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Market Neutral Portfolio Selection: A Pedagogic Illustration

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Abstract

This paper considers market neutral portfolio selection, which is an advanced investment topic. It draws on an idea in the investment literature that short selling a stock in practice is like investing in an artificially constructed security. Such an idea allows this paper to extend textbook coverage of portfolio selection without short sales to a realistic long-short setting. Spreadsheet illustrations are provided, with and without using the derived analytical results. Thus, the pedagogic materials as covered in this paper can accommodate investment courses with different levels of analytical rigor.

Keywords

market neutral investment, long-short equity strategy, single index model, beta neutrality, dollar neutrality

Cover Page Footnote

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1 Introduction

Market neutral strategies, as advanced investment tools for volatile equity markets, have gained considerable attention among investment practitioners and sophisticated investors. In essence, market neutrality is about achieving zero correlations between the returns of an investment portfolio and indices of equity markets or some specific economic sectors. With the remaining risk being unsystematic, what is sought in a market neutral investment is a return in excess of the risk-free interest rate. Among various market neutral strategies, the simplest one to comprehend from a pedagogic perspective is the long-short equity strategy, which seeks to exploit potential mispricing of securities (stocks), by buying and short selling securities that are considered to be undervalued and overvalued, respectively. In a long-short equity portfolio, market neutrality is achieved by offsetting completely the opposite responses, of long and short sides of the portfolio, to index movements.[1]

Implementation of the long-short equity strategy in practice requires, most importantly, the ability of the investor involved to identify correctly a set of mispriced securities. To achieve market neutrality and optimal portfolio performance in terms of risk-return trade-off, also required is a quantitative approach to allocate the available investment capital among the identified securities. This is where portfolio selection models can play an important role. For a model to be useful for constructing a market neutral portfolio in practice, it must be able to capture adequately the reality of long-short investing. Thus, before considering a specific model, let us first describe briefly the reality of long-short investing in the following:

Short selling involves the sale of a borrowed security via a brokerage firm. In U.S. equity markets, for example, a margin deposit of at least 50% of the share value is required for each security — regardless of whether the security is purchased or sold short — to be held at an account with the brokerage firm, in order to satisfy the Federal Reserve Board's Regulation T. Cash, interest bearing treasury bills, and some other securities that the short seller owns can be used to provide the deposit. Any interest generated from the deposit will be earned by the short seller. The short-sale proceeds are held as collateral for the borrowed security at the brokerage firm. The short seller may get a rebate from the brokerage firm for the interest that it earns from the short-sale proceeds; this is commonly called the short interest rebate. Further, the short seller is responsible

[1] See, for example, Nicholas (2000) and Jacobs and Levy (2005) for descriptions of various market neutral strategies.

to reimburse the lender of the security for any dividend payments.

In a long-short portfolio with matching values on long and short sides, suppose that all purchased securities are with 100% cash. As the margin requirement to satisfy Regulation T is 50% of the security value, any unused margin from the long side can provide the margin for the short side. Accordingly, for each dollar of investment capital, the investor can have as much as two dollars' worth of securities in a long-short portfolio, with one dollar's worth on each side. In practice, however, to avoid the risk of an insufficient margin due to adverse subsequent price movements, a cash reserve — also known as the liquidity buffer — is usually provided, thus reducing the available investment funds for the two sides of a long-short portfolio.

As indicated in Alexander (1993), short selling a security can be viewed as an investment in an artificially constructed security. The investment income has three major components. They include the negative of the price change of the security, the short interest rebate, and the interest earned on the margin deposit.[2] The first component — which is crucial for the success or failure of the investment — is risky, but the remaining two components are risk-free. In the context of a market neutral portfolio, a prorated portion of the interest earned on the liquidity buffer can be viewed as the interest from the margin deposit associated with a shorted security.

Drawing on Alexander's insight, Kwan (1999) has derived a market neutral portfolio selection model, along with an algorithm for portfolio construction. For analytical convenience, the model formulation in that study uses a well-known single index model to characterize the covariance structure of security returns.[3] More recently, Jacobs, Levy, and Markowitz (2005, 2006) have considered practical long-short portfolio optimization for various covariance structures and for different combinations of neutrality conditions.[4] These two recent studies, which have special emphases on computational efficiency, have extended some available fast algorithms for practical long-only portfolio construction to long-short cases.

In view of the practical relevance of market neutral portfolio selection models, this paper presents a basic version of such models, which is suitable for coverage in investment courses. As in Kwan

[2] Analytically, any dividend income and repayment (to lenders) for securities in long and short positions, respectively, can be incorporated into the price change of each security. The cost of borrowing shares to facilitate short-sale transactions can also be accounted for implicitly, by reducing the short interest rebate. What remain unaccounted for are the brokerage fees involved in equity trading. However, if such fees represent only a small proportion of the investment capital, whether they are accounted for or ignored in a portfolio selection model will not affect significantly the portfolio allocation results.

[3] See, for example, Elton, Gruber, Brown, and Goetzmann (2010, Chapter 7) for a description of the single index model.

[4] Bruce Jacobs and Kenneth Levy are well-known investment practitioners, and Harry Markowitz is a 1990 Nobel winner in economics for profound contributions of his pioneering work in modern portfolio theory.

(1999), the version here also uses the single index model to characterize the covariance structure. It shows how a market neutral equity portfolio can be constructed by extending investment textbook materials on long-only portfolio selection to a realistic long-short setting, with Microsoft Excel^{TM} playing an important pedagogic role.[5] As explained below, the analytical materials in this paper can be used in different ways, depending on the desired analytical details for the individual investment courses involved.

For investment courses where analytical details are de-emphasized, the analytical coverage of the topic is best focused on the corresponding investment concepts, including how the insight of Alexander (1993) can facilitate the formulation of a market neutral portfolio selection model. With portfolio selection formulated as a constrained optimization problem, Excel Solver can be used directly to provide a numerical solution for each set of input parameters. For such courses, as the corresponding analytical solution and its derivation are unimportant, the numerical illustrations that Solver provides will serve to complement the analytical coverage of the topic.

For investment courses where analytical models are presented, an Excel implementation based on the corresponding analytical results, in addition to the Solver-based approach, is useful. If the model derivation is sketched or omitted entirely in such courses, then it is also useful for the analytical coverage of the topic to include an intuitive explanation of the derived criteria for portfolio selection. A comparison of the derived analytical results with the corresponding textbook materials on long-only portfolio selection will enhance learning.[6]

Given the scope of this study, the issue of algorithmic efficiency for large portfolio construction as considered in the above-referenced studies is not as important. Thus, this paper uses various Excel functions to illustrate the computations involved in a small-scale case, for which the issue of algorithmic efficiency need not be addressed. Further, noting that the traditional approach of using slack variables to accommodate inequality constraints may be unfamiliar to many business students, this paper presents a model derivation without requiring their use. Here, an inequality constraint pertains to the requirement that the investment capital as allocated to each security be either zero or of a particular sign. In essence, the analytical tools for the derivations in this paper are confined to taking partial derivatives, using the Lagrangian approach to accommodate linear equality constraints, and solving systems of linear equations.[7]

[5] Hereafter, the software is simply referred to as Excel, with its trademark implicitly recognized.

[6] In two advanced investment courses for senior undergraduate and M.B.A. students, currently taught by the author of this paper, the derivation of the same market neutral portfolio selection model is covered. Experience has shown that we can greatly reduce the analytical burden for students, by noting the close similarities between the derivations of long-only and long-short portfolio models, and between the corresponding analytical results.

[7] See Kwan (2007) for an intuitive explanation of the Lagrangian approach in the context of portfolio selection.

For long-short portfolio construction on Excel, based on the analytical solution of a model, requires individual securities to be ranked and relabeled repeatedly in accordance with some ranking criteria. Although Excel has a menu item for manually sorting data in a worksheet, the procedure involved is inconvenient for the model considered. We are able to bypass such a manual procedure, by using instead various Excel functions, some of which are originally intended for other purposes. The rationale and the technical details will be provided later during the Excel illustrations.

The remainder of this paper is organized as follows: Section 2 formulates a basic version of market neutral portfolio selection models. Its analytical solution is derived from a pedagogic perspective in Section 3. A summary of the model formulation and the derived analytical solution, along with the key expressions in the above two sections, are provided in Section 4. This summary section is intended for readers who are primarily interested in applying Excel tools to market neutral portfolio construction, without the encumbrance of indirect analytical details. Excel illustrations, with and without relying on the derived analytical solution, are presented in Section 5. Finally, Section 6 provides some concluding remarks.

2 Formulation of a Market Neutral Portfolio Selection Model

Suppose that two disjoint sets of securities have been identified for potential holdings in long and short positions for a market neutral portfolio. Let us label the two sets as L and S and the individual securities considered as $1, 2, \ldots, n_L$ and $[1], [2], \ldots, [n_S]$, where n_L and n_S — which are the corresponding numbers of securities in the two sets — need not be the same. In a single-period setting, let P_i and $P_{[j]}$ be the beginning-of-period prices of securities i and $[j]$, respectively, for $i = 1, 2, \ldots, n_L$ and $j = 1, 2, \ldots, n_S$. The corresponding end-of-period prices, which are random, are labeled as $\widetilde{P}_i$ and $\widetilde{P}_{[j]}$.

The portfolio is formed at the beginning of the period. Suppose that each security on the long side is purchased with 100% cash and that the cash reserve in the portfolio has been predetermined to be a constant proportion $\lambda > 0$ of the share values of the shorted securities. Suppose also that the short seller earns a risk-free return on this deposit and is entitled to receive a rebate of a constant proportion $0 \leq \nu < 1$ of the risk-free interest that the brokerage firm earns on the short-sale proceeds. Let R_f be the risk-free interest rate over the period.

On a prorated basis, short selling each share of security $[j]$ corresponds to a beginning-of-period dollar investment of $\lambda P_{[j]}$. The corresponding end-of-period dollar return is $-\widetilde{P}_{[j]} + (1 + R_f)\lambda P_{[j]} + (1 + \nu R_f) P_{[j]}$. The first term, $-\widetilde{P}_{[j]}$, represents the end-of-period random price that the short seller will pay for buying the share in the market to terminate the short-sale arrangement. The second term, $(1 + R_f)\lambda P_{[j]}$, represents the cash deposit plus interest. The third term, $(1 + \nu R_f)P_{[j]}$, represents the short-sale proceeds that will be returned to the short seller by the brokerage firm plus the short interest rebate.

Now, denote $N_i \geq 0$ and $-N_{[j]} \geq 0$ as the numbers of shares of security i and security $[j]$ that are held in the portfolio, respectively, for $i = 1, 2, \ldots, n_L$ and $j = 1, 2, \ldots, n_S$. Holdings of fractional shares are assumed to be permissible, and we follow the common convention that a negative holding indicates the short sale of a security. With the allocation of the beginning-of-period investment capital being

$$W = \sum\nolimits_{i=1}^{n_L} N_i P_i + \sum\nolimits_{j=1}^{n_S} \left(-N_{[j]}\right) \lambda P_{[j]}, \tag{1}$$

the end-of-period value of the portfolio, which is random, is

$$\widetilde{W} = \sum\nolimits_{i=1}^{n_L} N_i \widetilde{P}_i + \sum\nolimits_{j=1}^{n_S} N_{[j]} \left[\widetilde{P}_{[j]} - (1 + R_f)\lambda P_{[j]} - (1 + \nu R_f) P_{[j]}\right]. \tag{2}$$

Therefore, the random rate of return (or, simply, the random return) of the portfolio is

$$\widetilde{R}_p = \frac{\widetilde{W} - W}{W} = \sum\nolimits_{i=1}^{n_L} \frac{N_i}{W} \left(\widetilde{P}_i - P_i\right) + \sum\nolimits_{j=1}^{n_S} \frac{N_{[j]}}{W} \left[\widetilde{P}_{[j]} - P_{[j]} - (\lambda + \nu) R_f P_{[j]}\right], \tag{3}$$

which can be written more succinctly as

$$\widetilde{R}_p = \sum\nolimits_{i=1}^{n_L} x_i \widetilde{R}_i + \sum\nolimits_{j=1}^{n_S} x_{[j]} \left[\widetilde{R}_{[j]} - (\lambda + \nu) R_f\right]. \tag{4}$$

Here, $\widetilde{R}_i = (\widetilde{P}_i - P_i)/P_i$ and $\widetilde{R}_{[j]} = (\widetilde{P}_{[j]} - P_{[j]})/P_{[j]}$ are the random returns of securities i and $[j]$, respectively, and $x_i = N_i P_i / W \geq 0$ and $x_{[j]} = N_{[j]} P_{[j]} / W \leq 0$ are the corresponding holdings of the two securities as proportions of the investment capital, with a negative proportion indicating the short sale of a security. These proportions are commonly known as portfolio weights. Equation (1) implies a budget constraint of

$$\sum\nolimits_{i=1}^{n_L} x_i - \lambda \sum\nolimits_{j=1}^{n_S} x_{[j]} = 1. \tag{5}$$

Equation (4) indicates that $\widetilde{R}_p$ is a linear combination of the $n_L + n_S$ random returns $\widetilde{R}_1, \widetilde{R}_2, \ldots, \widetilde{R}_{n_L}, \widetilde{R}_{[1]}, \widetilde{R}_{[2]}, \ldots, \widetilde{R}_{[n_S]}$. Now, let $\mu_1, \mu_2, \ldots, \mu_{n_L}, \mu_{[1]}, \mu_{[2]}, \ldots, \mu_{[n_S]}$ be the corresponding expected returns. By definition, the variance of $\widetilde{R}_i$ is the expected value of $(\widetilde{R}_i - \mu_i)^2$ and the covariance of $\widetilde{R}_i$ and $\widetilde{R}_j$ is the expected value of $(\widetilde{R}_i - \mu_i)(\widetilde{R}_j - \mu_j)$, where i and j can be any of the above $n_L + n_S$ subscripts. With σ_{ik}, $\sigma_{i[j]}$, and $\sigma_{[j][k]}$ denoting the individual covariances of returns, it is implicit that $\sigma_{ii} = \sigma_i^2$ and $\sigma_{[j][j]} = \sigma_{[j]}^2$ are the variances of returns of securities i and $[j]$, respectively. The portfolio's expected return and variance of returns can be expressed as

$$\mu_p = \sum\nolimits_{i=1}^{n_L} x_i\mu_i + \sum\nolimits_{j=1}^{n_S} x_{[j]} \left[\mu_{[j]} - (\lambda+\nu)\, R_f\right] \tag{6}$$

and

$$\sigma_p^2 = \sum\nolimits_{i=1}^{n_L} \sum\nolimits_{k=1}^{n_L} x_i x_k \sigma_{ik} + 2\sum\nolimits_{i=1}^{n_L} \sum\nolimits_{j=1}^{n_S} x_i x_{[j]}\sigma_{i[j]} + \sum\nolimits_{j=1}^{n_S} \sum\nolimits_{k=1}^{n_S} x_{[j]} x_{[k]}\sigma_{[j][k]}, \tag{7}$$

respectively.[8]

The portfolio's expected performance — when stated as its expected return in excess of the risk-free interest rate, per unit of risk exposure — is the Sharpe ratio

$$\theta = \frac{\mu_p - R_f}{\sigma_p} \tag{8}$$

in an *ex ante* context. With θ being the objective function of an optimization problem, the corresponding decision variables are the n_L+n_S portfolio weights $x_1, x_2, \ldots, x_{n_L}, x_{[1]}, x_{[2]}, \ldots, x_{[n_S]}$. Combining equations (5) and (6) leads to

$$\mu_p - R_f = \sum\nolimits_{i=1}^{n_L} x_i\left(\mu_i - R_f\right) + \sum\nolimits_{j=1}^{n_S} x_{[j]}(\mu_{[j]} - \nu R_f), \tag{9}$$

which is a convenient expression of the numerator of θ for use in its maximization under constraints.

2.1 Imposition of market neutrality

To facilitate the imposition of market neutrality on portfolio selection, we rely on the single index model to characterize the covariance structure of security returns. Specifically, we assume that the random return of each security varies linearly with the random return of a market index, labeled as $\widetilde{R}_m$. The variance of $\widetilde{R}_m$ is labeled as σ_m^2. The slopes of the individual linear relationships, labeled as β_i and $\beta_{[j]}$, for $i = 1, 2, \ldots, n_L$ and $j = 1, 2, \ldots, n_S$, are commonly known as the beta coefficients or, simply, the betas. The part of the random return of each security that is unexplained by the corresponding linear relationship is assumed to be correlated with neither $\widetilde{R}_m$ nor the unexplained returns of any other securities. The variances of the unexplained returns, labeled as σ_{ei}^2 and $\sigma_{e[j]}^2$, also for $i = 1, 2, \ldots, n_L$ and $j = 1, 2, \ldots, n_S$, are called the residual variances.

[8] The expected value and the variance of a linear combination of n random variables of the form $\sum_{i=1}^n a_i(\widetilde{Y}_i + b_i)$, where a_i and b_i are parameters and $\widetilde{Y}_i$ is a random variable, for $i = 1, 2, \ldots, n$, are $\sum_{i=1}^n a_i[E(\widetilde{Y}_i) + b_i]$ and $\sum_{i=1}^n \sum_{j=1}^n a_i a_j Cov(\widetilde{Y}_i, \widetilde{Y}_j)$, respectively. Here, $E(\widetilde{Y}_i)$ is the expected value of $\widetilde{Y}_i$ and $Cov(\widetilde{Y}_i, \widetilde{Y}_j)$ is the covariance of $\widetilde{Y}_i$ and $\widetilde{Y}_j$. The expressions of μ_p and σ_p^2 in equations (6) and (7), respectively, are based on such analytical results for the case where $n = n_L + n_S$. Each summation covering $n_L + n_S$ terms is equivalent to two summations covering, separately, n_L and n_S terms.

Under the single index model, the variances and covariances of security returns are given by

$$\sigma_i^2 = \beta_i^2\sigma_m^2 + \sigma_{ei}^2, \tag{10}$$

$$\sigma_{ik} = \beta_i\beta_k\sigma_m^2, \text{ for } i,k = 1,2,\ldots,n_L \text{ and } i \neq k; \tag{11}$$

$$\sigma_{[j]}^2 = \beta_{[j]}^2\sigma_m^2 + \sigma_{e[j]}^2, \tag{12}$$

$$\sigma_{[j][\ell]} = \beta_{[j]}\beta_{[\ell]}\sigma_m^2, \text{ for } j,\ell = 1,2,\ldots,n_S \text{ and } j \neq \ell; \tag{13}$$

and

$$\sigma_{i[j]} = \beta_i\beta_{[j]}\sigma_m^2, \text{ for } i = 1,2,\ldots,n_L \text{ and } j = 1,2,\ldots,n_S. \tag{14}$$

For analytical convenience, we assume β_i and $\beta_{[j]}$ for all i and $[j]$ to be positive. The use of the single index model allows the variance of returns — the total risk — of each security to be decomposed into systematic and unsystematic components, with the security's beta coefficient capturing its systematic risk. It also allows equation (7) to be written succinctly as

$$\sigma_p^2 = \beta_p^2\sigma_m^2 + \sigma_{ep}^2. \tag{15}$$

Here,

$$\beta_p = \sum\nolimits_{i=1}^{n_L} x_i\beta_i + \sum\nolimits_{j=1}^{n_S} x_{[j]}\beta_{[j]} \tag{16}$$

is a weighted average of the individual beta coefficients, and

$$\sigma_{ep}^2 = \sum\nolimits_{i=1}^{n_L} x_i^2\sigma_{ei}^2 + \sum\nolimits_{j=1}^{n_S} x_{[j]}^2\sigma_{e[j]}^2 \tag{17}$$

is the portfolio's residual variance, in terms of the individual residual variances.

We are now ready to impose market neutrality on the maximization of θ, in the form of beta neutrality and dollar neutrality, in addition to the budget constraint that equation (5) provides. Beta neutrality is about having a portfolio that is insensitive to movements in the market index, and dollar neutrality is about having matching security values on long and short sides of the portfolio. Both are equality constraints, and they are captured analytically by

$$\sum\nolimits_{i=1}^{n_L} x_i\beta_i + \sum\nolimits_{j=1}^{n_S} x_{[j]}\beta_{[j]} = 0 \tag{18}$$

and

$$\sum\nolimits_{i=1}^{n_L} x_i + \sum\nolimits_{j=1}^{n_S} x_{[j]} = 0. \tag{19}$$

As equation (18) implies $\sigma_p^2 = \sigma_{ep}^2$, the risk of a beta neutral portfolio is entirely unsystematic.

3 Model Derivation

For ease of pedagogic exposition, the derivation of the market neutral portfolio selection model is presented as a three-step derivation, after replicating some investment textbook materials from simpler models. In step 1 of the derivation, market neutrality is not imposed. Drawing on the analytical similarities of this preliminary model and the corresponding textbook materials, we establish some ranking criteria for selecting securities for long and short sides of the portfolio.

Specifically, such textbook materials start with portfolio selection with frictionless short sales.[9] Then, under the single-index characterization of the covariance structure of security returns, how the model involved can be transformed directly into a model for portfolio selection without short sales is explained. The relevance of such textbook materials will become clear in step 1 of the derivation. In step 2, we impose beta neutrality to the model to revise the ranking criteria from step 1. In the step 3, we also impose dollar neutrality to revise the ranking criteria even further. Each additional equality constraint in steps 2 and 3 is accommodated by using the well-known Lagrangian approach in differential calculus.

3.1 Preparation for Step 1

To prepare for step 1 of the derivation, we first replicate some related textbook materials, such as those in Elton, Gruber, Brown, and Goetzmann (2010, Chapters 6 and 9). For portfolio selection with frictionless short sales based on n securities, the input parameters include the expected returns and the covariances of returns of the individual securities. They are labeled as μ_i and σ_{ij}, for $i, j = 1, 2, \ldots, n$. Portfolio selection is via constrained maximization of $\theta = \left(\mu_p - R_f\right)/\sigma_p$. With the decision variables $x_1, x_2, \ldots, x_n$ being the portfolio weights, each of which can be of either sign, the only constraint is

$$\sum\nolimits_{i=1}^{n} x_i = 1. \tag{20}$$

As we can write

$$\mu_p - R_f = \sum\nolimits_{i=1}^{n} x_i \left(\mu_i - R_f\right) \tag{21}$$

and

$$\sigma_p = \sqrt{\sum\nolimits_{i=1}^{n} \sum\nolimits_{j=1}^{n} x_i x_j \sigma_{ij}}, \tag{22}$$

θ is a homogeneous function of $x_1, x_2, \ldots, x_n$ of degree zero. That is, the value of θ is unaffected by the substitution of $x_1, x_2, \ldots, x_n$ by $cx_1, cx_2, \ldots, cx_n$, where c is an arbitrary non-zero constant.

[9] Under the assumption of frictionless short sales, the short seller not only provides no deposit for the shorted security, but also has immediate access to the short-sale proceeds for investing in other securities.

Thus, a convenient way to reach the solution is to ignore the constraint in equation (20) first, but to scale the results from $\partial\theta/\partial x_i = 0$, for $i = 1, 2, \ldots, n$, in order to ensure that the constraint is satisfied eventually.

Specifically, setting the n first partial derivatives of θ to zeros leads to

$$\sum\nolimits_{j=1}^{n} \sigma_{ij} z_j = \mu_i - R_f, \text{ for } i = 1, 2, \ldots, n, \tag{23}$$

where

$$z_j = \left(\frac{\mu_p - R_f}{\sigma_p^2}\right) x_j. \tag{24}$$

As equation (23) represents a set of n linear equations, the unknown variables $z_1, z_2, \ldots, z_n$ can easily be solved. Combining equations (20) and (24) yields

$$\frac{\mu_p - R_f}{\sigma_p^2} = \sum\nolimits_{j=1}^{n} z_j. \tag{25}$$

Thus, the optimal portfolio weights $x_1, x_2, \ldots, x_n$ can be deduced from scaling $z_1, z_2, \ldots, z_n$ via

$$x_i = \frac{z_i}{\sum_{j=1}^{n} z_j}, \text{ for } i = 1, 2, \ldots, n. \tag{26}$$

In view of equation (25), the computation of σ_p is straightforward; specifically, we can write

$$\sigma_p = \sqrt{\frac{\mu_p - R_f}{\sum_{j=1}^{n} z_j}} = \sqrt{\frac{\sum_{i=1}^{n} x_i \left(\mu_i - R_f\right)}{\sum_{j=1}^{n} z_j}}. \tag{27}$$

Under the single-index characterization of the covariance structure of security returns, with parameters being analogous to those in equations (10) and (11), equation (23) becomes

$$\sum\nolimits_{j=1}^{n} \beta_i \beta_j \sigma_m^2 z_j + \sigma_{ei}^2 z_i = \mu_i - R_f, \text{ for } i = 1, 2, \ldots, n. \tag{28}$$

Letting

$$\phi = \sigma_m^2 \sum\nolimits_{j=1}^{n} \beta_j z_j, \tag{29}$$

we can re-arrange the terms in equation (28) to obtain

$$z_i = \frac{\beta_i}{\sigma_{ei}^2} \left(\frac{\mu_i - R_f}{\beta_i} - \phi\right), \text{ for } i = 1, 2, \ldots, n. \tag{30}$$

Here, $\left(\mu_i - R_f\right)/\beta_i$ — the ratio of each security's expected return, in excess of the risk-free interest rate, to its systematic risk or, simply, the excess-return-to-beta ratio — is an expected performance measure, and ϕ serves as a benchmark, which is commonly known as the cutoff rate of security performance.[10]

[10] Notice that, for a portfolio p, the ratio $(\mu_p - R_f)/\beta_p$ is the Treynor ratio in an *ex ante* context.

Equations (29) and (30) can be treated as $n+1$ linear equations, with $z_1, z_2, \ldots, z_n$ and ϕ being the unknown variables. Solving these equations yields

$$\phi = \frac{\sigma_m^2 \sum_{j=1}^n (\mu_j - R_f)\beta_j/\sigma_{ej}^2}{1 + \sigma_m^2 \sum_{j=1}^n \beta_j^2/\sigma_{ej}^2}. \tag{31}$$

Equations (30) and (31), when combined, allow us to establish some ranking criteria for portfolio selection without short sales.

For such a purpose, let us relabel the n securities so that we have

$$\frac{\mu_1 - R_f}{\beta_1} \geq \frac{\mu_2 - R_f}{\beta_2} \geq \cdots \geq \frac{\mu_n - R_f}{\beta_n}. \tag{32}$$

In terms of the ranking hierarchy of securities, security 1 is the highest, security 2 is the second highest, and so on. If any security k is selected for a long holding in the portfolio, so are securities $1, 2, \ldots, k-1$. Likewise, if any security k is not selected for a long holding, neither are securities $k+1, k+2, \ldots, n$. For security k, but not security $k+1$, to be selected for long holdings, we must have

$$\frac{\mu_k - R_f}{\beta_k} > \phi(k) \geq \frac{\mu_{k+1} - R_f}{\beta_{k+1}}. \tag{33}$$

Here,

$$\phi(k) = \frac{\sigma_m^2 \sum_{j=1}^k (\mu_j - R_f)\beta_j/\sigma_{ej}^2}{1 + \sigma_m^2 \sum_{j=1}^k \beta_j^2/\sigma_{ej}^2} \tag{34}$$

is the expression of ϕ in equation (31) based on relabeled securities $1, 2, \ldots, k$.

Given the above analytical features, we can construct successively a series of portfolios based on the k highest ranking securities, for $k = 1, 2, \ldots, n$, by using equations (30) and (31), where n is substituted by k for each portfolio. As augmenting a portfolio with an additional security, if feasible, represents an improvement in terms of risk-return trade-off, the optimal portfolio without short sales must be the portfolio based on the k highest ranking securities, satisfying the inequalities in condition (33). From a pedagogic perspective, an attractive feature of this textbook approach is that it allows us to bypass the formality pertaining to optimization under inequality constraints of $x_i \geq 0$, for $i = 1, 2, \ldots, n$.[11]

[11] As shown in Elton, Gruber, and Padberg (1976), if the portfolio selection problem is formulated formally as a no-short-sale case, there is an additive term δ_i/σ_{ei}^2 on the right hand side of equation (30). Here, δ_i is a slack variable corresponding to z_i, satisfying the conditions of $z_i \geq 0$, $\delta_i \geq 0$, and $z_i\delta_i = 0$, for $i = 1, 2, \ldots, n$. The cutoff rate ϕ in the revised equation (30) is given by $\phi(k)$ in equation (34), where k is the security that satisfies the inequalities in condition (33).

3.2 Step 1

In step 1 of the derivation, we exploit the analytical similarities between equations (9) and (21), and between equations (7) and (22), for the purpose of constrained maximization of $\theta = \left(\mu_p - R_f\right)/\sigma_p$. With σ_p and $\mu_p - R_f$ provided by equations (7) and (9), respectively, θ as a function of the $n_L + n_S$ portfolio weights $x_1, x_2, \ldots, x_{n_L}, x_{[1]}, x_{[2]}, \ldots, x_{[n_S]}$ is also homogeneous of degree zero. Let us assume for now that each of portfolio weights can have either sign. Under such an assumption, we can deduce the optimal portfolio weights via $\partial\theta/\partial x_i = 0$ and $\partial\theta/\partial x_{[j]} = 0$, for $i = 1, 2, \ldots, n_L$ and $j = 1, 2, \ldots, n_S$. The analytical results thus obtained will then be scaled, so that equation (5) is satisfied.

Analogous to how equation (23) is derived, we now have

$$\sum\nolimits_{\ell=1}^{n_L} \sigma_{i\ell} z_\ell + \sum\nolimits_{\ell=1}^{n_S} \sigma_{i[\ell]} z_{[\ell]} = \mu_i - R_f, \text{ for } i = 1, 2, \ldots, n_L, \tag{35}$$

and

$$\sum\nolimits_{\ell=1}^{n_L} \sigma_{[j]\ell} z_\ell + \sum\nolimits_{\ell=1}^{n_S} \sigma_{[j][\ell]} z_{[\ell]} = \mu_{[j]} - \nu R_f, \text{ for } j = 1, 2, \ldots, n_S, \tag{36}$$

instead. As in equation (24), z_ℓ is proportional to x_ℓ, and $z_{[\ell]}$ is proportional to $x_{[\ell]}$, with $(\mu_p - R_f)/\sigma_p^2$ being the proportionality constant. With equations (35) and (36) representing a system of $n_L + n_S$ linear equations, the unknown variables $z_1, z_2, \ldots, z_{n_L}, z_{[1]}, z_{[2]}, \ldots, z_{[n_S]}$ can easily be solved. However, the result will not be meaningful if any of the solved $z_1, z_2, \ldots, z_{n_L}$ $(z_{[1]}, z_{[2]}, \ldots, z_{[n_S]})$ turn out to be negative (positive). Based on the same idea from the previous subsection, a simple way to ensure that all solved variables are eventually of correct signs is by using the single index model to characterize the covariance structure of security returns. The detail is as follows:

We first substitute the expressions of all variances and covariances of returns from equations (10)-(14) into equations (35) and (36). The resulting equations are

$$\beta_i \sigma_m^2 \left(\sum\nolimits_{\ell=1}^{n_L} \beta_\ell z_\ell + \sum\nolimits_{\ell=1}^{n_S} \beta_{[\ell]} z_{[\ell]}\right) + \sigma_{ei}^2 z_i = \mu_i - R_f, \text{ for } i = 1, 2, \ldots, n_L, \tag{37}$$

and

$$\beta_{[j]} \sigma_m^2 \left(\sum\nolimits_{\ell=1}^{n_L} \beta_\ell z_\ell + \sum\nolimits_{\ell=1}^{n_S} \beta_{[\ell]} z_{[\ell]}\right) + \sigma_{e[j]}^2 z_{[j]} = \mu_{[j]} - \nu R_f, \text{ for } j = 1, 2, \ldots, n_S. \tag{38}$$

Letting

$$\phi = \sigma_m^2 \left(\sum\nolimits_{\ell=1}^{n_L} \beta_\ell z_\ell + \sum\nolimits_{\ell=1}^{n_S} \beta_{[\ell]} z_{[\ell]}\right), \tag{39}$$

we can write equations (37) and (38) as

$$z_i = \frac{\beta_i}{\sigma_{ei}^2} \left(\frac{\mu_i - R_f}{\beta_i} - \phi\right), \text{ for } i = 1, 2, \ldots, n_L, \tag{40}$$

and

$$z_{[j]} = \frac{\beta_{[j]}}{\sigma^2_{e[j]}} \left(\frac{\mu_{[j]} - \nu R_f}{\beta_{[j]}} - \phi \right), \text{ for } j = 1, 2, \ldots, n_S, \tag{41}$$

respectively.

Equations (39)-(41) can be considered as n_L+n_S+1 linear equations with $z_1, z_2, \ldots, z_{n_L}, z_{[1]}, z_{[2]}, \ldots, z_{[n_S]}$, and ϕ being the unknown variables. Solving these equations yields

$$\phi = \frac{\sigma^2_m \left[\sum_{\ell=1}^{n_L} (\mu_\ell - R_f)\beta_\ell/\sigma^2_{e\ell} + \sum_{\ell=1}^{n_S} (\mu_{[\ell]} - \nu R_f)\beta_{[\ell]}/\sigma^2_{e[\ell]} \right]}{1 + \sigma^2_m \left(\sum_{\ell=1}^{n_L} \beta^2_\ell/\sigma^2_{e\ell} + \sum_{[\ell]=1}^{n_S} \beta^2_{[\ell]}/\sigma^2_{e[\ell]} \right)}. \tag{42}$$

Equations (40)-(42) reveal some ranking criteria for portfolio selection, once the $n_L + n_S$ securities are relabeled to satisfy the conditions of

$$\frac{\mu_1 - R_f}{\beta_1} \geq \frac{\mu_2 - R_f}{\beta_2} \geq \cdots \geq \frac{\mu_{n_L} - R_f}{\beta_{n_L}} \tag{43}$$

and

$$\frac{\mu_{[1]} - \nu R_f}{\beta_{[1]}} \leq \frac{\mu_{[2]} - \nu R_f}{\beta_{[2]}} \leq \cdots \leq \frac{\mu_{[n_S]} - \nu R_f}{\beta_{[n_S]}}. \tag{44}$$

The ratio $(\mu_{[j]} - \nu R_f)/\beta_{[j]}$ can be interpreted intuitively. The expected dollar return from short selling one dollar's worth of security $[j]$, requiring a $\$\lambda$ deposit, is $-\$\mu_{[j]} + \$\lambda R_f + \$\nu R_f$, which is the sum of three return components as explained earlier. If the short sale is substituted by investing the same $\$\lambda$ in the risk-free security instead, the dollar return is $\$\lambda R_f$. The excess return, which is the difference between these two amounts, is $-\$\mu_{[j]} + \$\nu R_f$. As for risk, short selling security $[j]$ with a positive $\beta_{[j]}$ is like holding a security with a negative beta — which is equal to $-\beta_{[j]}$ — in a long position. Investing in a negative-beta security has a stabilizing, risk reducing effect on the portfolio. For two securities with positive excess returns and the same negative beta, the one with a higher excess return, which corresponds to a more negative (and thus lower) excess-return-to-beta ratio, is more attractive for holding.[12] As the excess-return-to-beta ratio of each security $[j]$ on the short side of the portfolio is $(-\mu_{[j]} + \nu R_f)/[-\beta_{[j]}] = (\mu_{[j]} - \nu R_f)/\beta_{[j]}$, a ranking hierarchy based on $(\mu_{[j]} - \nu R_f)/\beta_{[j]}$ is justified.

According to equation (40), if security h is selected for the long side, so are securities $1, 2, \ldots, h-1$; if security h is not selected for the long side, neither are securities $h+1, h+2, \ldots, n_L$. Likewise, according to equation (41), if security $[k]$ is selected for the short side, so are securities $[1], [2], \ldots, [k-1]$; if security $[k]$ is not selected for the short side, neither are securities $[k+1], [k+$

[12] For a comparison, see Elton, Gruber, and Padberg (1976) for ranking criteria of portfolio selection without short sales in the presence of securities with negative betas.

$2], \ldots, [n_S]$. For the portfolio consisting of h and k securities on long and short sides, respectively, to be optimal, the conditions of

$$\frac{\mu_h - R_f}{\beta_h} > \phi(h, [k]) \geq \frac{\mu_{h+1} - R_f}{\beta_{h+1}} \tag{45}$$

and

$$\frac{\mu_{[k]} - \nu R_f}{\beta_{[k]}} < \phi(h, [k]) \leq \frac{\mu_{[k+1]} - \nu R_f}{\beta_{[k+1]}} \tag{46}$$

must be satisfied. Here,

$$\phi(h, [k]) = \frac{\sigma_m^2 \left[\sum_{\ell=1}^{h} (\mu_\ell - R_f)\beta_\ell / \sigma_{e\ell}^2 + \sum_{\ell=1}^{k} (\mu_{[\ell]} - \nu R_f)\beta_{[\ell]} / \sigma_{e[\ell]}^2\right]}{1 + \sigma_m^2 \left(\sum_{\ell=1}^{h} \beta_\ell^2 / \sigma_{e\ell}^2 + \sum_{[\ell]=1}^{k} \beta_{[\ell]}^2 / \sigma_{e[\ell]}^2\right)} \tag{47}$$

is the expression of ϕ in equation (42) based on relabeled securities $1, 2, \ldots, h$ and $[1], [2], \ldots, [k]$.

Given the above two ranking hierarchies, the optimal portfolio where all securities have correct signs for $z_1, z_2, \ldots, z_{n_L}, z_{[1]}, z_{[2]}, \ldots, z_{[n_S]}$ can easily be constructed. Specifically, none of $z_1, z_2, \ldots, z_{n_L}$ can be negative and none of $z_{[1]}, z_{[2]}, \ldots, z_{[n_S]}$ can be positive. In principle, we can initiate a portfolio with the pair of highest ranking securities — which have been relabeled as securities 1 and [1] — and then augment the portfolio successively, with the remaining securities one at a time, in accordance with the above two ranking hierarchies, until it is no longer feasible to do so. Feasibility of each portfolio consisting of h and k highest ranking securities on long and short sides, respectively, can be verified with the signs of $(\mu_h - R_f)/\beta_h - \phi(h, [k])$ and $(\mu_{[k]} - \nu R_f)/\beta_{[k]} - \phi(h, [k])$.

In accordance with the two ranking hierarchies, as there are n_L and n_S ways to select securities for long and short sides, respectively, including infeasible cases, there are $n_L n_S$ ways in total to construct long-short portfolios. The optimal portfolio is the one that satisfies conditions (45) and (46). Once securities h and $[k]$ satisfying these conditions have been identified, we simply set $z_{h+1}, z_{h+2}, \ldots, z_{n_L}$ and $z_{[k+1]}, z_{[k+2]}, \ldots, z_{[n_S]}$ to zeros. Given the budget constraint in equation (5), the optimal portfolio weights are

$$x_i = \frac{z_i}{\sum_{\ell=1}^{n_L} z_\ell - \lambda \sum_{\ell=1}^{n_S} z_{[\ell]}}, \text{ for } i = 1, 2, \ldots, n_L, \tag{48}$$

and

$$x_{[j]} = \frac{z_{[j]}}{\sum_{\ell=1}^{n_L} z_\ell - \lambda \sum_{\ell=1}^{n_S} z_{[\ell]}}, \text{ for } j = 1, 2, \ldots, n_S. \tag{49}$$

Analogous to the case in equation (27), we can write

$$\sigma_p = \sqrt{\frac{\mu_p - R_f}{\sum_{i=1}^{n_L} z_i - \lambda \sum_{j=1}^{n_S} z_{[j]}}}, \tag{50}$$

where

$$\mu_p - R_f = \sum\nolimits_{i=1}^{n_L} x_i\,(\mu_i - R_f) + \sum\nolimits_{i=1}^{n_S} x_{[j]}(\mu_{[j]} - \nu R_f). \tag{51}$$

3.3 Step 2

Having completed step 1 of the derivation, we are ready to introduce beta neutrality to the same portfolio selection problem. Recall that $\theta = \left(\mu_p - R_f\right)/\sigma_p$ as a function of the $n_L + n_S$ portfolio weights $x_1, x_2, \ldots, x_{n_L}, x_{[1]}, x_{[2]}, \ldots, x_{[n_S]}$ is homogeneous of degree zero. In addition, the beta neutrality condition that equation (18) provides allows these portfolio weights to be scaled arbitrarily. Thus, we can ignore the budget constraint that equation (5) provides, for now. Suppose that, also for now, there are no restrictions on the signs of these portfolio weights.

With ω being a Lagrange multiplier, the Lagrangian is

$$\mathbf{L} = \theta - \omega\left(\sum\nolimits_{i=1}^{n_L} x_i\beta_i + \sum\nolimits_{j=1}^{n_S} x_{[j]}\beta_{[j]}\right). \tag{52}$$

From $\partial\mathbf{L}/\partial x_i = 0$, we have

$$\mu_i - R_f - \left(\frac{\mu_p - R_f}{\sigma_p^2}\right)\left(\sum\nolimits_{\ell=1}^{n_L} \sigma_{i\ell}x_\ell + \sum\nolimits_{\ell=1}^{n_S} \sigma_{i[\ell]}x_{[\ell]}\right) - \eta\beta_i = 0, \text{ for } i = 1, 2, \ldots, n_L, \tag{53}$$

where $\eta = \omega\sigma_p$. Likewise, from $\partial\mathbf{L}/\partial x_{[j]} = 0$, we have

$$\mu_{[j]} - \nu R_f - \left(\frac{\mu_p - R_f}{\sigma_p^2}\right)\left(\sum\nolimits_{\ell=1}^{n_L} \sigma_{[j]\ell}x_\ell + \sum\nolimits_{\ell=1}^{n_S} \sigma_{[j][\ell]}x_{[\ell]}\right) - \eta\beta_{[j]} = 0, \text{ for } j = 1, 2, \ldots, n_S. \tag{54}$$

Under the single index model, where the covariance structure of security returns is given by equations (10)-(14), we can write equations (53) and (54) as

$$\begin{aligned} & \mu_i - R_f - \left(\frac{\mu_p - R_f}{\sigma_p^2}\right)\left[\beta_i\sigma_m^2\left(\sum\nolimits_{\ell=1}^{n_L} \beta_\ell x_\ell + \sum\nolimits_{\ell=1}^{n_S} \beta_{[\ell]}x_{[\ell]}\right) + \sigma_{ei}^2 x_i\right] - \eta\beta_i \\ = \; & \mu_i - R_f - \left(\frac{\mu_p - R_f}{\sigma_p^2}\right)\sigma_{ei}^2 x_i - \eta\beta_i = 0, \text{ for } i = 1, 2, \ldots, n_L, \end{aligned} \tag{55}$$

and

$$\begin{aligned} & \mu_{[j]} - \nu R_f - \left(\frac{\mu_p - R_f}{\sigma_p^2}\right)\left[\beta_{[j]}\sigma_m^2\left(\sum\nolimits_{\ell=1}^{n_L} \beta_\ell x_\ell + \sum\nolimits_{\ell=1}^{n_S} \beta_{[\ell]}x_{[\ell]}\right) + \sigma_{e[j]}^2 x_{[j]}\right] - \eta\beta_{[j]} \\ = \; & \mu_{[j]} - \nu R_f - \left(\frac{\mu_p - R_f}{\sigma_p^2}\right)\sigma_{e[j]}^2 x_{[j]} - \eta\beta_{[j]} = 0, \text{ for } j = 1, 2, \ldots, n_S. \end{aligned} \tag{56}$$

As in equation (24), we let z_k and $z_{[k]}$ be proportional to x_k and $x_{[k]}$, respectively, with $(\mu_p - R_f)/\sigma_p^2$ being the proportionality constant. Then, equations (55) and (56) reduce to

$$z_i = \frac{\beta_i}{\sigma_{ei}^2}\left(\frac{\mu_i - R_f}{\beta_i} - \eta\right), \text{ for } i = 1, 2, \ldots, n_L, \tag{57}$$

and

$$z_{[j]} = \frac{\beta_{[j]}}{\sigma^2_{e[j]}} \left(\frac{\mu_{[j]} - \nu R_f}{\beta_{[j]}} - \eta \right), \text{ for } j = 1, 2, \ldots, n_S, \tag{58}$$

respectively.

The beta neutrality condition in equation (18) is equivalent to

$$\sum\nolimits_{i=1}^{n_L} z_i \beta_i + \sum\nolimits_{j=1}^{n_S} z_{[j]} \beta_{[j]} = 0. \tag{59}$$

Thus, we can consider equations (57)-(59) as $n_L + n_S + 1$ linear equations for the unknown variables $z_1, z_2, \ldots, z_{n_L}, z_{[1]}, z_{[2]}, \ldots, z_{[n_S]}$ and η. Solving these equations leads to

$$\eta = \frac{\sum_{\ell=1}^{n_L} (\mu_\ell - R_f)\beta_\ell / \sigma^2_{e\ell} + \sum_{\ell=1}^{n_S} (\mu_{[\ell]} - \nu R_f)\beta_{[\ell]} / \sigma^2_{e[\ell]}}{\sum_{\ell=1}^{n_L} \beta^2_\ell / \sigma^2_{e\ell} + \sum_{\ell=1}^{n_S} \beta^2_{[\ell]} / \sigma^2_{e[\ell]}}. \tag{60}$$

Except for the difference in how ϕ and η are computed, the expressions of z_i and $z_{[j]}$ in equations (40) and (41), for $i = 1, 2, \ldots, n_L$ and $j = 1, 2, \ldots, n_S$, are the same as those in equations (57) and (58). Thus, the ranking criteria for portfolio selection in step 2 apply here as well.

All that is required is to substitute $\phi(h, [k])$ in conditions (45) and (46) with $\eta(h, [k])$, the expression of η in equation (60) based on relabeled securities $1, 2, \ldots, h$ and $[1], [2], \ldots, [k]$. As we can write

$$\eta(h, [k]) = \frac{\sum_{\ell=1}^{h} \left[(\mu_\ell - R_f) / \beta_\ell \right] \beta^2_\ell \sigma^2_m / \sigma^2_{e\ell} + \sum_{\ell=1}^{k} [(\mu_{[\ell]} - \nu R_f) / \beta_{[\ell]}] \beta^2_{[\ell]} \sigma^2_m / \sigma^2_{e[\ell]}}{\sum_{\ell=1}^{h} \beta^2_\ell \sigma^2_m / \sigma^2_{e\ell} + \sum_{\ell=1}^{k} \beta^2_{[\ell]} \sigma^2_m / \sigma^2_{e[\ell]}}, \tag{61}$$

the expression is also a weighted average of the excess-return-to-beta ratios. The weights are provided by the corresponding ratios of systematic risk to unsystematic risk of the individual securities that are selected for the portfolio. The use of σ^2_m here is for providing an intuitive interpretation of the weights in the weighted average; however, the volatility of the market itself has no impact on how securities are selected for a market neutral portfolio.

3.4 Step 3

In step 2, no attention is paid to the dollar balance between long and short sides of the portfolio. If the cash reserve that λ represents is low and $\sum_{i=1}^{n_L} x_i + \sum_{j=1}^{n_S} x_{[j]}$ from step 2 turns out to be negative, then the unused margin from the long side may be inadequate to satisfy the margin requirement for the short side. If so, an additional cash deposit may be required. Further, as the dollar balance between the two sides is a practical feature of market neutral portfolios, it becomes necessary to impose, on the portfolio selection model involved, the condition that equation (19) provides as well.

The Lagrangian $\mathbf{L}$ of the same optimization problem in step 2, with dollar neutrality also imposed, will have an extra additive term. Specifically, we can write

$$\mathbf{L} = \theta - \omega \left(\sum\nolimits_{i=1}^{n_L} x_i \beta_i + \sum\nolimits_{j=1}^{n_S} x_{[j]} \beta_{[j]} \right) - \gamma \left(\sum\nolimits_{i=1}^{n_L} x_i + \sum\nolimits_{j=1}^{n_S} x_{[j]} \right), \tag{62}$$

where γ is a Lagrange multiplier. The presence of this extra term will cause the expression of $\partial \mathbf{L} / \partial x_i$ to carry also an extra additive term, $-\gamma \sigma_p$; the result is that $\mu_i - R_f$ will become $\mu_i - R_f - \gamma \sigma_p$, for $i = 1, 2, \ldots, n_L$. Likewise, it will also cause the expression of $\partial \mathbf{L} / \partial x_{[j]}$ to change in a similar way, with $\mu_{[j]} - \nu R_f$ becoming $\mu_{[j]} - \nu R_f - \gamma \sigma_p$, for $j = 1, 2, \ldots, n_S$. As the algebraic form of each expression remains unchanged, except for the addition of $-\gamma \sigma_p$ to each term of expected return, there are no changes in the algebraic steps leading to the portfolio solution.

Therefore, by letting $\tau = \gamma \sigma_p$, we can write

$$z_i = \frac{\beta_i}{\sigma_{ei}^2} \left(\frac{\mu_i - R_f - \tau}{\beta_i} - \eta \right), \text{ for } i = 1, 2, \ldots, n_L, \tag{63}$$

and

$$z_{[j]} = \frac{\beta_{[j]}}{\sigma_{e[j]}^2} \left(\frac{\mu_{[j]} - \nu R_f - \tau}{\beta_{[j]}} - \eta \right), \text{ for } j = 1, 2, \ldots, n_S, \tag{64}$$

where

$$\eta = \frac{\sum_{\ell=1}^{n_L} (\mu_\ell - R_f - \tau) \beta_\ell / \sigma_{e\ell}^2 + \sum_{\ell=1}^{n_S} (\mu_{[\ell]} - \nu R_f - \tau) \beta_{[\ell]} / \sigma_{e[\ell]}^2}{\sum_{\ell=1}^{n_L} \beta_\ell^2 / \sigma_{e\ell}^2 + \sum_{\ell=1}^{n_S} \beta_{[\ell]}^2 / \sigma_{e[\ell]}^2}. \tag{65}$$

Again, the way each of z_i and $z_{[j]}$ is connected to the corresponding portfolio weight remains the same as that in steps 1 and 2. However, with τ being initially unknown, equations (63)-(65) are not yet ready for use in portfolio selection.

As equations (63)-(65) indicate, the additive term, $-\tau$, represents a uniform adjustment to the expected returns of individual securities. If τ is positive, the adjustment will make the set of n_L securities collectively less attractive for purchasing and the set of n_S securities collectively more attractive for short selling. The adjustment will result in a decrease (an increase) of the proportion of investment capital for the long (short) side. If τ is negative instead, the effects on the two sides will be the opposite.

The corresponding change of $\sum_{i=1}^{n_L} z_i + \sum_{j=1}^{n_S} z_{[j]}$ in response to a change in τ being monotonic, a simple way to determine τ is via a numerical search. If the portfolio selection results from step 2 provide a positive (negative) $\sum_{i=1}^{n_L} z_i + \sum_{j=1}^{n_S} z_{[j]}$, the correct τ will also be positive (negative). Let us denote τ^* as an attempted value of τ. Based on this τ^*, the $n_L + n_S$ securities can be ranked and relabeled such that

$$\frac{\mu_1 - R_f - \tau^*}{\beta_1} \geq \frac{\mu_2 - R_f - \tau^*}{\beta_2} \geq \cdots \geq \frac{\mu_{n_L} - R_f - \tau^*}{\beta_{n_L}} \tag{66}$$

and

$$\frac{\mu_{[1]} - \nu R_f - \tau^*}{\beta_{[1]}} \leq \frac{\mu_{[2]} - \nu R_f - \tau^*}{\beta_{[2]}} \leq \cdots \leq \frac{\mu_{[n_S]} - \nu R_f - \tau^*}{\beta_{[n_S]}}. \tag{67}$$

For each attempted τ^*, the ranking approach for portfolio construction analogous to that in step 2 is applicable as well.

Beta neutrality is assured for the portfolio consisting of relabeled securities $1, 2, \ldots, h$ and [1], [2], $\ldots$, $[k]$. Analytically, the conditions of

$$\frac{\mu_h - R_f - \tau^*}{\beta_h} > \eta(h, [k]) \geq \frac{\mu_{h+1} - R_f - \tau^*}{\beta_{h+1}} \tag{68}$$

and

$$\frac{\mu_{[k]} - \nu R_f - \tau^*}{\beta_{[k]}} < \eta(h, [k]) \leq \frac{\mu_{[k+1]} - R_f - \tau^*}{\beta_{[k+1]}} \tag{69}$$

are satisfied. Here,

$$\eta(h, [k]) = \frac{\sum_{\ell=1}^{h} (\mu_\ell - R_f - \tau^*)\beta_\ell/\sigma_{e\ell}^2 + \sum_{\ell=1}^{k} (\mu_{[\ell]} - \nu R_f - \tau^*)\beta_{[\ell]}/\sigma_{e[\ell]}^2}{\sum_{\ell=1}^{h} \beta_\ell^2/\sigma_{e\ell}^2 + \sum_{\ell=1}^{k} \beta_{[\ell]}^2/\sigma_{e[\ell]}^2} \tag{70}$$

is the cutoff rate η based on the h and k highest ranking securities on long and short sides of the portfolio, respectively. Implicitly, we set $z_{h+1}, z_{h+2}, \ldots, z_{n_L}$ and $z_{[k+1]}, z_{[k+2]}, \ldots, z_{[n_S]}$ to be all zeros. If the resulting $\sum_{i=1}^{n_L} z_i + \sum_{j=1}^{n_S} z_{[j]}$ is positive (negative), the value of τ^* ought to be increased (decreased). In principle, if incrementally higher (lower) values of τ^* are attempted, there will be a specific value of τ^* that corresponds to $\sum_{i=1}^{n_L} z_i + \sum_{j=1}^{n_S} z_{[j]}$ being zero.

For each attempted τ^*, the optimal portfolio weights and the corresponding $z_1, z_2, \ldots, z_{n_L}, z_{[1]}$, $z_{[2]}, \ldots, z_{[n_S]}$ are related in the same manner as that in equations (48) and (49). The expression of σ_p is the same as that in equation (50); however, instead of equation (51), we have,

$$\mu_p - R_f = \sum\nolimits_{i=1}^{n_L} x_i (\mu_i - R_f - \tau^*) + \sum\nolimits_{i=1}^{n_S} x_{[j]}(\mu_{[j]} - \nu R_f - \tau^*). \tag{71}$$

Once the dollar neutrality condition is satisfied, equations (51) and (71) are equivalent.

4 Summary and Key Expressions

The construction of a long-short portfolio p has been formulated in Section 2 as constrained maximization of expected portfolio performance, $\theta = (\mu_p - R_f)/\sigma_p$, under the single index characterization of the covariance structure of security returns. As already defined when first introduced, R_f is the risk-free interest rate and μ_p and σ_p are the portfolio's expected return and standard deviation of returns, respectively. The portfolio selection is based on n_L securities on the long side and n_S securities on the short side, with the two sets of securities being disjoint.

The input parameters are as follows: On the long side, μ_i, β_i, and σ_{ei}^2 are the expected return, the beta coefficient, and the residual variance of security i, respectively, for $i = 1, 2, \ldots, n_L$. On the short side, the corresponding symbols are $\mu_{[j]}$, $\beta_{[j]}$, and $\sigma_{e[j]}^2$, for $j = 1, 2, \ldots, n_S$. In addition, σ_m^2 is the variance of market returns, $0 \leq \nu < 1$ is the proportion of interest rebate on the short-sale proceeds, and $\lambda > 0$ is the cash reserve as a proportion of the values of shorted securities.

The expression of $\mu_p - R_f$ is provided by equation (9); the expression of σ_p is based on equations (15)-(17). Here, $x_i \geq 0$, for $i = 1, 2, \ldots, n_L$, and $x_{[j]} \leq 0$, for $j = 1, 2, \ldots, n_S$, are portfolio weights to be determined. The constraints for the optimization problem, besides the signs of the individual portfolio weights, also include equations (5), (18), and (19). They correspond to the budget constraint, the beta neutrality condition, and the dollar neutrality condition, respectively.

The key expressions in the analytical solution, as derived in Section 3, are provided by equations (63), (64), and (70). The role of the cutoff rate η, which is also denoted as $\eta(h, [k])$, will soon be clear. Together, these equations facilitate a ranking approach for solving the portfolio selection problem. The unknown variables in these equations, $z_1, z_2, \ldots, z_{n_L}, z_{[1]}, z_{[2]}, \ldots, z_{[n_S]}$, are proportional to the corresponding unknown portfolio weights, with $(\mu_p - R_f)/\sigma_p^2$ being the proportionality constant. Such variables can be solved by using (i) the ranking properties of securities that equations (63) and (64) provide and (ii) the requirements that none of $z_1, z_2, \ldots, z_{n_L}$ are negative and none of $z_{[1]}, z_{[2]}, \ldots, z_{[n_S]}$ are positive.

Here is how a ranking approach for portfolio construction is implemented: For each attempted value τ^* of the unknown parameter τ in equations (63), (64), and (70), the individual securities considered are ranked explicitly. On the long side, the ranking is based on $(\mu_i - R_f - \tau)/\beta_i$, an expected performance measure; the higher the ratio, the more attractive is the security for portfolio holding. On the short side, it is based on $(\mu_{[j]} - \nu R_f - \tau^*)/\beta_{[j]}$ instead; in contrast, the lower (the more negative) the ratio, the more attractive is the security.

For each attempted τ^*, the cutoff rate η in equations (63) and (64) is denoted as $\eta(h, [k])$ to indicate the inclusion of the h and k highest ranking securities from long and short sides of the portfolio, respectively. For $h = 1, 2, \ldots, n_L$ and $k = 1, 2, \ldots, n_S$, a total of $n_L n_S$ portfolios can be constructed by following the two ranking hierarchies. As none of $z_1, z_2, \ldots, z_{n_L}$ can be negative and none of $z_{[1]}, z_{[2]}, \ldots, z_{[n_S]}$ can be positive, only some of the $n_L n_S$ portfolios thus constructed are feasible. Equation (18), the beta neutrality condition, is always satisfied for each feasible portfolio.

For each attempted τ^*, the feasible η with the highest number of included securities from each side is optimal.

There is a monotonic relationship between τ^* and the departure of $\sum_{i=1}^{n_L} z_i + \sum_{j=1}^{n_S} z_{[j]}$ from zero. Thus, a simple numerical search will lead to a specific value τ^* for equation (19) to hold as well. Once such a τ^* and the corresponding optimal η are determined, so are $z_1, z_2, \ldots, z_{n_L}, z_{[1]}, z_{[2]}, \ldots, z_{[n_S]}$ via equations (63) and (64); any of these values of the wrong signs are set to be zeros. Then, we can compute $x_1, x_2, \ldots, x_{n_L}, x_{[1]}, x_{[2]}, \ldots, x_{[n_S]}$ with equations (48) and (49), μ_p with equation (71), σ_p with equation (50), and θ with equation (8).

4.1 Remarks

Before illustrating the Excel implementation, here are some relevant analytical issues from pedagogic and expositional perspectives: For investment courses where either analytical details are de-emphasized or model derivations are omitted, the presentation of the required analytical materials can start directly with the derived expressions of $\mu_p - R_f$ and σ_p^2 in equations (9) and (15), respectively. For an intuitive explanation of these expressions, we can compare them with the corresponding expressions from investment textbooks. Such a comparison will enable students to see the effects of different analytical treatments of short sales on the formulation of a portfolio selection problem.

For portfolio selection without short sales or with short sales under two alternative simplifying assumptions, including frictionless short sales and Lintner's (1965) assumption, the common expressions — which correspond to those in equations (7) and (9) — are $\sigma_p^2 = \sum_{i=1}^{n} \sum_{k=1}^{n} x_i x_k \sigma_{ik}$ and $\mu_p - R_f = \sum_{i=1}^{n} x_i(\mu_i - R_f)$ for an n-security case.[13] Under the single index characterization of the covariance structure of security returns, equation (15) also holds there, but with $\beta_p = \sum_{i=1}^{n} x_i \beta_i$ and $\sigma_{ep}^2 = \sum_{i=1}^{n} x_i^2 \sigma_{ei}^2$ instead. In the case of σ_p^2, the connection of the expressions in this section and the corresponding textbook materials is obvious; by letting $n = n_L + n_S$, we can write each summation with n terms as two summations with n_L and n_S terms each. To explain equation (9) intuitively, a crucial point is that, as $0 \leq \nu < 1$, the short interest rebate, if any, is only partial. Each term in the summation $\sum_{j=1}^{n_S} x_{[j]}(\mu_{[j]} - \nu R_f)$ in equation (9), with νR_f being a partial interest rebate, represents the contribution of the corresponding shorted security to $\mu_p - R_f$.

The assumption that the sets of securities for long and short holdings are disjoint is initially

[13] Under Lintner's assumption, the short seller is required to provide a 100% deposit and has no immediate access to the short-sale proceeds. However, the short seller earns risk-free interests on both the deposit and the short-sale proceeds. See, for example, Elton, Gruber, Brown, and Goetzmann (2010, Chapter 6) for analytical details.

for analytical convenience. Given the ranking criteria in equations (63) and (64), relaxing the assumption will not affect the analytical solution. To see this, let us consider a security i that is selected for the long side. It follows from $(\mu_i - R_f - \tau)/\beta_i > \eta$ and $(\mu_i - \nu R_f - \tau)/\beta_i > (\mu_i - R_f - \tau)/\beta_i$ (given $0 \leq \nu < 1$) that $(\mu_i - \nu R_f - \tau)/\beta_i > \eta$. Accordingly, security i will not also be held short in the same portfolio. Likewise, for a security $[j]$ that is selected for the short side, as $(\mu_{[j]} - \nu R_f)/\beta_{[j]} < \eta$ and $(\mu_{[j]} - R_f)/\beta_{[j]} < (\mu_{[j]} - \nu R_f)/\beta_{[j]}$, we have $(\mu_{[j]} - R_f)/\beta_{[j]} < \eta$, which precludes the security from holding on the short side. Thus, even if the sets of securities for long and short holdings are not disjoint, no security considered will be selected for both long and short positions in the same portfolio.

5 Excel Illustrations

We now illustrate, with two alternative approaches, how a long-short equity portfolio can be constructed. To be accessible to more readers, the Excel file accompanying this paper has been saved as a 1997-2003 version, which uses .xls as an extension of the filename. The Excel file contains the three worksheets that correspond to the individual figures in this paper. It also contains an additional worksheet, which shows some variants in the Excel implementation. For a balance between expositional clarity and conciseness, representative Excel formulas in each worksheet, as displayed explicitly in the subsections below, are confined to those that may not be immediately obvious to readers.

Five securities are considered for each side of a long-short equity portfolio. A common set of input parameters is used for the two alternative approaches, so that consistency of the portfolio results can be verified. Figure 1 illustrates a purely Solver-based approach where the analytical solution is not utilized. As σ_m^2 does not appear in the analytical solution of the portfolio selection problem, it is part of the the input parameters only for the Solver-based approach in Figure 1. The illustrations in Figures 2 and 3 are based on the analytical solution; a ranking approach for portfolio selection is illustrated. Figure 2, which ignores the dollar neutrality condition, is intended to show how an Excel worksheet can be set up for Figure 3. A simple numerical search for the unknown τ^*, which can be achieved via either Solver or Goal Seek, is illustrated in Figure 3.

5.1 A Solver-Based Approach without Using the Analytical Solution

As shown in Figure 1, the input parameters are stored in cells that are slightly shaded. They include $R_f = 0.03$, $\sigma_m^2 = 0.05$, $\nu = 0.75$, and $\lambda = 0.10$ in B1:B4, denoted as “rf,” “market var,”

	A	B	C	D	E	F	G
1	rf	0.03					
2	market var	0.05					
3	rebate	0.75					
4	lambda	0.10					
5							
6							
7	sec L, sec S	exp ret	beta	res var	excess ret		
8	1	0.11	1.1	0.08	0.08		
9	2	0.12	0.6	0.06	0.09		
10	3	0.15	0.8	0.09	0.12		
11	4	0.01	1.0	0.06	-0.02		
12	5	0.08	1.8	0.07	0.05		
13	1	0.09	1.0	0.07	0.0675		
14	2	0.08	0.6	0.06	0.0575		
15	3	-0.01	1.2	0.06	-0.0325		
16	4	0.12	1.4	0.08	0.0975		
17	5	-0.03	1.1	0.09	-0.0525		
18							
19							
20	sec L, sec S	excess ret	weight x	x*excess ret	x, -lambda*x	x*beta	x sq * res var
21	1	0.08	0.19945299	0.015956239	0.199452993	0.2193983	0.00318252
22	2	0.09	0.16126258	0.014513632	0.161262577	0.0967575	0.00156034
23	3	0.12	0.28578245	0.034293894	0.285782449	0.228626	0.00735044
24	4	-0.02	0	0	0	0	0
25	5	0.05	0.26259289	0.013129644	0.262592889	0.4726672	0.00482685
26	1	0.0675	0	0	0	0	0
27	2	0.0575	-0.0543244	-0.00312365	0.005432441	-0.0325946	0.00017707
28	3	-0.0325	-0.446112	0.014498641	0.044611204	-0.5353344	0.01194096
29	4	0.0975	0	0	0	0	0
30	5	-0.0525	-0.4086545	0.021454359	0.040865446	-0.4495199	0.01502986
31							
32		sum	0	0.110722756	1	1.275E-09	0.04406804
33							
34		ex ret p	var p	st dev p	theta		
35		0.14072276	0.04406804	0.209923893	0.527442373		

Figure 1 An Excel Worksheet Illustrating a Solver-Based Approach for Long-Short Portfolio Selection under Beta Neutrality and Dollar Neutrality Conditions

"rebate," and "lambda," respectively, as well as original security labels in A8:A17 and security-specific input parameters in B8:D17, under the headings of "sec L, sec S," "exp ret," "beta," and "res var." The latter parameters are as follows: $\mu_1 = 0.11$, $\mu_2 = 0.12$, $\mu_3 = 0.15$, $\mu_4 = 0.01$, $\mu_5 = 0.08$, $\mu_{[1]} = 0.09$, $\mu_{[2]} = 0.08$, $\mu_{[3]} = -0.01$, $\mu_{[4]} = 0.12$, and $\mu_{[5]} = -0.03$; $\beta_1 = 1.1$, $\beta_2 = 0.6$, $\beta_3 = 0.8$, $\beta_4 = 1.0$, $\beta_5 = 1.8$, $\beta_{[1]} = 1.0$, $\beta_{[2]} = 0.6$, $\beta_{[3]} = 1.2$, $\beta_{[4]} = 1.4$, and $\beta_{[5]} = 1.1$; $\sigma^2_{e1} = 0.08$, $\sigma^2_{e2} = 0.06$, $\sigma^2_{e3} = 0.09$, $\sigma^2_{e4} = 0.06$, $\sigma^2_{e5} = 0.07$, $\sigma^2_{e[1]} = 0.07$, $\sigma^2_{e[2]} = 0.06$, $\sigma^2_{e[3]} = 0.06$, $\sigma^2_{e[4]} = 0.08$, and $\sigma^2_{e[5]} = 0.09$.

The corresponding $\mu_i - R_f$ and $\mu_{[j]} - \nu R_f$ are displayed in E8:E17, under the heading of "excess ret." These values are replicated in B21:B30 to facilitate the computations that follow. The specific task is to maximize $\theta = (\mu_p - R_f)/\sigma_p$ by finding the corresponding $x_1, x_2, \ldots, x_5 \geq 0$ and $x_{[1]}, x_{[2]}, \ldots, x_{[5]} \leq 0$ under three further constraints according to equations (5), (18), and (19). For any given set of portfolio weights, $\mu_p - R_f$ can be computed directly from equation (9), and σ_p can be deduced via equations (15)-(17).

To facilitate the search for the portfolio solution via Solver, an arbitrary set of initial portfolio weights is provided for C21:C30, under the heading of "weight x." For each initial x_i, the corresponding $x_i(\mu_i - R_f)$, x_i, $x_i\beta_i$, and $x_i^2\sigma^2_{ei}$ are displayed in the same row; for each initial $x_{[j]}$, the corresponding $x_{[j]}(\mu_{[j]} - \nu R_f)$, $-\lambda x_{[j]}$, $x_{[j]}\beta_{[j]}$, and $x^2_{[j]}\sigma^2_{e[j]}$ are also displayed in an analogous manner. These values are shown in D21:G30, under the headings of "x * excess ret," "x, -lambda * x," "x * beta," and "x sq * res var." The individual column sums of C21:G30 are shown in C32:G32. The sum in D32 is $\mu_p - R_f$ based on equation (9), and the sum in G32 is σ^2_{ep} based on equation (17). The sum in E32, if equal to one, will indicate that the allocation constraint as specified by equation (5) is satisfied. The sums in F32 and C32, if both equal to zeros, will indicate that both beta neutrality and dollar neutrality conditions in equations (18) and (19) are satisfied.

The sums in D32 and F32:G32, together with R_f in B1, allow μ_p, σ^2_p, σ_p, and θ to be computed and displayed in B35:E35, under the headings of "ex ret p," "var p," "st dev p," and "theta," respectively. With the initial portfolio weights being arbitrary, the corresponding portfolio results, inevitably, are suboptimal or even infeasible. To reach the optimal solution via Solver, the target cell, \$E\$35, is to be maximized, by changing \$C\$21:\$C\$30, under the constraints of \$C\$21:\$C\$25>=0, \$C\$26:\$C\$30<=0, \$C\$32=0, \$E\$32=1, and \$F\$32=0. The display in Figure 1 is based on the Solver solution. To make it easier to compare the end results here and those based on the analytical solution later, the cells showing $x_1, x_2, \ldots, x_5$, $x_{[1]}, x_{[2]}, \ldots, x_{[5]}$, μ_p, σ_p, and θ — which are C21:C30, B35, and D35:E35 — are displayed in a slightly darker shade.

In this example, four of the five securities considered on the long side are selected, with $x_1 = 0.1995$, $x_2 = 0.1613$, $x_3 = 0.2858$, and $x_5 = 0.2626$ (as rounded to four significant figures). On the short side, three of the five securities considered are selected, with $x_{[2]} = -0.0543$, $x_{[3]} = -0.4461$, and $x_{[5]} = -0.4087$. For the three excluded securities, the corresponding portfolio weights are $x_4 = x_{[1]} = x_{[4]} = 0$. As confirmed by the two zeros (or numbers with trivially small magnitudes) in F32 and C32, both beta neutrality and dollar neutrality conditions are satisfied. The remaining portfolio results are $\mu_p = 0.1407$, $\sigma_p = 0.2099$, and $\theta = 0.5274$. These numerical results will be used later to compare with those based on the analytical solution.

5.2 An Approach Based on the Analytical Solution with Beta Neutrality

Figure 2 shows the computational details based on the analytical solution from step 2 of the derivation in the previous section. In this step, the dollar neutrality condition is ignored. In view of the analytical materials in step 3, this step is equivalent to setting $\tau = 0$ in equations (63)-(65), instead of also solving for its value to ensure equation (19) to hold as well.

The cells containing the input parameters, as displayed in Figure 2, are also slightly shaded. They include R_f, ν, λ, and τ (which is set to be 0 as intended) in B1:B4, i, μ_i, β_i, and σ^2_{ei}, for $i = 1, 2, \ldots, 5$, in A7:A11, C7:C11, and E7:F11, and $[j]$, $\mu_{[j]}$, $\beta_{[j]}$, and $\sigma^2_{e[j]}$, for $j = 1, 2, \ldots, 5$, in A14:A18, C14:C18, and E14:F18. For the long side, the computed $\mu_i - R_f - \tau$, β_i^2/σ^2_{ei}, and $(\mu_i - R_f - \tau)/\beta_i$ are displayed in D7:D11 and G7:H11, under the headings of "excess ret," "wgt," and "ratio L," respectively. For the short side, the computed $\mu_{[j]} - \nu R_f - \tau$, $\beta^2_{[j]}/\sigma^2_{e[j]}$, and $(\mu_{[j]} - \nu R_f - \tau)/\beta_{[j]}$ are displayed in D14:D18 and G14:H18, under the headings of "excess ret," "wgt," and "ratio S," respectively.

To implement the ranking approach for portfolio selection, all security-specific data must be sorted first. For the current case where the dollar neutrality condition is ignored, manual sorting — which is intended to be performed on the data only once — is straightforward. However, to facilitate portfolio selection later under both beta neutrality and dollar neutrality, which requires repetitive sorting of the data as τ varies, the use of Excel functions RANK and VLOOKUP instead is more convenient. The idea is to use RANK to relabel the securities based on the ranking hierarchies that equations (63) and (64) provide and then to use VLOOKUP to place the sorted data in successive rows of a worksheet.

Complications will arise if there are tied ranks to prevent VLOOKUP from performing its intended task. For example, if the formula =RANK(H7,H$7:H$11) for B7 is copied to B7:B11, the ranks of securities 1, 2, 3, 4, and 5, as shown there in a descending order of the excess-return-to-beta

	A	B	C	D	E	F	G	H
1	rf	0.03						
2	rebate	0.75						
3	lambda	0.10						
4	tau	0						
5								
6	sec L	rank	exp ret	excess ret	beta	res var	wgt	ratio L
7	1	3	0.11	0.08	1.1	0.08	15.125	0.0727273
8	2	1	0.12	0.09	0.6	0.06	6	0.15
9	3	2	0.15	0.12	0.8	0.09	7.111111	0.15
10	4	5	0.01	-0.02	1.0	0.06	16.66667	-0.02
11	5	4	0.08	0.05	1.8	0.07	46.28571	0.0277778
12								
13	sec S	rank	exp ret	excess ret	beta	res var	wgt	ratio S
14	1	3	0.09	0.0675	1.0	0.07	14.28571	0.0675
15	2	5	0.08	0.0575	0.6	0.06	6	0.0958333
16	3	2	-0.01	-0.0325	1.2	0.06	24	-0.0270833
17	4	4	0.12	0.0975	1.4	0.08	24.5	0.0696429
18	5	1	-0.03	-0.0525	1.1	0.09	13.44444	-0.0477273
19								
20	sorted	exp ret	excess ret	beta	res var	wgt	ratio L	wgt*ratio
21		0.12	0.09	0.6	0.06	6	0.15	0.9
22		0.15	0.12	0.8	0.09	7.111111	0.15	1.0666667
23		0.11	0.08	1.1	0.08	15.125	0.072727	1.1
24		0.08	0.05	1.8	0.07	46.28571	0.027778	1.2857143
25		0.01	-0.02	1	0.06	16.66667	-0.02	-0.3333333
26								
27	sorted	exp ret	excess ret	beta	res var	wgt	ratio S	wgt*ratio
28		-0.03	-0.0525	1.1	0.09	13.44444	-0.047727	-0.6416667
29		-0.01	-0.0325	1.2	0.06	24	-0.027083	-0.65
30		0.09	0.0675	1	0.07	14.28571	0.0675	0.9642857
31		0.12	0.0975	1.4	0.08	24.5	0.069643	1.70625
32		0.08	0.0575	0.6	0.06	6	0.095833	0.575
33								
34	ratio S	-0.047727	-0.027083	0.0675	0.069643	0.095833		
35								
36		cutoff						
37		0.013286	-0.009015	0.009919	0.027713	0.032346		
38		0.049895	0.013352	0.025282	0.037447	0.041121		
39		0.058181	0.027025	0.034256	0.042555	0.045449		
40		0.042183	0.027336	0.031881	0.038018	0.040231		
41		0.032278	0.021203	0.025831	0.032242	0.034442		

Figure 2 An Excel Worksheet Illustrating Long-Short Portfolio Selection with Beta Neutrality Based on the Corresponding Analytical Solution

	A	B	C	D	E	F	G	H
42								
43		feasible cutoff						S count
44		0.013286	-0.009015					2
45		0.049895	0.013352					2
46		0.058181	0.027025					2
47			0.027336					1
48								0
49								
50	L count	3	4	0	0	0		
51								
52		optimal cutoff						
53		0.027336						
54						beta	dollar	
55	sorted	z L		x L		L side		
56	1	0.624129		0.197944		0.734287	0.936385	
57	2	1.22664		0.389032				
58	3	1.090346		0.345806				
59	4	0		0				
60	5	0.011359		0.003603				
61								
62	sorted	z S		x S		S side		
63	1	0		0		-0.734287	-0.636153	
64	2	0		0				
65	3	-1.088387		-0.345185				
66	4	0		0				
67	5	-0.91744		-0.290969				
68						sum		
69						0	0.300231	
70	z normalization factor							
71		3.153057						
72								
73	ex ret p	0.14902						
74	st dev p	0.194287						
75	theta	0.612597						

Figure 2 An Excel Worksheet Illustrating Long-Short Portfolio Selection with Beta Neutrality Based on the Corresponding Analytical Solution (Continued)

ratios, will appear to be 3, 1, 1, 5, 4, respectively. Due to the tie between securities 2 and 3 for the top rank, VLOOKUP will fail to sort properly the data pertaining these two securities. As a precautionary measure, tie breaking is automatically performed, if required. Two securities of a tied rank will either be both selected to the portfolio or be both excluded from it. Thus, whether a security ends up having a higher rank or a lower rank, as the result of tie breaking, does not affect the desirability of the security for portfolio holding.

Here is how tie breaking can be accomplished: While the formula for B7 is still =RANK(H7,H$7:H$11), the formula for B8, which is also copied to B8:B11, is =RANK(H8,H$7:H$11)+COUNTIF(H$7:H7,H8) instead. The idea of using the Excel function COUNTIF is to see if the same rank has appeared previously; if so, the number of previous appearances will be added to the initial rank of the security in question. In the current example, the ranks of securities 1, 2, 3, 4, and 5 after tie breaking are 3, 1, 2, 5, 4, respectively. The same idea applies to the five securities that are considered for the short side. The formula for B14 is =RANK(H14,H$14:H$18,1); the formula for B15, which is copied to B15:B18, is =RANK(H15,H$14:H$18,1)+COUNTIF(H$14:H14,H15). The third argument in the Excel function RANK here, which is non-zero, indicates that the ranking is intended to be in an ascending order.

The Excel file accompanying this paper also contains a worksheet that shows an alternative approach for tie breaking. Given the close similarity between such a worksheet and that for Figure 2, there is no need to display it as a separate figure here. In essence, instead of ranking the data in H7:H11 and in H14:H18 directly, we rank the data in I7:I11 and in I14:I18, which contain the same data plus a small random number in each cell. Specifically, the formula =H7+RAND()/1000000 for I7 is copied to I7:I11 and I14:I18. The formula for B7, which is =RANK(I7,I$7:I$11) instead, is copied to B7:B11; likewise, the formula for B14, which is =RANK(I14,I$14:I$18,1) instead, is copied to B14:B18. Notice that the contaminated data in I7:I11 and I14:I18 are intended only for establishing unique ranks of the securities considered and that such data will not be used for any subsequent computations.

The security-specific data in C7:H11 are sorted with VLOOKUP according to the ranks in B7:B11, with the results displayed in B21:G25. Specifically, the formulas =VLOOKUP($A7,$B$7:$H$11,2,FALSE) for B21 is copied to B21:G21, with minor changes. All that is required is to change the 2 there for C21, D21, $\ldots$, G21 to $3, 4, \ldots, 7$, respectively, to indicate the corresponding columns of data in VLOOKUP. Subsequently, the formulas in B21:G21 are copied to B21:G25. The sorted data in B28:G32 are obtained in the same manner, but with the starting formula

=VLOOKUP($A14,$B$14:$H$18,2,FALSE) for B28.

As the data in C7:H11 and C14:H18 are all numerical data, the function SUMIF can also be used to sort them. Although SUMIF is commonly used to provide the sum of the data involved under certain criteria, it is also suitable for use to sort numerical data if each sum pertains to a single item. The individual ranks among the five values in B7:B11, as well as those in B14:B18, are always distinct, so that SUMIF can work well as intended for the current setting. To implement such an alternative approach, the formula =SUMIF($B$7:$B$11,$A7,C$7:C$11) for B21 is copied to B21:G25, and the formula =SUMIF($B$14:$B$18,$A14,C$14:C$18) for B28 is copied to B28:G32. The corresponding details can be found in the worksheet in the Excel file accompanying this paper, which also illustrates the alternative approach for tie breaking as described above.

To facilitate the computation of $\eta(h,[k])$, for $h,k=1,2,\ldots,5$, the products of $(\mu_i - R_f - \tau)/\beta_i$ and β_i^2/σ_{ei}^2, for $i=1,2,\ldots,5$, are stored in H21:H25, and the products of $(\mu_{[j]} - \nu R_f - \tau)/\beta_{[j]}$ and $\beta_{[j]}^2/\sigma_{e[j]}^2$, for $j=1,2,\ldots,5$, are stored in H28:H32, both under the heading of "wgt*ratio." All 25 cases of $\eta(h,[k])$ are stored in B37:F41, under the heading of "cutoff." They are generated by copying the cell formula for B37, which is =(SUM($H$21:$H21,$H$28:$H$28)/(SUM($F$21:$F21,$F$28:$F$28))), to B37:F41. With B37:F41 treated as a 5×5 matrix, its (h,k) element is $\eta(h,[k])$.

Among these 25 matrix elements, only those corresponding to feasible portfolios are displayed in B44:F48, under the heading of "feasible cutoff." Feasibility requires that the condition of $(\mu_i - R_f - \tau)/\beta_i > \eta(h,[k]) \geq (\mu_{[j]} - \nu R_f - \tau)/\beta_{[j]}$ is satisfied. For this task, the cell formula =IF(OR($G21<=B37, B$34>B37),"",B37) for B44 is copied to B44:F48. The optimal value of $\eta(h,[k])$ is the one consisting of the highest numbers of securities from long and short sides of the portfolio. The counts of non-blank row and column entries in B44:F48 are shown in H44:H48 and B50:F50, respectively, by means of the Excel function COUNT. The cell formula =OFFSET(A43,MAX(B50:F50),MAX(H44:H48)) for B53 displays the optimal $\eta(h,[k])$ for the case where the dollar neutrality condition is ignored.

The values of z_i, for $i=1,2,\ldots,5$, based on the optimal $\eta(h,[k])$ are displayed in B56:B60. This task is accomplished by copying the cell formula =IF(H7-$B$53>0,E7/F7*(H7-B$53),0) for B56 to B56:B60. Likewise, the values of $z_{[j]}$, for $j=1,2,\ldots,5$, are displayed in B63:B67; the task involves copying the cell formula =IF(H14-$B$53<0,E14/F14*(H14-B$53),0) for B63 to B63:B67. The normalization factor, $\sum_{i=1}^{5} z_i - \lambda \sum_{j=1}^{5} z_{[j]}$, is displayed in B71. Once the individual values of z_i and $z_{[j]}$ are scaled by such a normalization factor, the set of optimal portfolio weights is reached. It is displayed in D56:D60 and D63:D67 with a darker shade.

The corresponding results of μ_p, σ_p, and θ, as provided in B73:B75, are also shaded. The cell formula for B73 is =MMULT(TRANSPOSE(D56:D60),D7:D11)+MMULT(TRANSPOSE(D63:D67), D14:D18)+B1, which requires the "Shift," "Ctrl," and "Enter" keys on the keyboard to be presses simultaneously. This cell formula for the computation of μ_p is based on equation (71), where each summation is now presented more succinctly in matrix forms.[14] The computations of σ_p and then θ are based on equations (50) and (8), respectively.

In this illustration, among the five securities considered for the long side, four are selected, with $x_1 = 0.1979$, $x_2 = 0.3890$, $x_3 = 0.3458$, and $x_5 = 0.0036$. On the short side, two of the five securities considered are selected, with $x_{[3]} = -0.3452$ and $x_{[5]} = -0.2910$. The portfolio weights for the four excluded securities are set to be zeros; that is, $x_4 = x_{[1]} = x_{[2]} = x_{[4]} = 0$. The remaining portfolio results are $\mu_p = 0.1490$, $\sigma_p = 0.1943$, and $\theta = 0.6126$.

As expected, the beta neutrality condition in equation (18) is satisfied; $\sum_{i=1}^{5} x_i\beta_i = 0.7343$ in F56 and $\sum_{j=1}^{5} x_{[j]}\beta_{[j]} = -0.7343$ in F63 sum to zero in F69. However, as displayed in G69, the sum of $\sum_{i=1}^{5} x_i = 0.9364$ in G56 and $\sum_{j=1}^{5} x_{[j]} = -0.6362$ in G63 is 0.3002, which is non-zero, indicating that the dollar neutrality condition in equation (19) is violated. This outcome is not a surprise, as the unknown parameter τ has been preset to be zero, rather than solved. Presetting $\tau = 0$ is analytically equivalent to ignoring the dollar neutrality condition. With one less constraint to be satisfied, a higher computed $\theta = 0.6126$ here, as compared to the corresponding $\theta = 0.5274$ in Figure 1, is as expected.

5.3 An Approach Based on the Analytical Solution with Both Beta Neutrality and Dollar Neutrality

The illustration in Figure 3 differs from that in Figure 2 only in the way the unknown parameter τ is treated. In Figure 3, the attempted τ^* in B4 is no longer a preset value; rather, it is a parameter that is allowed to vary until the dollar neutrality condition in equation (19) is satisfied. As the relationship between the attempted τ^* and the departure of $\sum_{i=1}^{n_L} z_i + \sum_{j=1}^{n_S} z_{[j]}$ from zero is monotonic, a numerical search for the optimal τ^* can easily be performed. Thus, the corresponding Excel worksheets for the two figures differ only in an additional numerical procedure. For the dialog box of either Solver or Goal Seek, which is used to perform this procedure, we simply set the target cell \$G\$69 equal to a value of zero, by changing \$B\$4, the cell containing any initial value of τ^*.

In Figure 2, where the dollar neutrality condition is ignored, the magnitude of the total invest-

[14] The summation $\sum_{i=1}^{n} a_i b_i$ can be written equivalently as the matrix product $\boldsymbol{A}'\boldsymbol{B}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are n-element column vectors with elements being $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, respectively. Here, the prime denotes matrix transposition.

	A	B	C	D	E	F	G	H
1	rf	0.03						
2	rebate	0.75						
3	lambda	0.10						
4	tau	0.096626						
5								
6	sec L	rank	exp ret	excess ret	beta	res var	wgt	ratio L
7	1	3	0.11	-0.016626	1.1	0.08	15.125	-0.0151146
8	2	2	0.12	-0.006626	0.6	0.06	6	-0.0110434
9	3	1	0.15	0.023374	0.8	0.09	7.111111	0.0292174
10	4	5	0.01	-0.116626	1.0	0.06	16.66667	-0.1166261
11	5	4	0.08	-0.046626	1.8	0.07	46.28571	-0.0259034
12								
13	sec S	rank	exp ret	excess ret	beta	res var	wgt	ratio S
14	1	4	0.09	-0.029126	1.0	0.07	14.28571	-0.0291261
15	2	3	0.08	-0.039126	0.6	0.06	6	-0.0652101
16	3	2	-0.01	-0.129126	1.2	0.06	24	-0.107605
17	4	5	0.12	0.000874	1.4	0.08	24.5	0.0006242
18	5	1	-0.03	-0.149126	1.1	0.09	13.44444	-0.1355691
19								
20	sorted	exp ret	excess ret	beta	res var	wgt	ratio L	wgt*ratio
21		0.15	0.023374	0.8	0.09	7.111111	0.029217	0.2077684
22		0.12	-0.006626	0.6	0.06	6	-0.011043	-0.0662606
23		0.11	-0.016626	1.1	0.08	15.125	-0.015115	-0.2286083
24		0.08	-0.046626	1.8	0.07	46.28571	-0.025903	-1.1989557
25		0.01	-0.116626	1	0.06	16.66667	-0.116626	-1.9437676
26								
27	sorted	exp ret	excess ret	beta	res var	wgt	ratio S	wgt*ratio
28		-0.03	-0.149126	1.1	0.09	13.44444	-0.135569	-1.8226518
29		-0.01	-0.129126	1.2	0.06	24	-0.107605	-2.5825211
30		0.08	-0.039126	0.6	0.06	6	-0.06521	-0.3912606
31		0.09	-0.029126	1	0.07	14.28571	-0.029126	-0.4160865
32		0.12	0.000874	1.4	0.08	24.5	0.000624	0.015294
33								
34	ratio S	-0.135569	-0.107605	-0.06521	-0.029126	0.000624		
35								
36		cutoff						
37		-0.078562	-0.094206	-0.090765	-0.077185	-0.055847		
38		-0.063307	-0.084336	-0.082307	-0.071583	-0.053028		
39		-0.045819	-0.068396	-0.068129	-0.061648	-0.047837		
40		-0.035340	-0.050830	-0.051561	-0.049138	-0.041360		
41		-0.048288	-0.059355	-0.059616	-0.056691	-0.048594		

Figure 3 An Excel Worksheet Illustrating Long-Short Portfolio Selection with Beta Neutrality and Dollar Neutrality Based on the Corresponding Analytical Solution

	A	B	C	D	E	F	G	H
42								
43		feasible cutoff						S count
44		-0.078562	-0.094206					2
45		-0.063307	-0.084336					2
46		-0.045819	-0.068396					2
47		-0.035340	-0.050830	-0.051561				3
48								0
49								
50	L count	4	4	1	0	0		
51								
52		optimal cutoff						
53		-0.051561						
54						beta	dollar	
55	rank	z L		x L		L side		
56	3	0.501142		0.199456		1.01745	0.909091	
57	2	0.405178		0.161262				
58	1	0.718033		0.28578				
59	5	0		0				
60	4	0.659774		0.262593				
61								
62	rank	z S		x S		S side		
63	4	0		0		-1.01745	-0.909091	
64	3	-0.136488		-0.054323				
65	2	-1.120876		-0.446113				
66	5	0		0				
67	1	-1.026763		-0.408655				
68						sum		
69						0	1.04E-07	
70	z normalization factor							
71		2.51254						
72								
73	ex ret p	0.140723						
74	st dev p	0.209924						
75	theta	0.527442						

Figure 3 An Excel Worksheet Illustrating Long-Short Portfolio Selection with Beta Neutrality and Dollar Neutrality Based on the Corresponding Analytical Solution (Continued)

ment capital on the long side is greater than that on the short side. Thus, a positive τ^* is required for dollar neutrality to hold. The numerical search result is $\tau^* = 0.0966$, as displayed in a darker shade in B4. As confirmed in F69:G69, both beta neutrality and dollar neutrality conditions are satisfied. The corresponding portfolio results, $x_1, x_2, \ldots, x_5$, $x_{[1]}, x_{[2]}, \ldots, x_{[5]}$, μ_p, σ_p, and θ, as displayed also in the same shade in D56:D60, D63:D67, and B73:B75, are numerically the same as those in Figure 1.

6 Concluding Remarks

Market neutral strategies are practical investment tools in volatile equity markets. Although market neutral strategies as adopted by investment practitioners have various levels of sophistication, a basic idea is still the pursuit of insensitivity of investment returns in response to movements of an equity market or of some specific economic sectors. To prepare business students adequately for the investment world, it is useful for them to acquire some essential knowledge of the materials involved in investment courses.

Among various market neutral strategies, a long-short equity strategy is the easiest to comprehend from a pedagogic perspective; it is about holding some securities in long positions and some other securities in short positions under realistic short-sale conditions. This paper has presented the formulation and then the derivation of a long-short portfolio selection model under beta neutrality and dollar neutrality conditions. The approach here is an extension of textbook materials for long-only portfolio construction.

The extension draws on an idea in the investment literature that short selling a security is like investing in an artificially constructed security. By recognizing the return components of such an investment, this paper has been able to apply the same analytical tools in investment textbooks to a long-short setting. For the task of model derivation, no advanced analytical tools are involved, as intended. Most notable is that, by following the investment textbook approach for analytical convenience without compromising the analytical results, this paper has also been able to bypass the use of formal optimization tools, which rely on slack variables to accommodate inequality constraints.

Excel has played an important pedagogic role in this paper. In order to accommodate investment courses with different levels of analytical rigor, Excel illustrations in this paper have been presented with and without requiring knowledge of the analytical solution. Some convenient technical features in the Excel illustrations of this paper are also worth noting. Technically, the numerical search for the optimal value of an unknown parameter, which pertains to the dollar neutrality condition, requires the individual securities to be relabeled repeatedly according to how they are ranked for each attempted value of such a parameter. The use of various combinations of Excel functions has allowed us to bypass the process of sorting the same numerical data manually. It is hoped that, with a significant reduction of the computational burden that is associated with the long-short equity model involved, business students can pay more attention to conceptual and practical issues arising from the model.

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