

10-18-2013

## The role of spreadsheets in an investigation of Fibonacci Numbers

John E. Baker

*Natural Maths*, john@naturalmaths.com.au

Steve Sugden

*Queensland University of Technology*, ssugden@bond.edu.au

Follow this and additional works at: <http://epublications.bond.edu.au/ejsie>



This work is licensed under a [Creative Commons Attribution-Noncommercial-No Derivative Works 4.0 License](https://creativecommons.org/licenses/by-nc-nd/4.0/).

---

### Recommended Citation

Baker, John E. and Sugden, Steve (2013) The role of spreadsheets in an investigation of Fibonacci Numbers, *Spreadsheets in Education (eJSiE)*: Vol. 7: Iss. 2, Article 2.

Available at: <http://epublications.bond.edu.au/ejsie/vol7/iss2/2>

This Regular Article is brought to you by the Bond Business School at [epublications@bond](mailto:epublications@bond.edu.au). It has been accepted for inclusion in *Spreadsheets in Education (eJSiE)* by an authorized administrator of [epublications@bond](mailto:epublications@bond.edu.au). For more information, please contact [Bond University's Repository Coordinator](#).

---

# The role of spreadsheets in an investigation of Fibonacci Numbers

## Abstract

We introduce a function  $Z(k)$  which measures the number of distinct ways in which a number can be expressed as the sum of Fibonacci numbers. Using a binary table and other devices, we explore the values that  $Z(k)$  can take and reveal a surprising relationship between the values of  $Z(k)$  and the Fibonacci numbers from which they were derived. The article shows the way in which standard spreadsheet functionalities makes it possible to reveal quite striking patterns in data.

## Keywords

Mathematics investigation, Fibonacci numbers, Zeckendorf

## Distribution License



This work is licensed under a [Creative Commons Attribution-Noncommercial-No Derivative Works 4.0 License](https://creativecommons.org/licenses/by-nc-nd/4.0/).

# The role of spreadsheets in an investigation of Fibonacci Numbers

John Baker

*Natural Maths*

[john@naturalmaths.com.au](mailto:john@naturalmaths.com.au)

Stephen Sugden

*Queensland University of Technology*

[stephen.sugden@qut.edu.au](mailto:stephen.sugden@qut.edu.au)

## Abstract

We introduce a function  $Z(k)$  which measures the number of distinct ways in which a number can be expressed as the sum of Fibonacci numbers. Using a binary table and other devices, we explore the values that  $Z(k)$  can take and reveal a surprising relationship between the values of  $Z(k)$  and the Fibonacci numbers from which they were derived. The article shows the way in which standard spreadsheet functionalities makes it possible to reveal quite striking patterns in data.

Keywords: Mathematics investigation, Fibonacci numbers, Zeckendorf

## 1. Introduction

This paper extends the introduction to the topic of expressing a positive number as the sum of Fibonacci numbers that was initiated in Baker and Sugden [2]. In that paper we introduced the notation that if

$$k = \sum_{i=1}^n e_i f(i) \text{ for all } k \in Z \text{ where } e_i = 0 \text{ or } 1$$

then  $Z(k)$  is defined to be the number of distinct ways in which  $k$  can be expressed as the sum of Fibonacci numbers. In this paper, we will call an instance of  $k$  being expressed as the sum of Fibonacci numbers a  $Z$ -expression.

In an earlier paper in this journal, Abramovich and Leonov [1] showed how effective a spreadsheet can be in the exploration of Fibonacci numbers and reflected on activities designed for a technology-rich mathematics education course for prospective teachers of secondary mathematics. It is argued that an appropriate experience with a mathematical frontier can motivate the teachers to teach through a guided discovery mode.

Our goal is similar in that we show how certain features of the spreadsheet, in particular conditional formatting and data table filtering, were used to explore the values of  $Z(k)$ . The process showed how results that Klarner [5] and [6] describes can be displayed visually and led to further results about the expression of a natural number as the sum of Fibonacci numbers.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1													
2		b	1	1	2	3	5	8	13	21	34	55	k
3		0	0	0	0	0	0	0	0	0	0	0	0
4		1	1	0	0	0	0	0	0	0	0	0	1
5		2	0	1	0	0	0	0	0	0	0	0	1
6		3	1	1	0	0	0	0	0	0	0	0	2
7		4	0	0	1	0	0	0	0	0	0	0	2
8		5	1	0	1	0	0	0	0	0	0	0	3
9		6	0	1	1	0	0	0	0	0	0	0	3
10		7	1	1	1	0	0	0	0	0	0	0	4
11		8	0	0	0	1	0	0	0	0	0	0	3
12		9	1	0	0	1	0	0	0	0	0	0	4
13		10	0	1	0	1	0	0	0	0	0	0	4
14		11	1	1	0	1	0	0	0	0	0	0	5
15		12	0	0	1	1	0	0	0	0	0	0	5
16		13	1	0	1	1	0	0	0	0	0	0	6
17		14	0	1	1	1	0	0	0	0	0	0	6
18		15	1	1	1	1	0	0	0	0	0	0	7
19		16	0	0	0	0	1	0	0	0	0	0	5

Figure 1: Z-expressions of k

n	1	2	3	4	5	6	7	8	9	11		n	1	2	3	4	5	6	7	8	9	11
f(n)	1	1	2	3	5	8	13	21	34	k		f(n)	1	1	2	3	5	8	13	21	34	k
	1	1	1	0	1	1	0	1	0	38		0	0	0	1	0	0	1	0	1	50	
	1	0	0	1	1	1	0	1	0	38		0	1	1	0	0	0	1	0	1	50	
	0	1	0	1	1	1	0	1	0	38		1	0	1	0	0	0	1	0	1	50	
	1	1	1	0	0	0	1	1	0	38		0	0	0	1	1	1	0	0	1	50	
	1	0	0	1	0	0	1	1	0	38		0	1	1	0	1	1	0	0	1	50	
	0	1	0	1	0	0	1	1	0	38		1	0	1	0	1	1	0	0	1	50	
	1	1	1	0	0	0	0	0	1	38		0	0	0	1	1	1	1	1	1	0	50
	1	0	0	1	0	0	0	0	1	38		0	1	1	0	1	1	1	1	1	0	50
	0	1	0	1	0	0	0	0	1	38		1	0	1	0	1	1	1	1	1	0	50

Figure 2: The binary tables for 38 and 50

The use of a binary table as shown in Figure 1 allowed data filtering to be applied in order to find all possible Z-expressions of  $k$ , and an example is given in Figure 2, where  $k = 38$  and  $k = 50$ . The binary table was also used to demonstrate particular instances of the following formula:

$$\sum_{i=1}^n f(n) = f(n + 2) - 1 \tag{Equation 1}$$

This is a well-known result (Identity 22 in [4]) and it is one that was used many times in our investigation.

The  $Z(k)$  values that were derived from the binary table indicated that

$$Z(f(2n)) = Z(f(2n + 1)) = n + 1$$

A proof of this was given in [2].

The above is a précis of the results discussed in the earlier article. Here we would like to take the reader further into our exploration of Fibonacci numbers to look at

1. maximum and minimum values of  $Z(k)$  within an interval
2. the symmetry of  $Z(k)$  within an interval
3. a formula for the maximum value of  $Z(k)$  within an interval
4. the distribution of Z-pairs, which are pairs of consecutive numbers that have the same  $Z$ -value.

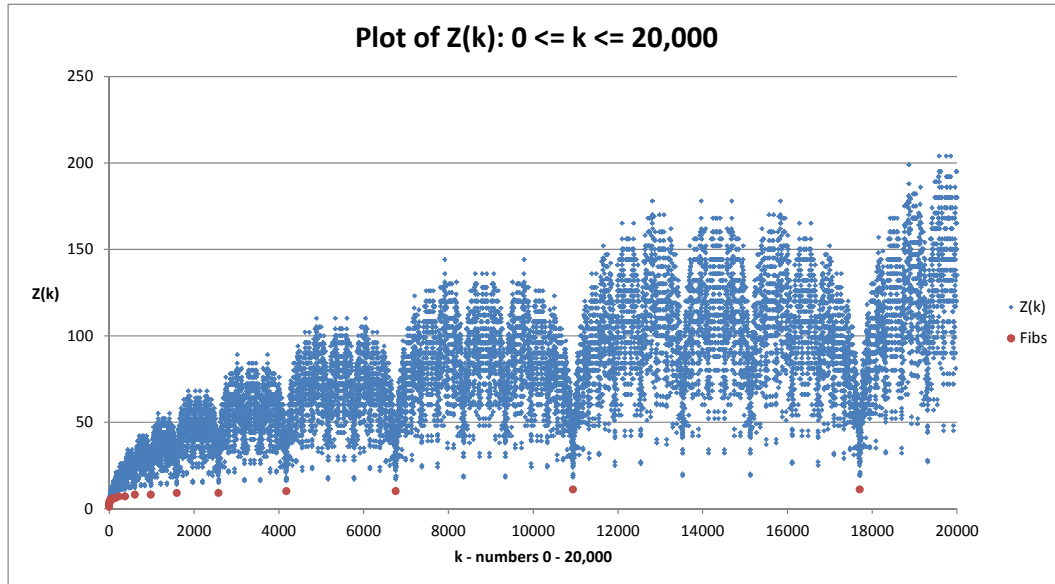


Figure 3: A plot of  $Z(k)$  for large values of  $k$

We will also show how the spreadsheet gave us the tools with which to undertake the investigation.

## 2. Max and Min values of $Z(k)$ within particular intervals

The binary table enabled us to create a list of  $Z$ -values for above ( $1 \leq k < 610$ ) and certain trends emerged from these data. In particular, it appeared that the  $Z$ -value were symmetric within intervals  $[f(n), f(n + 1) - 1]$ . To check that the trends continue for larger values of  $k$  a Visual Basic program enabled us to extend the values of  $k$  to 20,000 and the resulting chart of  $Z(k)$  values, with Fibonacci numbers highlighted, is given in Figure 3 and shows that the trends do continue.

As well as highlighting the presence of the  $Z(f(n))$  values at the minimum positions, this chart also shows that within each interval there are a few points that have a  $Z(k)$  value that is clearly greater than the rest. These extrema were then located in each interval for which  $n \leq 21$  and used to generate the table below (see Figure 4) which shows that the value of these maximum  $Z(k)$  is a Fibonacci number in every other row. Formally this can be stated as:

$$\max(Z(k)) = f(n + 2) \text{ when } f(2n) \leq k < f(2n + 1)$$

The values for  $\max(Z(k))$  that lie in the remaining intervals are not Fibonacci numbers, but they do exhibit a similar pattern. Taking the difference between the numbers in the third column for row  $2n + 1$  and row  $2n$ , we find that the differences are themselves Fibonacci numbers. Thus:

$$\max(Z(k)) = f(n + 2) + f(n - 1) \text{ when } f(2n + 1) \leq k < f(2n + 2)$$

The graph also shows that

$$\min(Z(k)) = f(n) \text{ when } f(n) \leq k \leq f(n + 1) - 1$$

These are two truly surprising results, as we had no inkling that the number of  $Z$ -expressions of a number by Fibonacci numbers would be in any way related to the Fibonacci numbers themselves.

$n$	$f(n)$	$Z(f(n))$	$Max(Z(k))$
1	1	2	2
2	1	2	2
3	2	3	3
4	3	3	3
5	5	3	4
6	8	4	5
7	13	4	6
8	21	5	8
9	34	5	10
10	55	6	13
11	89	6	16
12	144	7	21
13	233	7	26
14	377	8	34
15	610	8	42
16	987	9	55
17	1597	9	68
18	2584	10	89
19	4181	10	110
20	6765	11	144
21	10946	11	178

Figure 4: Table of Max values

### 3. Symmetry within an Interval

The graph of  $Z(k)$  shown in Figure 3 also suggested that the Fibonacci numbers themselves formed the starting points of intervals within which the  $Z(k)$  values are symmetric. For example, the numbers 38 and 50 lie in the interval  $[34, 54]$  and both have a  $Z$ -value of 9. A close examination of the two tables in Figure 2 shows that every  $Z$ -expression of 38 has a corresponding  $Z$ -expression of 50 which is its complement.

The binary tables give the clue to a proof that, indeed, the  $Z$ -values are symmetrical within an interval between two Fibonacci numbers. Let  $k$  lie in the interval  $[f(n), f(n + 1) - 1]$ . Then

$$k = \sum_{i=0}^n e_i f(i)$$

Define  $e'_i = 1 - e_i$  and consider

$$k' = \sum_{i=1}^n e'_i f(i)$$

$$k + k' = \sum_{i=1}^n (e_i + e'_i)f(i) = \sum_{i=1}^n f(i) = f(n + 2) - 1 = f(n) + f(n + 1) - 1$$

Thus  $k$  and  $k'$  are symmetrically placed within the interval  $[f(n), f(n + 1) - 1]$ .

#### 4. Lower and Upper Maxima

In the demonstration  $Z(k)$  is symmetrical between  $f(n)$  and  $f(n + 1) - 1$ , we used a method of locating the symmetrical pairs by finding the complementary sum formed by  $e'_i$  in place of  $e_i$ . This process enables us to locate pairs of numbers, let us call them  $L(n)$  and  $U(n)$  where  $L(n)$  is the smallest value of  $k$  in the  $n$ th interval for which  $Z(k)$  is maximal.  $U(n)$  is then found as the complement of  $L(n)$  and will be the largest value of  $k$  for which  $Z(k)$  is maximal.

At this point we need to make the following assertion. Let  $k$  be a number in the  $n$ th interval, whose  $Z$ -expression is

$$k = \sum_{i=2}^n e_i f(i)$$

Then the  $Z$ -expression must include either  $f(n)$  or  $f(n - 1)$  but not both, because

$$\sum_{i=2}^{n-2} e_i f(i) = f(n) - 1$$

which is not in the  $n$ th interval. Also,  $f(n)$  and  $f(n - 1)$  cannot both be part of the  $Z$ -expression as that would make  $k$  too large to be in the interval.

As an example of  $L(n)$  and  $U(n)$  we turn to the binary table for  $L(n)$  in the 10th interval, which is [55, 88]. The binary table for  $L(10) = 64$  is shown in Figure 5 and the binary table for  $U(n) = 79$  is shown in Figure 6.

As expected, the table for 64 shows 34 or 55 in each of the  $Z$ -expressions. For those that include 34, the residue is a sub-table for the number 30, while those whose  $Z$ -expressions include 55 form a sub-table for 9. Thus we can show this as:

$$Z(L(10)) = Z(64) = Z(30) + Z(9)$$

The table for  $U(10) = 79$  is somewhat different. Again, the  $Z$ -expressions contain either 34 or 55, but there is a block of 1s in rows 1 - 5. This is because each  $Z$ -expression of 79 that includes 34, must also include 21 and 13, since  $79 - 34 - 21 = 24$  which cannot be achieved without including 13. But  $79 - 34 - 21 - 13 = 11$  and that can be achieved without using 8.

In a general sense, then, we can state that a formula for  $Z(k)$ , where  $f(n) < k \leq f(n + 1) - 1$  has two parts. The first part is  $Z$ -expressions that include  $f(n)$ , while the second part is a  $Z$ -expression that includes  $f(n - 2)$  and as many immediately prior Fibonacci numbers as leave

$$k - f(n - 2) - f(n - 3) - \dots - f(n - r) < f(n - r)$$

n	1	2	3	4	5	6	7	8	9	10	11	
F(n)	1	1	2	3	5	8	13	21	34	55	89	k
1	1	1	1	0	1	1	1	0	1	0	0	64
2	1	0	0	1	1	1	1	0	1	0	0	64
3	0	1	0	1	1	1	1	0	1	0	0	64
4	1	1	1	0	1	0	0	1	1	0	0	64
5	1	0	0	1	1	0	0	1	1	0	0	64
6	0	1	0	1	1	0	0	1	1	0	0	64
7	1	0	0	0	0	1	0	1	1	0	0	64
8	0	1	0	0	0	1	0	1	1	0	0	64
9	1	1	1	0	1	0	0	0	0	1	0	64
10	1	0	0	1	1	0	0	0	0	1	0	64
11	0	1	0	1	1	0	0	0	0	1	0	64
12	1	0	0	0	0	1	0	0	0	1	0	64
13	0	1	0	0	0	1	0	0	0	1	0	64

Figure 5: Binary table for 64

n	1	2	3	4	5	6	7	8	9	10	11	
F(n)	1	1	2	3	5	8	13	21	34	55	89	k
1	1	0	1	1	1	0	1	1	1	0	0	79
2	0	1	1	1	1	0	1	1	1	0	0	79
3	1	0	1	0	0	1	1	1	1	0	0	79
4	0	1	1	0	0	1	1	1	1	0	0	79
5	0	0	0	1	0	1	1	1	1	0	0	79
6	1	0	1	1	1	0	1	0	0	1	0	79
7	0	1	1	1	1	0	1	0	0	1	0	79
8	1	0	1	0	0	1	1	0	0	1	0	79
9	0	1	1	0	0	1	1	0	0	1	0	79
10	0	0	0	1	0	1	1	0	0	1	0	79
11	1	0	1	0	0	0	0	1	0	1	0	79
12	0	1	1	0	0	0	0	1	0	1	0	79
13	0	0	0	1	0	0	0	1	0	1	0	79

Figure 6: Binary table for 79

The general expression for  $Z(k)$  is:

$$Z(k) = Z(k - f(n)) + Z\left(k - \sum_{i=n-r}^{n-2} f(i)\right)$$

where  $r$  is chosen so that the residue after  $f(n - 2), \dots, f(n - r)$  have been subtracted from  $k$  is less than  $f(n - r)$ .

Using this formula,

$$Z(U(10)) = Z(79) = Z(24) + Z(11)$$

Figure 7 gives a table of the values of  $L(n)$  and  $U(n)$  which shows a remarkable pattern. Based on the data in the table, we have the following recurrences that generate the  $Z$ -values for  $L(n)$  and  $U(n)$ .

$$Z(L(n + 4)) = Z(U(n + 2)) + Z(L(n))$$

$$Z(U(n + 4)) = Z(L(n + 2)) + Z(U(n))$$

From their definitions, we know that  $Z(L(n)) = Z(U(n))$  for all  $n$ , and that the form of these recurrences leads to Fibonacci numbers in the even-numbered intervals and to the sum of two Fibonacci numbers in the odd-numbered intervals.



Interval	Range	L(n)	U(n)	Calculation of Z(k)
2	1 – 1	1	1	Z(1) = 2
3	2 – 2	2	2	Z(2) = 2
4	3 – 4	3	4	Z(3) = 3 Z(4) = 3
5	5 – 7	6	6	Z(1) + Z(1) = 2 + 2 Z(1) + Z(1) = 2 + 2
6	8 – 12	9	11	Z(1) + Z(4) = 2 + 3 Z(3) + Z(1) = 3 + 2
7	13 – 20	14	19	Z(1) + Z(6) = 2 + 4 Z(6) + Z(1) = 2 + 4
8	21 – 33	24	30	Z(11) + Z(3) = 5 + 3 Z(4) + Z(9) = 3 + 5
9	34 – 54	40	48	Z(6) + Z(19) = 4 + 6 Z(14) + Z(6) = 6 + 4
10	55 – 88	64	79	Z(11) + Z(30) = 5 + 8 Z(24) + Z(11) = 8 + 5
11	89 – 143	103	129	Z(14) + Z(48) = 6 + 10 Z(40) + Z(19) = 10 + 6
12	144 – 232	168	208	Z(24) + Z(79) = 8 + 13 Z(64) + Z(30) = 13 + 8
13	233 – 376	293	336	Z(40) + Z(129) = 10 + 16 Z(103) + Z(48) = 16 + 10
14	377 – 610	441	545	Z(64) + Z(208) = 13 + 21 Z(168) + Z(79) = 21 + 13

Figure 7: Calculation of Z(L(n)) and Z(U(n))

The following steps suggest a line of argument that furnishes a proof that the above formulae are correct. We have seen that every Z-expression of L(n) must include either f(n) or f(n - 1). We also note that the length of the interval [f(n), f(n + 1) - 1] is f(n - 1) = f(n - 3) + f(n - 2). Figure 8 shows what is left of L(n) once f(n) has been subtracted.

If L(n) is close enough to f(n) then the residue is less than f(n - 3) and so we will be looking in interval n - 4. In that interval, L(n - 4) and U(n - 4) have the greatest Z-value.

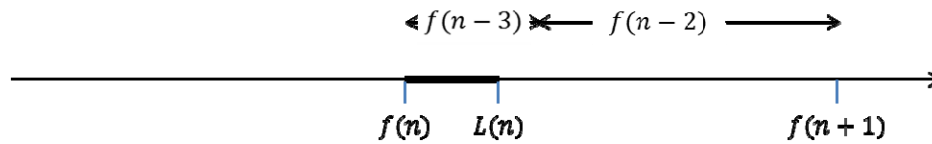


Figure 8: Residue when f(n) is subtracted

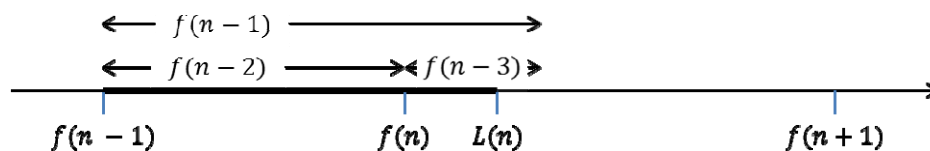


Figure 9: Residue when f(n - 1) is subtracted

For those  $Z$ -expressions that include  $f(n - 1)$ , a different diagram is needed to show what is left of  $L(n)$  after  $f(n - 1)$  has been subtracted (see Figure 9).

This time the residue is less than  $f(n - 1)$  and so we will be looking in interval  $n - 2$ . In that interval  $L(n - 2)$  and  $U(n - 2)$  have the greatest value.

Thus the recurrences:

$$Z(L(n + 4)) = Z(U(n + 2)) + Z(L(n))$$

$$Z(U(n + 4)) = Z(L(n + 2)) + Z(U(n))$$

use the optimal choice of  $Z$ -values from the intervals in which the residues lie after  $f(n)$  and  $f(n - 1)$  have been subtracted from  $L(n)$ .

The diagrams show  $L(n)$  lying closer to  $f(n)$  than  $f(n) + f(n - 3)$ . If  $L(n)$  were further from  $f(n)$  then it can be shown that the residues after  $f(n)$  and  $f(n - 1)$  have been subtracted are both less than  $f(n - 2)$  with the result that the residues will both lie in interval  $n - 3$ . Any choice of two values in interval  $n - 3$  will have a combined  $Z$ -value of less than that of  $L(n)$  as the data that has been generated confirms.

Therefore, the iterative formulas above are optimal for general values of  $n$  and do match the initial data that has been generated.

#### 4.1. Values of $L(n)$ and $U(n)$

The previous section suggests a recurrence relation that can be used to find the maximum values that  $Z(k)$  can take in the intervals between two Fibonacci numbers. In this section we focus on what those values of  $k$  might be. In the 10<sup>th</sup> interval that starts at 55, the graph of  $Z(k)$  shows that  $L(10) = 64$  and  $U(10) = 79$  (see Figure 10).

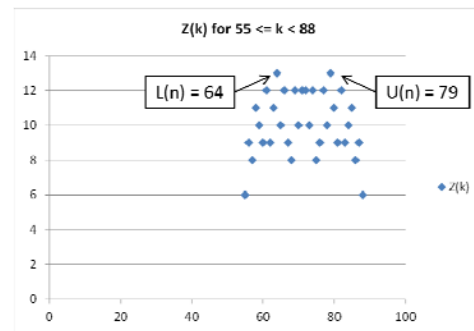


Figure 10: Plot of  $Z(k)$

A table of values was constructed for the first 20 intervals and the values of  $L(n)$  recorded. As a check, the values of  $U(n)$  were read off the charts and found to be exactly those expected, as derived from

$$L(n) + U(n) = f(n) + f(n + 1) - 1$$

However, it did not appear that the values of  $L(n)$  followed a discernible pattern. An extra column was added to the table in which the distance of  $L(n)$  from the starting point of the interval,  $L(n) - f(n)$ , was recorded. The resulting table is shown in Figure 11:

$n$	$f(n)$	$L(n)$	$L(n) - f(n)$
2	1	1	
3	2	1	
4	3	3	0
5	5	6	1
6	8	9	1
7	13	14	1
8	21	24	3
9	34	40	6
10	55	64	9
11	89	103	14
12	144	168	24
13	233	273	40
14	377	441	64
15	610	713	103
16	987	1155	168
17	1597	1870	273
18	2584	3025	441
19	4181	4894	713
20	6765	7920	1155

Figure 11: Table of  $L(n)$  values

The fact that this difference column is an offset of the  $L(n)$  column can be seen from the table and leads to the following recurrence for  $L(n)$ , while the recurrence for  $U(n)$  follows from the symmetry argument.

$$L(n) = f(n) + L(n - 4) \quad \text{Equation 4}$$

$$U(n) = f(n) + f(n + 1) - 1 - L(n) \quad \text{Equation 5}$$

A proof of this conjecture can be built on our previous result:

$$Z(L(n + 4)) = Z(U(n + 2)) + Z(L(n))$$

But before describing the method of proof, let us first examine in detail a binary table that gives the  $Z$ -expressions of 168, which is the  $L(n)$  value for the 12<sup>th</sup> interval [144, 232].

In Figure 12, the rows have been numbered for easy identification. In the last 8 rows, i.e. rows 14 – 21 where  $f(n)$  is part of each  $Z$ -expression of  $L(n)$ , the remaining Fibonacci numbers in each of those  $Z$ -expressions add to 24, which is the value of  $L(n)$  in the 8<sup>th</sup> interval (21, 33), i.e.,  $L(n - 4)$ .

In rows 6 – 13, we see that the numbers  $f(n - 1)$  and  $f(n - 2)$ , i.e. 89 and 55, are both used, but we can also see that the residues for these rows are exactly the same as the residues for rows 14 – 21. The above formula can be seen to match with the  $Z$ -expressions in rows 6 – 13.

	1	1	2	3	5	8	13	21	34	55	89	144	233	377	k
1	1	0	1	1	1	0	1	1	1	0	1	0	0	0	168
2	0	1	1	1	1	0	1	1	1	0	1	0	0	0	168
3	1	0	1	0	0	1	1	1	1	0	1	0	0	0	168
4	0	1	1	0	0	1	1	1	1	0	1	0	0	0	168
5	0	0	0	1	0	1	1	1	1	0	1	0	0	0	168
6	1	0	1	1	1	0	1	0	0	1	1	0	0	0	168
7	0	1	1	1	1	0	1	0	0	1	1	0	0	0	168
8	1	0	1	0	0	1	1	0	0	1	1	0	0	0	168
9	0	1	1	0	0	1	1	0	0	1	1	0	0	0	168
10	0	0	0	1	0	1	1	0	0	1	1	0	0	0	168
11	1	0	1	0	0	0	0	1	0	1	1	0	0	0	168
12	0	1	1	0	0	0	0	1	0	1	1	0	0	0	168
13	0	0	0	1	0	0	0	1	0	1	1	0	0	0	168
14	1	0	1	1	1	0	1	0	0	0	0	1	0	0	168
15	0	1	1	1	1	0	1	0	0	0	0	1	0	0	168
16	1	0	1	0	0	1	1	0	0	0	0	1	0	0	168
17	0	1	1	0	0	1	1	0	0	0	0	1	0	0	168
18	0	0	0	1	0	1	1	0	0	0	0	1	0	0	168
19	1	0	1	0	0	0	0	1	0	0	0	1	0	0	168
20	0	1	1	0	0	0	0	1	0	0	0	1	0	0	168
21	0	0	0	1	0	0	0	1	0	0	0	1	0	0	168

Figure 12: Binary table for 168

In rows 1 – 5, the Z-expressions all include  $f(n - 1), f(n - 3)$  and  $f(n - 4)$ , given that

$$f(n - 4) + f(n - 3) = f(n - 2)$$

and that

$$f(n - 2) + f(n - 1) = f(n)$$

We see that in rows 1 – 5, the subtraction of  $f(n - 1), f(n - 3)$  and  $f(n - 4)$  is equivalent to subtracting  $f(n)$  and that the residue is again  $L(n - 4)$  in each case.

We have shown above that, in the case of a particular example, every Z-expression of  $L(n)$  can be expressed as  $f(n)$  and a residue that is equal to  $L(n)$ .

From Equation 4, we know that each of the Z-expressions of  $L(n)$  that include  $f(n)$  do have  $L(n - 4)$  as the residue. If the Z-expression is of the form:

$$L(n) = \sum_{i=1}^{n-3} e_i f(i) + f(n - 2) + f(n - 1)$$

then we know that the residue is  $L(n - 4)$ . If, however, the  $f(n - 2)$  term is missing, we need to show that the Z-expression must be of the form:

$$L(n) = \sum_{i=1}^{n-4} e_i f(i) + f(n - 3) + f(n - 1)$$

and that  $e_{n-4} \neq 0$ . If  $e_{n-4} = 0$ , the residue cannot be greater than

$$\sum_{i=1}^{n-5} f(i) = f(n - 3) - 1$$

In that case,  $L(n) \leq f(n - 3) - 1 + f(n - 3) + f(n - 1) < f(n)$ , implying that

$$L(n) \text{ does not belong to } [f(n), f(n + 1) - 1]$$

which is false. Thus  $e_{n-4} = 1$  and the required outcome is proven.

### 5. Zeckendorf Pairs

In the context of prime numbers, one of the outstanding unproved mathematical conjectures is that there are infinitely many twin primes, such as 41 and 43 which are two primes that differ by 2. We can borrow this concept and relate it to our investigation of  $Z$ -values by making the definition:

$\{k, k + 1\}$  is called a Zeckendorf pair or  $Z$ -pair if  $Z(k) = Z(k + 1)$

We have seen that for Fibonacci numbers,

$$Z(f(2n)) = n + 1$$

$$Z(f(2n + 1)) = n + 1$$

Also, within the interval  $[f(2n), f(2n + 1) - 1]$  we have shown that the  $Z$ -values have symmetry. In particular, we know that  $Z(f(2n)) = Z(f(2n + 1) - 1)$ . Thus it is immediately clear that  $f(2n + 1) - 1$  and  $f(2n + 1)$  form a  $Z$ -pair, because their  $Z$ -values are both  $n + 1$ .

This shows that there are infinitely many  $Z$ -pairs, but the question remains as to how they are distributed. We looked at the data that spreadsheet generates for  $Z$ -pairs that do not include a Fibonacci numbers. The first few of these are shown in Figure 14, where the values of  $Z(k)$  are given for each pair. In Column 4, we have included size of the interval between the values of  $k$ , which is consistently either 13 or 21.

Another feature of the table is that the  $Z$ -values for  $k$ ,  $Z(k)$ , are all multiples of three. A possible reason for this is that binary tables for each value appear to end in 3 for  $k$  and 4 for  $k + 1$ . The tables for 3 and 4 are shown in Figure 13 below and we find that those patterns are repeated whenever two consecutive numbers form a  $Z$ -pair.

		1	1	2	3	k
1		1	0	1	0	3
2		0	1	1	0	3
3		0	0	0	1	3
		1	1	2	3	k
1		1	1	1	0	4
2		1	0	0	1	4
3		0	1	0	1	4

Figure 13: Binary tables for 3 and 4

$k$	$k + 1$	$Z(k)$	$k_n - k_{n-1}$
3	4	3	
16	17	6	13
37	38	9	21
50	51	9	13
71	72	12	21
92	93	12	21
105	106	15	13
126	127	15	21
139	140	12	13
160	161	18	21
181	182	18	21
194	195	18	13
215	216	18	21
236	237	15	21
249	250	21	13
270	271	24	21
283	284	21	13
304	305	24	21
325	326	21	21
338	339	24	13
359	360	21	21
372	373	15	13
393	394	24	21
414	415	27	21
427	428	27	13
448	449	30	21
469	470	24	21
482	483	30	13
503	504	30	21

Figure 14. Table of Z-pairs up to  $k = 503$

For the values of  $k$  given in Figure 14, the binary tables for  $k$  and  $k + 1$  were explored as were the binary tables for  $k - 3$ . The tables suggested that  $k$  can be expressed as

$$k = 3 + \left( \sum_{i=5}^n e_i f(i) \right) \text{ where } e_i = 0 \text{ or } 1 \text{ for all } i.$$

$$k + 1 = 4 + \left( \sum_{i=5}^n e_i f(i) \right) \text{ where } e_i = 0 \text{ or } 1 \text{ for all } i.$$

That is, the numbers that form Z-pairs have a Zeckendorf representation that starts with the fifth Fibonacci number, 5, or greater and each representation begins with one of the three representations of 3 or 4.

Using the binary table, but this time starting at  $f(5)$ , we searched for numbers,  $p$ , that could be expressed as

$$p = \left( \sum_{i=5}^n e_i f(i) \right) \text{ where } e_i = 0 \text{ or } 1 \text{ for all } i.$$

For  $p = 47$  and  $p = 68$  we find that  $k = p + 3$  and  $k + 1 = p + 4$  are indeed a Z-pair because  $Z(k) = Z(k + 1) = 9$ . However, there are other values of  $p$  for which  $Z(p + 3) \neq Z(p + 4)$ . For example, the binary tables for  $p = 29, 42$  and  $55$  are shown in Figure 15.

Note that none of the above include a 5, a feature that distinguishes them from tables for which  $\{p + 3, p + 4\}$  is a Z-pair. For example, the binary tables for 47 and 68 are shown in Figure 16.

Let us now look at the full binary tables for the numbers  $p + 3$  and  $p + 4$  when  $p = 42$ . Figures 17 shows how rows A and B of the representation of 42 generate rows 1 – 3 and 4 – 6 of each of tables for 45 and 46. But when  $f(6)$  is changed to  $f(4) + f(5)$  there are two ways in which the remaining 3 can be made while there is only 1 way in which the remaining 4 can be made. Rows 7 and 8 of the table for 45 begin with  $f(1) + f(3) = 3$  and  $f(2) + f(3) = 3$ , while there is only row 7 of the table for 46 which begins with  $f(1) + f(2) + f(3) = 4$ . The same is true for rows 9 and 10 of 45 and row 8 of 46. As a result,  $Z(45) \neq Z(46)$  and  $\{45, 46\}$  is not a Z-pair.

	5	8	13	21	34	55	89	144	233	377	610	987	1597	2584	k
	0	1	0	1	0	0	0	0	0	0	0	0	0	0	29
	0	1	1	1	0	0	0	0	0	0	0	0	0	0	42
	0	1	0	0	1	0	0	0	0	0	0	0	0	0	42
	0	1	1	0	1	0	0	0	0	0	0	0	0	0	55
	0	0	0	1	1	0	0	0	0	0	0	0	0	0	55
	0	0	0	0	0	1	0	0	0	0	0	0	0	0	55

Figure 15: Representations of  $p$  as the sum of Fibonacci numbers from  $f(5)$  onwards

	5	8	13	21	34	55	89	144	233	377	610	987	1597	2584	k
	1	1	1	1	0	0	0	0	0	0	0	0	0	0	47
	1	1	0	0	1	0	0	0	0	0	0	0	0	0	47
	0	0	1	0	1	0	0	0	0	0	0	0	0	0	47
	1	1	0	1	1	0	0	0	0	0	0	0	0	0	68
	0	0	1	1	1	0	0	0	0	0	0	0	0	0	68
	1	1	0	0	0	1	0	0	0	0	0	0	0	0	68
	0	0	1	0	0	1	0	0	0	0	0	0	0	0	68

Figure 16: Representations of  $p$  as the sum of Fibonacci numbers from  $f(5)$  onwards for which  $\{p + 3, p + 4\}$  is a Z-pair.

					5	8	13	21	34	55	89	144	233	377	k
				A	0	1	1	1	0	0	0	0	0	0	42
				B	0	1	0	0	1	0	0	0	0	0	42

	1	1	2	3	5	8	13	21	34	55	89	144	233	377	k
1	1	0	1	0	0	1	1	1	0	0	0	0	0	0	45
2	0	1	1	0	0	1	1	1	0	0	0	0	0	0	45
3	0	0	0	1	0	1	1	1	0	0	0	0	0	0	45
4	1	0	1	0	0	1	0	0	1	0	0	0	0	0	45
5	0	1	1	0	0	1	0	0	1	0	0	0	0	0	45
6	0	0	0	1	0	1	0	0	1	0	0	0	0	0	45
7	1	0	1	1	1	0	1	1	0	0	0	0	0	0	45
8	0	1	1	1	1	0	1	1	0	0	0	0	0	0	45
9	1	0	1	1	1	0	0	0	1	0	0	0	0	0	45
10	0	1	1	1	1	0	0	0	1	0	0	0	0	0	45

	1	1	2	3	5	8	13	21	34	55	89	144	233	377	k
1	1	1	1	0	0	1	1	1	0	0	0	0	0	0	46
2	1	0	0	1	0	1	1	1	0	0	0	0	0	0	46
3	0	1	0	1	0	1	1	1	0	0	0	0	0	0	46
4	1	1	1	0	0	1	0	0	1	0	0	0	0	0	46
5	1	0	0	1	0	1	0	0	1	0	0	0	0	0	46
6	0	1	0	1	0	1	0	0	1	0	0	0	0	0	46
7	1	1	1	1	1	0	1	1	0	0	0	0	0	0	46
8	1	1	1	1	1	0	0	0	1	0	0	0	0	0	46

Figure 17: Representations of  $p + 3$  and  $p + 4$  as the sum of Fibonacci numbers from  $f(1)$  onwards for which  $\{p + 3, p + 4\}$  is not a Z-pair.

					5	8	13	21	34	55	89	144	233	377	k
				A	1	1	1	1	0	0	0	0	0	0	47
				B	1	1	0	0	1	0	0	0	0	0	47
				C	0	0	1	0	1	0	0	0	0	0	47

	1	1	2	3	5	8	13	21	34	55	89	144	233	377	k
1	1	0	1	0	1	1	1	1	0	0	0	0	0	0	50
2	0	1	1	0	1	1	1	1	0	0	0	0	0	0	50
3	0	0	0	1	1	1	1	1	0	0	0	0	0	0	50
4	1	0	1	0	1	1	0	0	1	0	0	0	0	0	50
5	0	1	1	0	1	1	0	0	1	0	0	0	0	0	50
6	0	0	0	1	1	1	0	0	1	0	0	0	0	0	50
7	1	0	1	0	0	0	1	0	1	0	0	0	0	0	50
8	0	1	1	0	0	0	1	0	1	0	0	0	0	0	50
9	0	0	0	1	0	0	1	0	1	0	0	0	0	0	50

	1	1	2	3	5	8	13	21	34	55	89	144	233	377	k
1	1	1	1	0	1	1	1	1	0	0	0	0	0	0	51
2	1	0	0	1	1	1	1	1	0	0	0	0	0	0	51
3	0	1	0	1	1	1	1	1	0	0	0	0	0	0	51
4	1	1	1	0	1	1	0	0	1	0	0	0	0	0	51
5	1	0	0	1	1	1	0	0	1	0	0	0	0	0	51
6	0	1	0	1	1	1	0	0	1	0	0	0	0	0	51
7	1	1	1	0	0	0	1	0	1	0	0	0	0	0	51
8	1	0	0	1	0	0	1	0	1	0	0	0	0	0	51
9	0	1	0	1	0	0	1	0	1	0	0	0	0	0	51

Figure 18: Representations of  $p + 3$  and  $p + 4$  as the sum of Fibonacci numbers from  $f(1)$  onwards for which  $\{p + 3, p + 4\}$  is a Z-pair.



This process will happen for every situation in which none of representations of  $p$  as

$$p = \left( \sum_{i=5}^n e_i f(i) \right) \text{ where } e_i = 0 \text{ or } 1 \text{ for all } i.$$

contain  $f(5)$ .

The situation for  $p = 47$  is different (see Figure 18). Here, either the first two numbers in the representation of  $p$  are  $f(5) + f(6)$  or the first non-zero number is  $f(7)$ .

In this case, rows A, B and C of the representation of 47 each generate three rows of 50 and 51 and as a result,  $\{50, 51\}$  is a Z-pair.

The examples given above can be generalised to the following conjecture about the existence of Z-pairs.

### 5.1. Conjecture

Let  $p$  be a number that can be expressed as

$$p = 5 + 8 + \left( \sum_{i=7}^n e_i f(i) \right) \text{ where } e_i = 0 \text{ or } 1 \text{ for all } i$$

then  $Z(p + 3) = Z(p + 4)$  and the pair  $\{p + 3, p + 4\}$  is a Z-pair. If  $p$  cannot be expressed in that format, then  $\{p + 3, p + 4\}$  is not a Z-pair.

## 6. Conclusion

The aim of this investigation was not to discover new features of the Fibonacci numbers. The fact that we chose what proved to be a very fertile source for mathematical exploration can only be described as a serendipity. Our hope was that we would be able to show that there are certain features of spreadsheet that provide powerful tools for the investigation of a mathematical topic. Of these, the two most important would be:

- Conditional Formatting
- Charting the Data

But there are other features that can be of great value. For example, the COUNTIF function can reveal trends in a way that would be most difficult to see otherwise as does the ability to filter a data set enable you to single out certain data for inspection. We also introduced the concept of a binary table as a mechanism for making sure that every possible outcome was catered for.

So often, a mathematical investigation requires that data be generated from which generalisations can be formed, or limits such as maxima and minima can be found. In this case it has been these types of activity that the spreadsheet has made possible and accessible.

## 7. References

1. Abramovich, S and Leonov, GA. (2009) "Spreadsheets and the discovery of new knowledge," *Spreadsheets in Education (eJSiE)*: Vol. 3: Iss. 2, Article 1. Available at: <http://epublications.bond.edu.au/ejsie/vol3/iss2/1>
2. Baker, JE and Sugden, SJ (2014) "Fun with Fibonacci" (*eJSiE*) Vol. 7; Iss 2, Article 1. Available at: <http://epublications.bond.edu.au/ejsie/vol7/iss2/1>.
3. Baker, JE (2007) "Excel and the Goldbach Comet," *Spreadsheets in Education (eJSiE)*: Vol. 2: Iss. 2, Article 2.
4. Chandra, P and Weisstein, EW. "Fibonacci Number." From *MathWorld--A Wolfram Web Resource*, retrieved 15 Feb 2014 from <http://mathworld.wolfram.com/FibonacciNumber.html>.
5. Klarner, D. (1968) "Partitions of N into Distinct Fibonacci Numbers" *Fibonacci Quarterly* Volume 6, Number 4
6. Klarner, D (1966) "Representation of N as a Sum of Distinct Elements from Special Sequences" *Fibonacci Quarterly* Volume 4, Number 4.
7. Zeckendorf, E. (1972). "Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas". *Bull. Soc. R. Sci. Liège* (in French) 41: 179–182.
8. Zeckendorf's Theorem [http://en.wikipedia.org/wiki/Zeckendorf's\\_theorem](http://en.wikipedia.org/wiki/Zeckendorf's_theorem), retrieved 15 Feb 2014.