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Solving a Diophantine problem using different expressions of the difference of two squares

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1. Introduction

The development of students' abilities to solve problems independently and to discover new information has an important position among the objectives of mathematics teaching. In the process of solving suitably selected problems, students can discover new knowledge which relates to previous knowledge. It is possible to organize the teaching process in a way which permits the students to discover this knowledge independently. The well-known constructivist approach may be used as the basis for this teaching method. Bruner's constructivist theory is a general framework for an instructional approach to teaching from which the concrete teaching strategies can be derived, one of them being guided discovery learning.

The teacher devises a series of statements or questions that guide the learner, step by step, making a series of discoveries that leads to a single predetermined goal. Guided discovery learning can promote meaningful learning of problem solving strategies by activating students' thinking. As Pressley et al. [15] note: "The teacher provides hints about the concepts the child is to discover, but the child has to make substantial effort to figure out the situation compared to when a teacher directly teaches how to do a task." (p. 155). According to Bogart and Flahive [5], guided discovery in mathematics teaching can be characterized as the method which covers "looking at special cases, trying to discover patterns, wandering up blind alleys, possibly being frustrated, but finally putting it all together into a solution of a problem or a proof of a theorem." (p. 1). The teacher guides students to discover patterns in data and processes through the use of guiding tasks and instructions. To quote Gerver and Sgroi's paper [8]: "Students sequentially uncover layers of mathematical information one step at a time and learn new mathematics." (p. 6).

Our paper describes how the fundamental principles of guided discovery learning can be used to design and develop a lesson plan focused on the exploration of relationships involving natural numbers and their squares. The important aspect of the lesson plan is a pluralistic approach to the teaching and learning based on different representations of the same concept and different problem solving strategies to solve a problem. Burn [6] emphasizes: "We have already seen the importance of meeting the same ideas in different examples, but it also seems to be important to meet the ideas in different representations." (p. 28). Spreadsheet methods and tools can help to highlight potential connections between numerical and graphical representations [10]. The students will be involved in interacting with relating numerical, graphical and symbolic representations of mathematical patterns and relations. They will be required to create and work with simple models in the form of tables, graph, expressions and equations under the guidance of the teacher. Lehrer and Lesh [12] highlight the importance of modelling in mathematics teaching: "Modelling emphasizes the need for a broad mathematical education that includes several forms of mathematical inquiry." (p. 358). Modelling activities offer such powerful means of developing mathematical understanding and retaining mathematical knowledge [4].

Abramovich [1] argues that students can use technology as a means of enhancing one's knowledge of mathematics. Technology provides suitable tools for promoting multi-representational approaches to problem solving. It also offers ways to promote students' modelling and problem solving competencies. Technology can provide an alternative to rote learning and automatic memorisation by supporting a guided discovery learning environment that allows the construction of definitions and algorithms by students [18]. It is convenient to use a spreadsheet program in certain phases of the teaching process in order to simplify the construction of tables, to create a graphical interpretation of the relations, to write formulas for calculations and to generalize relationships between numbers in tables. Sugden [16] also emphasizes that spreadsheet can be used as a medium that gives immediate feedback on modelling steps and it lends support for quick tabulation and graphing. A spreadsheet program makes intrinsic use of variables in calculations and it allows various experimentations with data.

Mathematical reasoning and knowledge should be dominant in the teaching process. Therefore, a Microsoft Excel workbook containing worksheets with problems and the necessary prepared data and graphical representations was created for our students.

The prepared workbook navigated the students through the lesson. The basic operations: entering and editing data, writing down a simple formula and copying it in a table are the only skills students require to be able to work with a spreadsheet. The students are expected to complete or create the tables, explore the prepared graph, eventually write down results obtained by means of logical considerations and solving of equations using paper and pencil techniques. We recommend including described guided discovery lessons in teaching after students have absorbed the concept of divisibility of natural numbers and the basic laws of divisibility of natural numbers by 2. The proposed lesson plan was experimentally tested in classroom practice for 17 of 15-year old students in a Slovak grammar school (first grade) during one double class lasting 90 minutes.

2. Starting points and objectives of designed Guided Discovery Lessons

Stimulating the curiosity of students and their innate inclinations to discover new things is an important factor in motivating a positive relation between mathematical discovery and, indeed, discovery, in general. It is very important for a teacher preparing a lesson on any mathematical subject to choose a problem capable of capturing the interest of students and motivating them to look for a way to find its solution. In guided discovery lessons the choice of the problem is important not only for its motivation value but also for its value as a potential source of new patterns which may be discovered on the way to a final solution. Described guided discovery lessons are focused on solving a sequence of partial problems leading up to a solution of the central problem in such a way that the necessary information is revealed step by step, thereby enabling the deeper understanding of the central problem.

The central problem, which was selected for our experimental guided discovery lessons, was one formulated by the famous Greek mathematician Diophantus of Alexandria: *Find a natural number n such that both n and n +100 are squares*.

The standard solution that can be found in some mathematical textbooks is based on expressing one of the variables as a function of the other one and applying appropriate divisibility criteria. The following solution of Diophantine problem is presented in a mathematics textbook [9] as an extending section on divisibility of natural numbers that is aimed for students interested in mathematics. *Let b ($b^2 = n + 100$) be greater than a ($a^2 = n$) and let the difference b – a be equal to m. Then*

$$(a+m)^2 = a^2 + 100 \tag{1}$$

Expressing the variable a from (1) we have:

$$a = \frac{100 - m^2}{2m} \tag{2}$$

Since the denominator is even, so also must be the numerator and thus m^2 is even. Hence, m is even and may be written as m = 2p, $p \in N$. After substituting 2p for m into the equation (2) and reducing the obtained fraction by a factor of 4, we have the equation:

$$a = \frac{25 - p^2}{p} \qquad (3)$$

Since p is a factor of p^2 *and* $a \in N$, *p must be a factor of 25 and thus may equal 1 or 5. For* $p = 5$ *we obtain* $a = 0$, *hence,* $p = 1$. *Then* $m = 2$ *and, using it in (2), we have* $a = 24$. *Therefore the required number n is 576.*

However, the presented solution may be too difficult for students, and it is hardly realizable without appropriate teacher guidance. Moreover, the way to solve a problem as a part of the teaching process will not always be as direct as this – it may call for analysing relationships derived from the problem formulation and the use of heuristic strategies to find a way to a solution of the problem. Abramovich and Sugden [2] have developed interesting ideas for applying technology to solve Diophantine equations. Kahn and Kyle [11] noted the following: "It is also apparent that learning mathematics involves students engaging with a process rather than simply absorbing mathematical products." (p. 205).

The main objective of our guided discovery lessons is to develop the students' ability to formulate the empirically discovered patterns and relations exactly and to increase their ability to use algebraic symbolism for generalizing the properties of natural numbers in the form of mathematical theorems.

The design of the lesson plan is based on an assumption that in the beginning of the cognitive process students will compute the squares of natural numbers and explore their differences. The constructed table will not only be used for looking for a solution of the central problem, but in the second part of the guided discovery lessons, it can help students to discover the relation between the sums of the first consecutive odd numbers and the squares of natural numbers. The thought activities based on abstraction and generalization should result in the formulation of a mathematical theorem. Additionally, various ways of solving the central problem enable deeper understanding of relations between natural numbers and the differences of their squares. Also, in *Principles and Standards for School Mathematics* [13], it is recommended that students should be exposed to "problems that draw on a variety of aspects of mathematics, that are solvable using a variety of methods, and that students can access in different ways." [p. 289].

3. Lesson plan

The lesson plan is split into two parts, which can be realized during two classes. At the beginning of the teaching process, the Diophantine problem is presented to the students. Our teaching experience shows that, although there are some students who try to formulate and solve corresponding equations with variables from the very beginning, the majority of them try to find the solutions by experimenting with particular numbers. For this kind of calculations, it is especially suitable to make use of a spreadsheet program. It enables students to recognize patterns more readily and quickly. Students will be guided in this direction to solve the Diophantine problem by exploring relations between concrete natural numbers and by using these relations in the modelling of tables. Therefore students will use the MS Excel spreadsheet by solving the first three partial problems. At the end of the first part students will be required to build an equation presenting another possible way of solving the Diophantine problem. It expresses the relationships between two natural numbers a, b ($a < b$) whose squares are the required numbers n and $n + 100$:

$$b^2 - a^2 = 100 \tag{4}$$

Students will solve this equation using paper and pencil techniques. After expressing the left part of (4) as the product of two numbers, it is suitable to look for factors of the number 100, which lead to a solution of the equation (4). The solution of equation (4) is also used for checking the correctness of results obtained from computations in the tables.

When solving partial problems, the students are guided to find one solution of Diophantine problem and so give reasons for the uniqueness of this solution.

In the second part students will be guided to discover a new property of natural numbers, and to formulate a mathematical theorem which is finally used to explain the empirical findings. This part is aimed to develop the students' ability for further inquiry and reasoning experience. The MS Excel spreadsheet will only be used for exploring created tables and for executing supplementary calculations. In solving the last problem, students will be required to choose an approach to problem solving based on finding solutions of the constructed equation or modelling in tables with the use of a spreadsheet.

3.1. Part A

The Diophantine problem is approached empirically with different tools such as calculations and graphical explorations in the spreadsheet. The process of solving starts with the construction of a table containing the computation of squares of several first consecutive natural numbers, and the exploration of their differences. Based on such a constructed table, the students can find an upper bound for the required natural numbers and, after expanding of the table with a new column, they can also find the solution of Diophantine problem. In this way, the students will solve the problem intuitively by modelling the squares and relations. The square determines a quadratic dependence, which may be suitably explored using graphs. The graphical representation of the quadratic relationship brings a different view to patterns and differences between squares. When computing tables in a spreadsheet, the variables are used in an intuitive way in connection with cells of the table. Later, the variables are introduced, and their relation is expressed as an equation.

Problem 1: *Complete a table containing the first ten natural numbers by computing their squares and the differences of squares of consecutive numbers. By adding other data into the table, try to find the solution of Diophantine problem.*

The worksheet entitled Estimate contains the instructions for creating the table with appropriate column headings. The cells with given data are highlighted. The students have to express the algorithm for computation in the form of formulas and use these formulas to compute the required numbers in the Square and Difference columns.

To compute a square, one has to enter the following formula into cell B2: =A2*A2 (or =A2^2). To determine the difference between the squares 2^2 and 1^2, there is the formula: =B3 – B2 in cell C3. To repeat calculation for the numbers in the Number column, one can simply copy formulas into the next rows of the table. Figure 1 shows the described table.

C3 fx =B3-B2

	A	B	C
1	Number	Square	Difference
2	1	1	
3	2	4	3
4	3	9	5
5	4	16	7
6	5	25	9
7	6	36	11
8	7	49	13
9	8	64	15
10	9	81	17
11	10	100	19

Figure 1: Calculation of the squares and differences

The differences of the squares of consecutive natural numbers are consecutive odd numbers. The students may observe that after inserting additional rows into the table, it is not possible to find two consecutive natural numbers such that the difference of their squares is equal to 100. The required squares (576 and 676) are not in two consecutive rows of the table, but some students might be able to detect these numbers in the table. The difference of the squares for the numbers 51 and 50 is already 101, thus it makes no sense to insert more rows into the table.

D4 fx =B4-B2

	A	B	C	D
1	Number	Square	Difference	Difference2
2	1	1		
3	2	4	3	
4	3	9	5	8
5	4	16	7	12
6	5	25	9	16
7	:	:	:	:
26	24	576	47	92
27	25	625	49	96
28	26	676	51	**100**

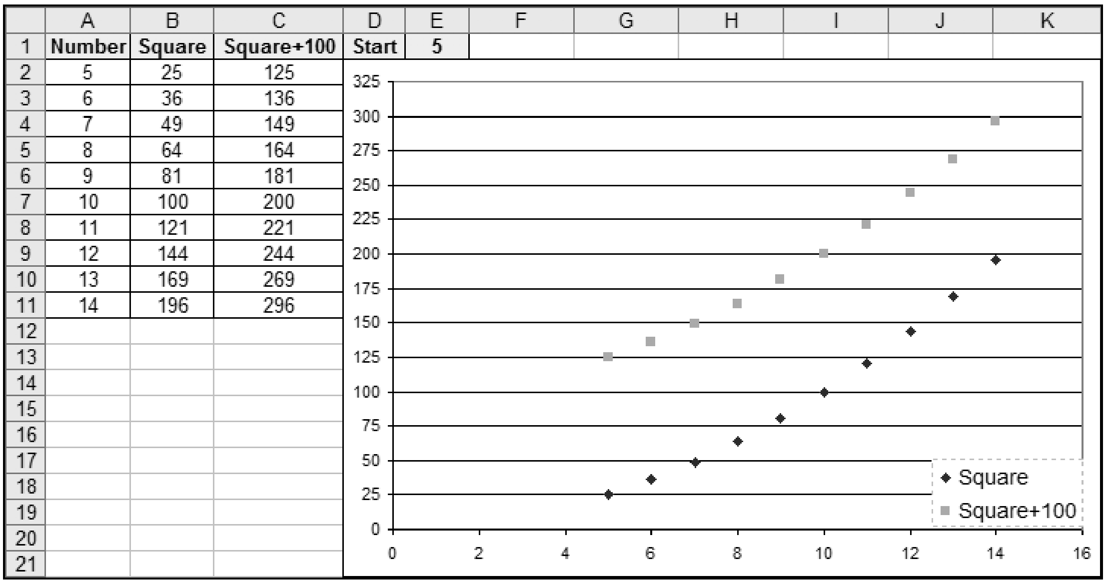
Figure 2: Extension of the table

Since the sum of two odd numbers is even, the students themselves might realize that it would be suitable to add an additional column into the table to determine differences between squares of natural numbers which differ by 2. In the new Difference2 column, one has to enter the formula: =B4 – B2 in cell D4. By copying this formula it is obvious that the difference of 100 is connected with the squares 676 and 576. The figure 2 shows selected rows in the extended table.

The next conclusion resulting from the constructed table is the estimate of the largest natural number up to which it is necessary to search for the squares and their differences. The Difference column shows that it is enough to construct the table up to the number 50 because, when squaring the greater consecutive numbers, their differences already are greater than 100. Similarly, according to calculations in the Difference2 column it is possible to say it is sufficient to compute differences of squares up to 26.

The students of our class were able to write down and copy formulas into the table, but seven of them did not succeed in finding a solution using table computations in time. It was caused by the fact that they either did not create the Difference2 column or they did not construct the differences between squares systematically but only for the selected pairs of natural numbers. We prepared guidance questions for them to focus their attention on the sums of two consecutive numbers in the Difference column: What remainder do you get when you divide the sums of two consecutive numbers in the Difference column by 2? How do you express these sums by means of squares of natural numbers? So we led them to add the new column to the old table. The students are supposed to only write down the solution they have found. But the question concerning the upper bound for tested natural numbers was discussed together with us. The students were required to find systematic ways to explore the differences of squares by solving the first problem. Looking for a way to organize data into well-arranged tables and schemes is an important aspect of the development of algorithmic thinking.

The spreadsheet environment can help the students to see the connections between numerical data in the table and their visual and symbolic representation [7]. It enables the students to experiment with the inputs and look for relations between data elements [3]. Since the larger the table the more difficult it is to find the necessary information the next worksheet called Graph, containing a smaller table together with a visual representation of the problem, was prepared for the students.



	A	B	C	D	E
1	Number	Square	Square+100	Start	5
2	5	25	125		
3	6	36	136		
4	7	49	149		
5	8	64	164		
6	9	81	181		
7	10	100	200		
8	11	121	221		
9	12	144	244		
10	13	169	269		
11	14	196	296		

Figure 3: Visual representation of the squares

In the graph, isolated points from the graph of the basic quadratic function $y = x^2$ (Square) and the graph of the quadratic function $y = x^2 + 100$ (Square + 100) are depicted.

The data source table contains only ten rows, so that not too many points were displayed in the graph at the same time. To prevent the students accessing the source table of the graph, the initial number which starts the sequence of numbers in the individual rows of the table, is taken from cell E1. The table is interconnected with cell E1 by entering the formula: =E1 into cell A2. When the value in cell E1 is changed the values in the table are recalculated and the graph is redrawn (see figure 3).

Problem 2: *Modify the table in worksheet named Graph and find the points of the graph that represent the solution of Diophantine problem.*

The students were required to modify the table as well as the graph by changing the initial value in cell E1 and studying the resulting graphs. If the second coordinate of some point from the graph of the function Square + 100 is the same as that of a point from the graph of the function Square, then the first coordinates of these points represent numbers whose squares differ by 100. If the second coordinates of the respective points are very close to each other, the students can use a feature of MS Excel whereby the coordinates of that point are displayed if the cursor is held close to a point in a graph.

The students of our class understood the relationship between a quadratic formula and its graph. However, they did not have enough experience with the comparison of the graphs of two quadratic functions and they were unable to find the relationship between points of the graphs of different functions. This problem may be considered as preparatory to later exploration of the parametric system of quadratic functions. Therefore, the students are encouraged to explore a graph for a while and it is explained then to them that a point of the graph Square + 100 is also a square of a natural number only if the horizontal line crossing this point contains also a point of the graph Square. After explaining the meaning of the points depicted on the graph, our students easily found a pair of points with second coordinate equal to 676. The first coordinates of these points are 24 and 26, hence it is possible to claim that the numbers $24^2 + 100$ and 26^2 are equal. In our opinion, in this phase, it is already time to introduce variables a, b ($a < b$) to represent the natural numbers whose squares are equal to n and $n + 100$ (the solutions to Diophantine problem).

After exploring the squares of natural numbers and their differences using tables and graphs, the natural question for the students at this stage of problem solving is the following: Is this the only solution of Diophantine problem? We have already found that it makes sense to look for solutions only in the finite set of the first natural numbers. In the next step, the advantages of spreadsheets are used and a table which will enable us to check all possibilities systematically to get other solutions of this problem is constructed. Since the worksheet Search contains only the formulation of the problem the students will now construct the whole table independently.

Problem 3: *Determine the sets of numbers that need to be tested to find all solutions of Diophantine problem for the variables a or b, based on previous estimates of upper bounds for* *tested natural numbers. For the given variable and all associated numbers determined in this way, construct the table for computing the corresponding values of the other variable.*

Design of a well-arranged table to find solutions of this Diophantine problem may be based on a more complex formula involving the use of logical function IF. The table for testing, whether the square roots of investigated natural numbers are integer, is shown in figure 4. In order to reduce the table, some table rows are hidden. The formula with logic function entered in column D is displayed in the formula bar.

D2   fx =IF(SQRT(C2)=INT(SQRT(C2));A2;"")

	A	B	C	D
1	a	$n = a^2$	$a^2 + 100$	test
2	1	1	101	
3	2	4	104	
4	3	9	109	
5	4	16	116	
6	5	25	125	
7	:	:	:	
21	20	400	500	
22	21	441	541	
23	22	484	584	
24	23	529	629	
25	24	576	676	24
26	25	625	725	
27	26	676	776	

Figure 4: The part of a table for solving the problem

Students in our classroom had only basic skills in writing formulas in MS Excel. We preferred the use of simple formulas and we expected that students would construct tables analogous to the table shown in figure 5.

	A	B	C	D
1	b	$b^2 = n+100$	$n = a^2$	a
2	11	121	21	4.582576
3	12	144	44	6.63325
4	13	169	69	8.306624
5	14	196	96	9.797959
6	15	225	125	11.18034
7	16	256	156	12.49
8	17	289	189	13.74773
9	18	324	224	14.96663
10	19	361	261	16.15549
11	20	400	300	17.32051
12	21	441	341	18.46619
13	22	484	384	19.59592
14	23	529	429	20.71232
15	24	576	476	21.81742
16	25	625	525	22.91288
17	**26**	**676**	**576**	**24**

Figure 5: The expected table of students' solving of the problem

For computing a^2 a simple formula in cell C2: =B2 – 100 can be used. The value of the variable a is computed in cell D2 using formula: =SQRT(C2). If the values of the variable a were entered in the first column of the table, similar formulas would be used, but to compute b^2, the number 100 would be added to a^2.

Due to considerations concerning the upper bound for tested numbers and the meaning of variables a, b ($b > a$), the majority of the students were constructing a table with given values for the variable b. At first some students entered also numbers 1, 2, ... as values of variable b. But they quickly realized that they would then have negative values for a^2. After several attempts, students' tables took on the form displayed in figure 5.

Two students found out also the special case: $b = 10$, $a = 0$ which provoked the further discussion. Similar to Diophantus′ approach, we also did not include the number 0 to the set of natural numbers, and therefore we did not consider the numbers 0 and 100 as a solution of this Diophantine problem. The calculations confirmed the hypothesis that Diophantine problem has only one solution in the domain of natural numbers – namely 24^2 and 26^2.

To confirm this result, logical considerations based on the relation between variables a, b are utilized at the end of Part A. The relations used in the definition of formulas in the table in figure 4 or 5 will easily lead to the equation (4). After factorisation of the left side of this equation, it is also possible to use divisors of number 100 and the laws of divisibility of sum and subtraction of two natural numbers by 2 to solve this equation in the domain of natural numbers. Another way of solving this equation can be based on solving systems of two linear equations.

Problem 4: *Express the relation between squares of variables a, b in the form of an equation. Find all decompositions of the number 100 into the product of two factors and determine those ones which yield the solution to the formulated equation in the set of natural numbers.*

Factorising the expression b^2 - a^2 we obtain the equation:

$$(b-a)(b+a)=100 \tag{5}$$

In solving this equation in the natural number domain, it is possible to use the properties of the factorisations of the number 100 into two natural numbers. There are exactly four factorisations of the number 100 which can be used as (b – a) and (b + a) in the left part of (5): (1, 100), (2, 50), (4, 25) and (5, 20). If $b – a = 1$ then exactly one of the two numbers a, b has to be odd. Then sum of the numbers a, b has to be odd too and it cannot be equal to 100. Thus the first and then also the fourth factorisations are eliminated. Similar reasoning rules out the third possibility.

At this stage it was easy for the students to build up the equation (5). Most students were able to find the above four factorisations and choose the only suitable one. Though the students were asked to try to solve the equation using the laws of divisibility of natural numbers by 2 to eliminate unsuitable factorisations, they mostly used the method of addition for solving a system of two linear equations. They were able to see that three of the factorisations were unsuitable since the number 2b would have to be odd for them. The only one left is the second factorisation which implies $b - a = 2$, $b + a = 50$. This system has exactly one solution, namely $a = 24$, $b = 26$.

3.2. Part B

In the second part of the proposed guided discovery lessons a more detailed analysis of the first table in the Estimate worksheet will be performed. By considering the table and looking for connections between the differences of squares of consecutive natural numbers, it may lead to the discovery of the relation between the sum of the first one, two, three, ... consecutive odd natural numbers, and the square of 1, 2, 3, ... (see the table in figure 6). After discovering this relation for several first consecutive odd natural numbers, students should generalize this property of odd numbers and exactly formulate it as a mathematical theorem. The students are required to apply this theorem to find another way of solving the Diophantine problem so that diversity in student thinking was promoted. To improve and deepen the understanding of new knowledge discovered in the teaching process, a supplementary problem is prepared at the end of Part B. This problem may also serve as a feedback and could enable the teacher to gain information concerning the extent of discovered mathematical knowledge.

For easier orientation in the table and the expression of discovered relations between numbers using a variable, the head of the table in figure 1 is changed to: k, k^2, and Difference. If the second odd number 3 (cell C3) is added to the first odd number 1 (cell B2), we get the result 4 (B3) which is 2^2. If the third odd number 5 is added to the number 4, we obtain the result 9 or 3^2. Using the described way of summation of numbers, one can continue at the table in the way showed in figure 6.

	A	B	C
1	k	k^2	Difference
2	1	1	
3	2	4	+ 3
4	3	9	+ 5
5	4	16	+ 7
6	:	:	:

Figure 5: The differences between the squares

A generalization of this method is the following theorem: For every natural number k the sum of the first k odd natural numbers equals k^2, i.e.

$$\forall k \in N : 1 + 3 + 5 + \cdots + (2k - 1) = k^2 \qquad (6)$$

The next problem should direct the students' attention to discover the formula (6).

Problem 5: *In the adjusted table from the Estimate worksheet, add together consecutive odd numbers step by step from 1 through 3, 5 etc., in the Difference column and determine partial sums and the sum of the first ten odd natural numbers. Try to discover a pattern and* *formulate exactly your hypothesis about the sum of the first k consecutive odd natural numbers.*

The students were asked to solve the problem by exploring patterns in the table. If they needed assistance in solving the first part of this problem, they could exploit the scheme shown in figure 6 which was placed on the next worksheet entitled Relationship. Eight students were able to formulate the general statement. The most common student mistake was the incomplete statement that the sum of the first k consecutive odd natural numbers is a square. Therefore, at the end of solving Problem 5, the theorem (6) was written down on the board, pointing out the numbers in the table and its validity for several first odd natural numbers.

The approved way of passing from this to the general statement would be by mathematical induction. Unfortunately, this method of proving general relationships is usually (in high school mathematics curriculum) explained after the divisibility of natural numbers. Therefore, the justification of the theorem (6) will be based on ideas used by Gauss in his youth [14, p. 7]. The method used by Gauss for the sum of the first hundred natural numbers, we apply for the sum of the first k odd natural numbers. For its explanation, the previous problem may be used. First, we write down the sum of the first ten odd numbers in a row. The second row contains the same numbers in an inverted order. The sum of the numbers in each column will equal 20. Since each number occurs twice in our sum the resulting sum equals 100.

1	+	3	+	5	+	7	+	9	+	11	+	13	+	15	+	17	+	19
19	+	17	+	15	+	13	+	11	+	9	+	7	+	5	+	3	+	1
20		20		20		20		20		20		20		20		20		20

Let us apply this method to calculating the sum of the first k odd natural numbers.

1	+	3	+	...	$2i-1$	...	+	$2k-3$	+	$2k-1$
$2k-1$	+	$2k-3$	+	...	$2k-(2i-1)$	...	+	3	+	1

At i-th place in the first row of the table, there is i-th odd number $2i-1$. In the second row, below this number, there is the odd number $2k-(2i-1)$. Then the sum in each column will be $2k$ and the resulting sum will be half of $k.2k$.

The theorem (6) can be used to write down the following equalities for the numbers a, b considered in solving Diophantine problem:

$$a^2 = 1 + 3 + 5 + \ldots + 2a - 1$$

$$b^2 = 1 + 3 + 5 + \ldots + (2a-1) + (2a+1) + \ldots + (2b-1)$$

Thus it is possible to express the difference $b^2 - a^2$ as the sum of several consecutive odd natural numbers and this sum must equal 100. Since the sum of two odd numbers is even and the sum of even numbers is even, to have a total sum even, an even number of summands is required. For two summands, it is possible to find the numbers 49 and 51 readily. In our notation these numbers equal $2a+1$, $2b-1$. Thus $a = 24$, $b = 26$. We could go on to investigate the case of four, six, eight or ten consecutive odd natural numbers. Further investigations are not required since the sum of the first ten odd natural numbers equals 100.

Students are asked to use these considerations in the application of the theorem (6) in solving the Diophantine problem.

Problem 6: *According to the discovered theorem, complete the following equalities:*

$a^2 = 1 + 3 + 5 + ...$

$b^2 = 1 + 3 + 5 + ... + (2a - 1) + ...$

Find the numbers whose sum is equal to the difference $b^2 - a^2$. Try sums with different numbers of summands and determine the one that yields the solution to Diophantine problem.

Though the expression $2a - 1$ in the second equality was provided as a help in the expansion of the terms, some students wrote up the last term $2b - 1$ to the second equality only, and they did not realize that the difference $b^2 - a^2$ may be expressed as the sum of the consecutive odd natural numbers from $2a + 1$ to $2b - 1$. These students did not know which summands need to be used in an expression of the number 100 and they solved the problem incorrectly. The next mistake was that some students have found only two summands 49 and 51, calculated a, b, but they did not test other cases. Probably, they were influenced by their knowledge that this Diophantine problem has only one solution. Almost half of the students explored also sums of four or more summands. For calculations, they used spreadsheet but also the summation of terms with variables. For example, if the equality for four summands is considered

$$(2a + 1) + (2a + 3) + (2a + 5) + (2a + 7) = 100,$$

then we arrive to a non-integer a.

	A	B	C	D
1		9	11	13
2		11	13	15
3		13	15	17
4		15	17	19
5		17	19	21
6		19	21	23
7	**sum**	**84**	**96**	**108**

Figure 6: Calculation of the sums of six consecutive odd natural numbers

Figure 7 presents a student's solution based on calculations in tables. This student showed that the sum of six consecutive odd natural numbers cannot equal 100. Some students showed that the number 100 cannot be expressed as a sum of 4, 6 or 8 consecutive odd numbers. They were able to use partial result of Problem 5, and when determining the maximum number of summands, they correctly stated that the number 100 can be expressed as the sum of at most 10 consecutive odd natural numbers because the sum of the first ten consecutive odd natural numbers already equals 100. Since the value of the variable a would be zero, this case does not yield a solution of Diophantine problem, as noted before.

This method of solving Diophantine problem is the final stage of the lesson plan wherein we attempted to solve the central problem using basic operations on natural numbers with special emphasis on divisibility by 2. The general theorem (6) together with the solution of the last problem helped the students understand why in the first table on the Estimate worksheet the differences between squares of consecutive natural numbers form the sequence of consecutive odd numbers. The differences between squares of natural numbers $(n + 2)^2$ and n^2 may be expressed as the sum of two consecutive odd numbers and therefore have to be even. The greater the difference between two natural numbers, the more consecutive odd numbers is needed to make up the difference between their squares.

At the end of the guided discovery lessons, an additional problem which is intended for independent work of students has been included. The solution of the additional problem can be based on similar methods as used in solving the above mentioned problems. The students may look for a solution designing a table in a spreadsheet environment or writing an equation and solving it in a similar way as the equation (4).

Problem 7: *The students have decided to make cardboard rulers in the shape of right triangles with integer sides. The length of the shorter leg of the triangle should be 12 cm. How many different rulers can they make?*

For right triangles whose legs measure 12, b and the hypotenuse, c holds $(c - b)(c + b) = 144$. There are four suitable factorisations of 144, namely (2, 72), (4, 36), (6, 24), (8, 18). Of these, only the first two yield right triangles with shorter leg 12. The students can therefore make triangles with the following dimensions: 12, 35, 37 and 12, 16, 20.

The students solved this problem at the end of the lesson. Seven students solved the problem using the method above - by constructing an equation based on the Pythagorean Theorem and looking for suitable factorisations of the number 144. Two of the students missed some factorisations of 144 and did not find both solutions. Five students solved the problem correctly. Ten students were seeking a solution with the help of a table designed in a spreadsheet environment. Most of them changed the longer leg and tried to find a row with an integer hypotenuse. One student created a small table and he found only one solution. His table is shown in figure 8.

a	b	c^2	c
12	13	313	17,69181
12	14	340	18,43909
12	15	369	19,20937
12	**16**	**400**	**20**
12	17	433	20,80865
12	18	468	21,63331
12	19	505	22,47221
12	20	544	23,32381
12	21	585	24,18677
12	22	628	25,05993
12	23	673	25,94224
12	24	720	26,83282
12	25	769	27,73085
12	26	820	28,63564
12	27	873	29,54657
12	28	928	30,46309
12	29	985	31,38471
12	30	1044	32,31099

Figure 7: An incomplete student's solution

The students did not devote much consideration to the maximum possible length of the longer leg and the number of rows in their tables ranged from eighteen (*b = 30*) to eighty eight (*b = 100*). They could see that the table should contain sixty rows (*b = 71*), because the difference $c - b$ is smaller than 1 in consecutive rows. The required number of rows in the table can be determined by solving the inequality:

$$\sqrt{144+b^2}-b \geq 1 \qquad (7)$$

Two students used an original, unexpected approach to solving this problem. They tried to create tables in which the values of the hypotenuse *c* were changed. Since they did not calculate the smallest length of the hypotenuse *c* they began calculations with $c = 1$. Their tables contained negative numbers and error messages by calculations of the lengths of the leg *b* in several first rows of the tables. Figure 9 shows a part of the student's table that was created in the described way.

c	c^2	b^2	b
:	:	:	:
12	144	0	0
13	169	25	5
14	196	52	7,211103
15	225	81	9
16	256	112	10,58301
17	289	145	12,04159
18	324	180	13,41641
19	361	217	14,73092
20	**400**	**256**	**16**
:	:	:	:

Figure 8: A student's solution based on the change of the hypotenuse *c*

Seven students found both solutions by means of calculations in tables. None of the students explored sums of consecutive odd natural numbers equal to 144 (the applicable sums: 71 + 73, 33 + 35 + 37 + 39).

4. Conclusion

In this paper we tried to propose and analyse guided discovery lessons designed to enhance the students' knowledge of the properties of natural numbers. The key elements of the designed lesson plan are the discovery and justification of the relationship about the sum of the first consecutive odd natural numbers and its application to the investigation of the differences of squares of natural numbers. At the same time the new knowledge may carry a preparatory function concerning the summing of terms of an arithmetic sequence.

Tabach and Friedlander [17] have distinguished three basic types of learning activities in which spreadsheet tools can be suitably implemented: generational activities, transformational activities, and global/meta-level activities. The above described sequence of classroom activities may be included in the first and third groups of learning activities. The students used simple formulas in a spreadsheet environment which provide potential to promote quantitative and graphical methods for problem solving and generalization of experimental findings. Solving of prepared sequence of problems requires dealing with various global/meta-level learning activities, mainly defining relations, modelling, analysing data, and justifying.

The described lesson plan enables students to acquire knowledge and skills through direct experience by doing various learning activities. Students can use spreadsheet tools for higher levels of exploration. Spreadsheet promotes to reduce mechanical work and offers opportunities for the use of different solution strategies. Spreadsheet tools enable students to work with different representations of data and facilitate transition from numerical sequences to corresponding symbolizations using variables. Analysis of arithmetical and graphical models allows students to discover patterns, make generalizations, and understand abstract concept.

The lesson plan includes also spreadsheet as a tool, which can help to solve mathematical problems and increase the motivation of students. The use of modern technologies in mathematics teaching influences the mathematics that is taught and enhances students' learning. It also develops the students' ability to adapt to the ever increasing technologically oriented workplace.

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