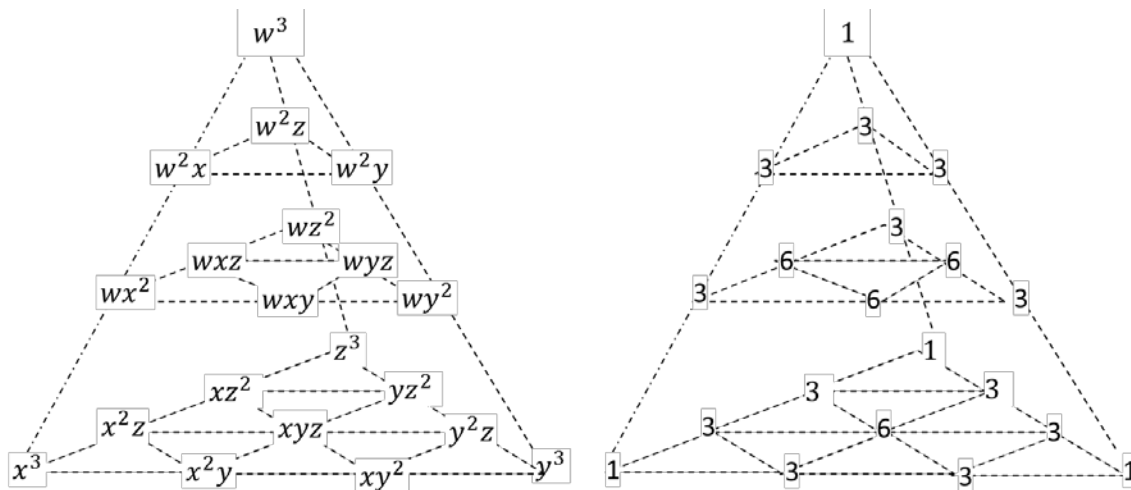
Figure 8: Terms in the expansion of $(w + x + y + z)^3$ and the corresponding Pascal Pyramid of coefficients

Thus the Pascal Pyramid for the expansion of the 3-variable expression, $(x + y + z)^n$ has one layer for each value of $n$, whereas the Pascal Pyramid for the expansion of the 4-variable expression, $(w + x + y + z)^n$, requires one whole pyramid for each value of $n$.

## 7. Limitations

We have seen that it is not appropriate to generalise the triangular-based pyramid to a square-based pyramid, as the ensuing figure does not have enough points at which coefficients can be positioned. Some of the required features are present, such as sides that match the rows of Pascal's triangle do apply and the numbers so generated do add to the required total, but the shortage of positions for the coefficients imply that a square-based pyramid cannot have all the Pascal Pyramid properties. An hexagonal-based pyramid can also be created in which the sides are rows of Pascal's triangle and where the numbers add to the correct total (a power of 6 in this case), but the shortage of positions in the resulting pyramid means that it cannot have all the required properties.

As the size of the expression to be expanded into terms and their coefficients generalising the current investigation into hyper-pyramids of higher dimensions may not be fruitful, as the coefficients can readily be obtained by using combinations. For example, to find the coefficient of the term $v^2 x^3$ in the expansion of $(v + w + x + y + z)^5$ can be found by evaluating $C(2,5) \times C(3,5)$. This is because there will be $C(2,5)$ ways of forming $v^2$ and $C(3,5)$ ways of forming $x^3$. This simple method of finding the required coefficients is probably much less demanding that trying to construct hyper-pyramids.

## 8. Conclusion

This article describes an exploration of a well-known mathematical object, Pascal's Triangle. The starting point for the exploration was the recurrence formula used to generate Pascal's Triangle:

$$C(n,r) = C(n-1, r-1) + C(n-1, r)$$

This is a rule for finding one row of Pascal's Triangle from the numbers in the row above it. The initial hypothesis was that a similar rule could be used to find the coefficients of the terms of $(x + y + z)^n$.

It was shown how a Pascal Pyramid for this 3-variable expression can be formed and the steps required to create a spreadsheet to generate the coefficients in a reasonably economical way were shown. A second type of Pascal Pyramid was then created for the coefficients of terms in the expansion of $(w + x + y + z)^n$. At each stage, the spreadsheet made it possible to include quick checks that the data being generated had the correct properties.

More generally, the authors believe that spreadsheets have a very important role to play in mathematical problem solving and, now that computers are to be found in every mathematics teaching room, there appears to be no reason for not using them with students to undertake mathematical explorations.