

8-1-2015

Pythagorean Theorem with Hippocrates' Lunes

Gunhan Caglayan

New Jersey City University, gcaglayan@njcu.edu

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Recommended Citation

Caglayan, Gunhan (2015) Pythagorean Theorem with Hippocrates' Lunes, *Spreadsheets in Education (eJSiE)*: Vol. 8: Iss. 2, Article 5. Available at: <http://epublications.bond.edu.au/ejsie/vol8/iss2/5>

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Pythagorean Theorem with Hippocrates' Lunes

Abstract

Is it possible to place a lune on the hypotenuse of a right triangle whose area is equal to the sum of the areas of the other two lunes placed on the legs of the triangle? In this article, I use dynamic geometry software and spreadsheets in an attempt to answer this question along with the conditions satisfying the existence of such a lune. This **in the classroom** article also offers a method of investigating a trigonometric equation involving two variables using spreadsheets and dynamic geometry snapshots that are presented in a manner that complements the analytic and the visual approaches. I conclude by reinforcing the idea that the Pythagorean Theorem is indeed a relationship of areas, with or without the restriction that the lunes placed on the sides of a right triangle be similar.

Keywords

Hippocrates of Chios, quadrature, lunes, dynamic geometry software, Pythagorean Theorem, spreadsheets

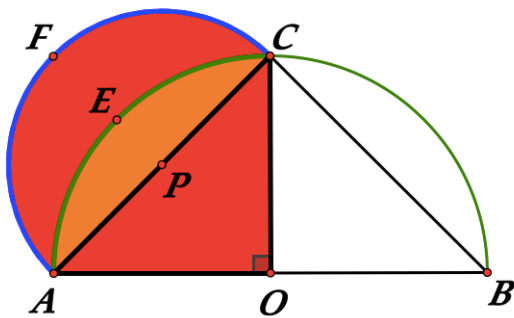
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Hippocrates' Lunes and the Pythagorean Theorem: A Spreadsheet Approach for Solving the Lune Condition

Hippocrates of Chios (c. 440 B.C.) successfully squared certain lunes, plane figures bounded by two circular arcs. Books III and IV, which details Euclid's work on the properties of circles, greatly influenced the development of Greek science and astronomy; "they became part of the Greek mathematician's toolbox for solving other problems... Hippocrates used results on circles in his quadrature of lunes" (Katz, 2009, p.66). "One of the earliest Greek mathematicians to attempt to treat the problem of the "quadrature of the circle" in pure geometric form, with the specific restriction that only compass and straightedge should be used, was Hippocrates of Chios" (NCTM, 1989, p.150). Hippocrates showed that certain lunes could be "squared," that is, the areas of such lunes would be equivalent to the areas of certain polygons (e.g., triangles, rectangles, etc.) (Midonick, 1965; Van Der Waerden, 1963). His proofs relied on a fundamental principle that intertwined similarity and area conservation: "the areas of circles are to one another as the squares on their diameters, a fact evidently known to the Babylonian scribes" (Katz, 2009, p.41). For example, if AOC is a quarter circular sector and another semicircle is constructed in such a way that the diameter of the semicircle coincides with chord AC , then it can be shown that triangle AOC and lune $AFCE$ (in red) bounded by the semicircle and the quarter circular sector have equal areas (Fig.1)



$$\begin{aligned} \text{Area } \widehat{CFA} &= 49.70 \text{ cm}^2 \\ \text{Area } \widehat{CEA} &= 18.06 \text{ cm}^2 \\ (\text{Area } \widehat{CFA}) - (\text{Area } \widehat{CEA}) &= 31.64 \text{ cm}^2 \\ \text{Area } \triangle OCA &= 31.64 \text{ cm}^2 \end{aligned}$$

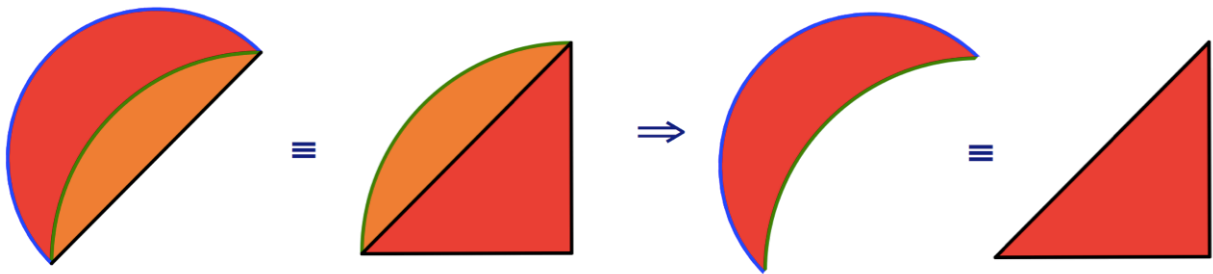


Figure 1: Lune formed on a quadrant of a circle

In what follows, I investigate the lunes formed on the three sides of a right triangle.

I. Lunes Formed on the Three Sides of a Right Triangle

Inscribing a right triangle ABC in a semicircle ACB , followed by the construction of lunes $AFCE$ and $BHCG$ on both legs (Fig.2), it can be shown that the conjectures (i) *the area of lune $AFCE$ is equal to the area of triangle AOC* ; (ii) *the area of lune $BHCG$ is equal to the area of triangle BOC* **are false**. However, calculations on Geometer's Sketchpad show that the **sum** of the areas of the two lunes is equal to two times the area of AOC (Fig.2). It is also worth noting that triangles AOC and BOC are both isosceles triangles (with legs as radii of the semicircle ACB) with the same area because they have the same length of base and height.

$$\text{Area } \widehat{CFA} = 36.38 \text{ cm}^2$$

$$\text{Area } \widehat{CEA} = 7.30 \text{ cm}^2$$

$$\text{Area } \triangle AOC = 46.79 \text{ cm}^2$$

$$(\text{Area } \widehat{CFA}) - (\text{Area } \widehat{CEA}) = 29.07 \text{ cm}^2$$

$$\text{Area } \widehat{BHC} = 148.50 \text{ cm}^2$$

$$\text{Area } \widehat{BGC} = 83.99 \text{ cm}^2$$

$$\text{Area } \triangle COB = 46.79 \text{ cm}^2$$

$$(\text{Area } \widehat{BHC}) - (\text{Area } \widehat{BGC}) = 64.50 \text{ cm}^2$$

$$(\text{Area } \triangle AOC) + (\text{Area } \triangle COB) = 93.58 \text{ cm}^2$$

$$((\text{Area } \widehat{CFA}) - (\text{Area } \widehat{CEA})) + ((\text{Area } \widehat{BHC}) - (\text{Area } \widehat{BGC})) = 93.58 \text{ cm}^2$$

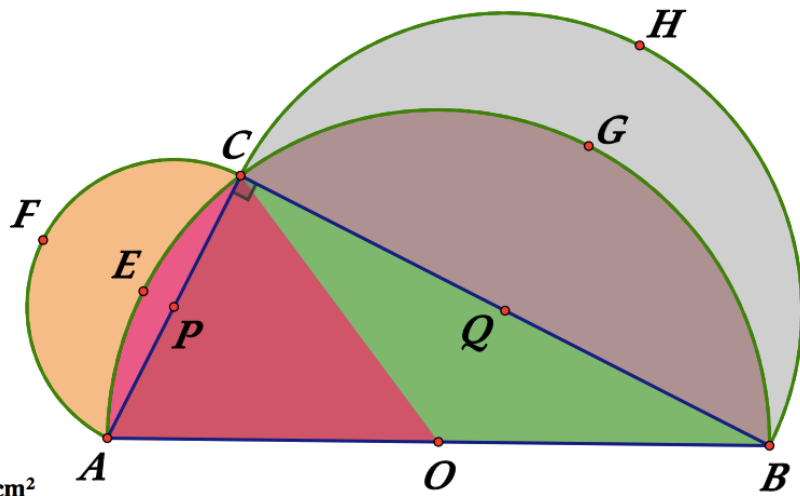


Figure 2: Lunes formed on the legs of a right triangle

Proof: Let $AB = c, BC = a, AC = b$. (i) The area of semicircle AFC is equal to $\pi \cdot \left(\frac{b}{2}\right)^2 = \frac{\pi b^2}{4}$; (ii) the area of semicircle BHC is equal to $\frac{\pi a^2}{4}$; and (iii) the area of right triangle ABC is equal to $\frac{ab}{2}$. The sum of these three areas must be equal to the area of the big semicircle ABC (which is $\frac{\pi c^2}{4}$) plus the sum of the areas of lunes $AFCE$ and $BHCG$. It can then be deduced that the sum of the areas of lunes $AFCE$ and $BHCG$ is equal to $\frac{\pi b^2}{4} + \frac{\pi a^2}{4} + \frac{ab}{2} - \frac{\pi c^2}{4} = \frac{\pi}{4}(b^2 + a^2 - c^2) + \frac{ab}{2}$, which implies that the **sum** of the areas of the two lunes is equal to the **sum** of the areas of triangles AOC and BOC , which is equal to the area of the original right triangle ABC (Fig.3).

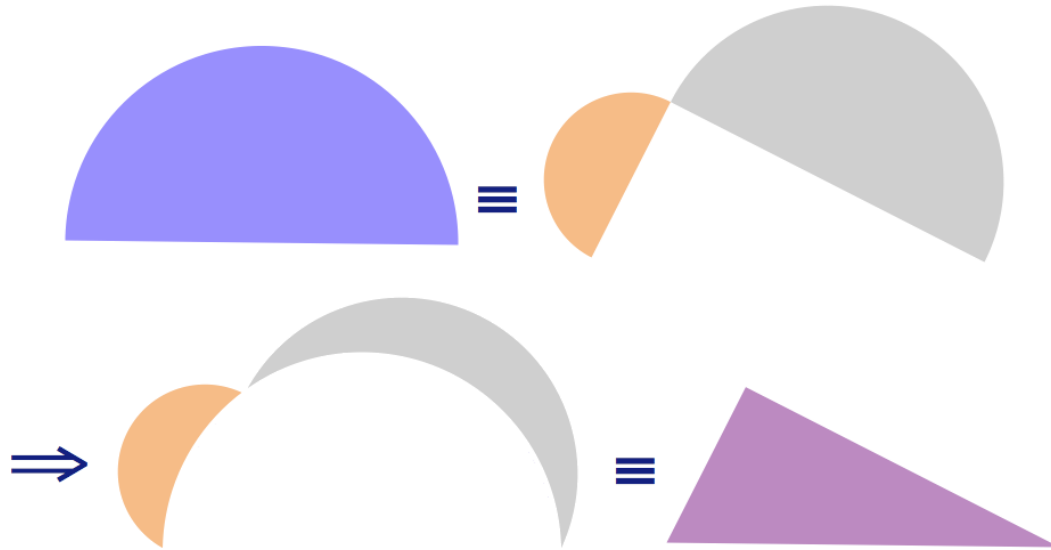


Figure 3: Area conservation principle

Problem Statement: Is it possible to place a lune (on the hypotenuse of triangle ABC) whose area is equal to the sum of the areas of the other two lunes (on the legs)? This problem has not been investigated before. Dynamic geometry software may be used in an attempt to answer this question along with the conditions satisfying the existence of such a lune.

Similarity Arguments: First of all, similarity arguments do not apply because the circular segments AEC and BGC are not similar: They are similar only when the right triangle ABC is isosceles, in which case the circular arc sectors containing these segments are the same fraction (i.e., one fourth) of their circles. In general, as evident by Fig.2, the circular segments AEC and BGC are not similar.

Area Conservation Principle: The area of circular segment AEC is equal to $\frac{1}{2}\pi\left(\frac{b}{2}\right)^2 - \frac{ab}{4}$; the area of circular segment BGC is equal to $\frac{1}{2}\pi\left(\frac{a}{2}\right)^2 - \frac{ab}{4}$. The sum of the areas of these circular segments is therefore equal to $\frac{1}{2}\pi\left(\frac{a^2+b^2}{4}\right) - \frac{ab}{2}$, which can be simplified as $\frac{1}{2}\pi\left(\frac{c^2}{4}\right) - \frac{ab}{2}$ by using the Pythagorean Theorem $a^2 + b^2 = c^2$ for triangle ABC (Fig.2). Namely the desired circular segment to be placed on side AB (the hypotenuse) should be of area $\frac{1}{2}\pi\left(\frac{c}{2}\right)^2 - \frac{ab}{2}$, which is equivalent to the area of semicircle ABC of diameter c minus the area of right triangle ABC .

Constructing a Circular Segment on the Hypotenuse: How to determine the center M of a circular segment with known area and chord length? How far should that center be from either endpoint of chord AB ? Figure 4 illustrates the circular segment ALB on the hypotenuse AB of triangle ABC with center M that is located on the perpendicular bisector of the hypotenuse AB . θ is the radian measure of the central angle $\angle AMB$ of the

corresponding arc sector. $R = AM = LM = BM$ is the radius of the same arc sector. By proportional reasoning, the area of the arc sector is $\frac{\theta R^2}{2}$, where R is the radius of the arc sector. Using the Heron's formula $\sqrt{s(s-R)(s-R)(s-c)}$ with semiperimeter $s = \frac{1}{2}(R + R + c)$, the area of the isosceles triangle AMB contained in the arc sector is given by $\frac{c}{2}\sqrt{R^2 - \frac{c^2}{4}}$. The area of the sought circular segment ALB is therefore $\frac{\theta R^2}{2} - \frac{c}{2}\sqrt{R^2 - \frac{c^2}{4}}$. Expressing θ in terms of c and R , the last expression becomes $R^2 \sin^{-1} \frac{c}{2R} - \frac{c}{2}\sqrt{R^2 - \frac{c^2}{4}}$. Setting this equal to $\frac{1}{2}\pi \left(\frac{c}{2}\right)^2 - \frac{ab}{2}$, the equation $R^2 \sin^{-1} \frac{c}{2R} - \frac{c}{2}\sqrt{R^2 - \frac{c^2}{4}} = \frac{1}{2}\pi \left(\frac{c}{2}\right)^2 - \frac{ab}{2}$ is obtained. This equation implies that R can be obtained once the lengths of sides a, b, c are known.

$$m\angle AMB = \theta, m\angle AMO = m\angle BMO = \alpha$$

$$\alpha = \frac{\theta}{2} = \arcsin\left(\frac{c}{2R}\right)$$

$$s = \frac{1}{2}(R + R + c) = R + \frac{c}{2}$$

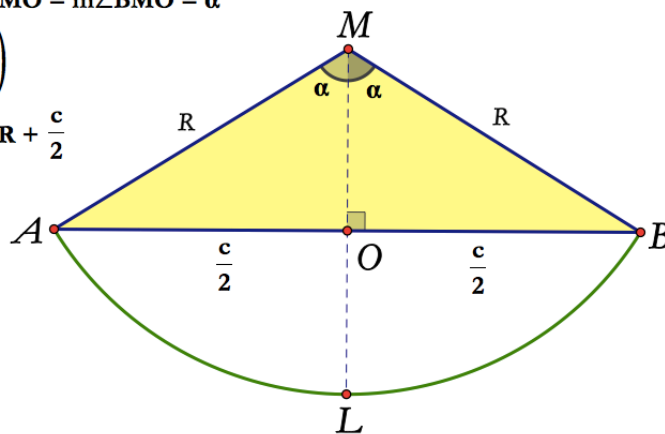


Figure 4: Attempts to locate the center of the circular segment with hypotenuse AB as chord

Illustrating the Equation: The last equation can be illustrated in a dynamic geometry software, such as Geometer's Sketchpad (Fig.5). The calculations also verify the above derived equation $R^2 \sin^{-1} \frac{c}{2R} - \frac{c}{2}\sqrt{R^2 - \frac{c^2}{4}} = \frac{1}{2}\pi \left(\frac{c}{2}\right)^2 - \frac{ab}{2}$. It is also worth noting that not only is it possible to place a **circular segment** AKB on the hypotenuse AB with the sought conditions, it is possible to place a **lune** $AKBL$ with the same sought conditions as well.

$$\begin{aligned} \text{Area } \widehat{CFA} &= 118.96 \text{ cm}^2 & \text{Area } \widehat{CEA} &= 68.08 \text{ cm}^2 \\ (\text{Area } \widehat{CFA}) - (\text{Area } \widehat{CEA}) &= 50.89 \text{ cm}^2 \\ \text{Area } \widehat{BHC} &= 27.80 \text{ cm}^2 & \text{Area } \widehat{BGC} &= 5.46 \text{ cm}^2 \\ (\text{Area } \widehat{BHC}) - (\text{Area } \widehat{BGC}) &= 22.34 \text{ cm}^2 \\ \text{Area } \widehat{BKA} &= 146.77 \text{ cm}^2 \\ R^2 \cdot \sin^{-1}\left(\frac{c}{2 \cdot R}\right) - \left(\frac{c}{2}\right) \cdot \sqrt{R^2 - \frac{c^2}{4}} &= 73.54 & a &= 8.41 \text{ cm} \\ & & c &= 19.33 \text{ cm} \\ & & b &= 17.41 \text{ cm} \\ & & R &= 11.37 \text{ cm} \\ & & \theta &= 2.03 \text{ radians} \\ \left(\frac{1}{2}\right) \cdot \pi \cdot \left(\frac{c}{2}\right)^2 - \frac{a \cdot b}{2} &= 73.54 \text{ cm}^2 \\ (\text{Area } \widehat{CEA}) + (\text{Area } \widehat{BGC}) &= 73.54 \text{ cm}^2 \\ \text{Area } \widehat{ALB} &= 73.54 \text{ cm}^2 \\ (\text{Area } \widehat{BKA}) - (\text{Area } \widehat{ALB}) &= 73.23 \text{ cm}^2 \\ ((\text{Area } \widehat{CFA}) - (\text{Area } \widehat{CEA})) + ((\text{Area } \widehat{BHC}) - (\text{Area } \widehat{BGC})) &= 73.23 \text{ cm}^2 \\ \text{Area } \triangle ACB &= 73.23 \text{ cm}^2 \end{aligned}$$

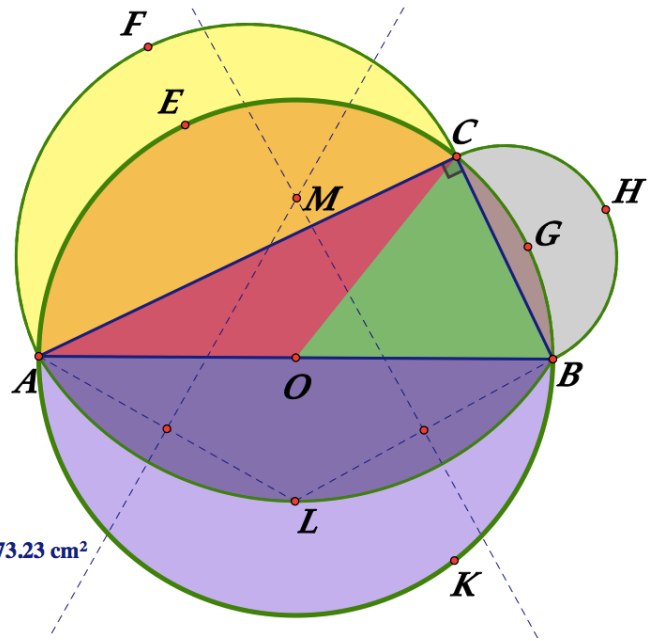


Figure 5: Verifying the equation

The Lune Condition: Using $\alpha = \theta/2 = \sin^{-1} \frac{c}{2R}$, the lune condition may be rewritten as $\alpha R^2 - \frac{c}{2} R \cos \alpha = \frac{1}{2} \pi \left(\frac{c}{2}\right)^2 - \frac{ab}{2}$. Multiplying both sides by $\frac{4}{c^2}$ gives $\alpha \left(\frac{2R}{c}\right)^2 - \frac{2R}{c} \cos \alpha = \frac{\pi}{2} - 2 \frac{a \cdot b}{c \cdot c} \Rightarrow \alpha \csc^2 \alpha - \cot \alpha = \frac{\pi}{2} - 2 \frac{a \cdot b}{c \cdot c} \Rightarrow \alpha \csc^2 \alpha - \cot \alpha - \frac{\pi}{2} + 2 \frac{a \cdot b}{c \cdot c} = 0$. Defining $\beta = m\angle BAC$, the last equation can be written in terms of α and β only: $\alpha \csc^2 \alpha - \cot \alpha - \frac{\pi}{2} + 2 \sin \beta \cos \beta = 0 \Rightarrow \alpha \csc^2 \alpha - \cot \alpha - \frac{\pi}{2} + \sin 2\beta = 0$, which is the trigonometric version of the lune condition.

II. Solving the Trigonometric Equation involving α and β

In this section, the conditions satisfying the existence of this lune, in particular, the trigonometric equation involving two variables will be explored using the spreadsheets feature of GeoGebra dynamic software.

Defining Quantities on Spreadsheets: *Column B* will be used to enter angle measures (in radians) for β , starting with $B1 = \frac{\pi}{180}$, $B2 = \frac{2\pi}{180}$, $B3 = \frac{3\pi}{180}$, up to B45. *Column C* will be used to determine the value of $\frac{\pi}{2} - \sin 2\beta$ for each β defined in *Column B*.

Graphics view will be used to graph the function $f(\alpha) = \alpha \csc^2 \alpha - \cot \alpha$ within the domain $\left[0, \frac{\pi}{2}\right]$ using the syntax `Function[x (csc(x))^2 - cot(x), 0, pi/2]`. The range of this function is identified from the graph as $\left[0, \frac{\pi}{2}\right]$. *Column D* will be used to define the horizontal lines via the syntax `D1 = Function[C1, 0, pi/2]` and dragging down to D45. These horizontal lines appear on the Graphics view as well (Figs.6-7).

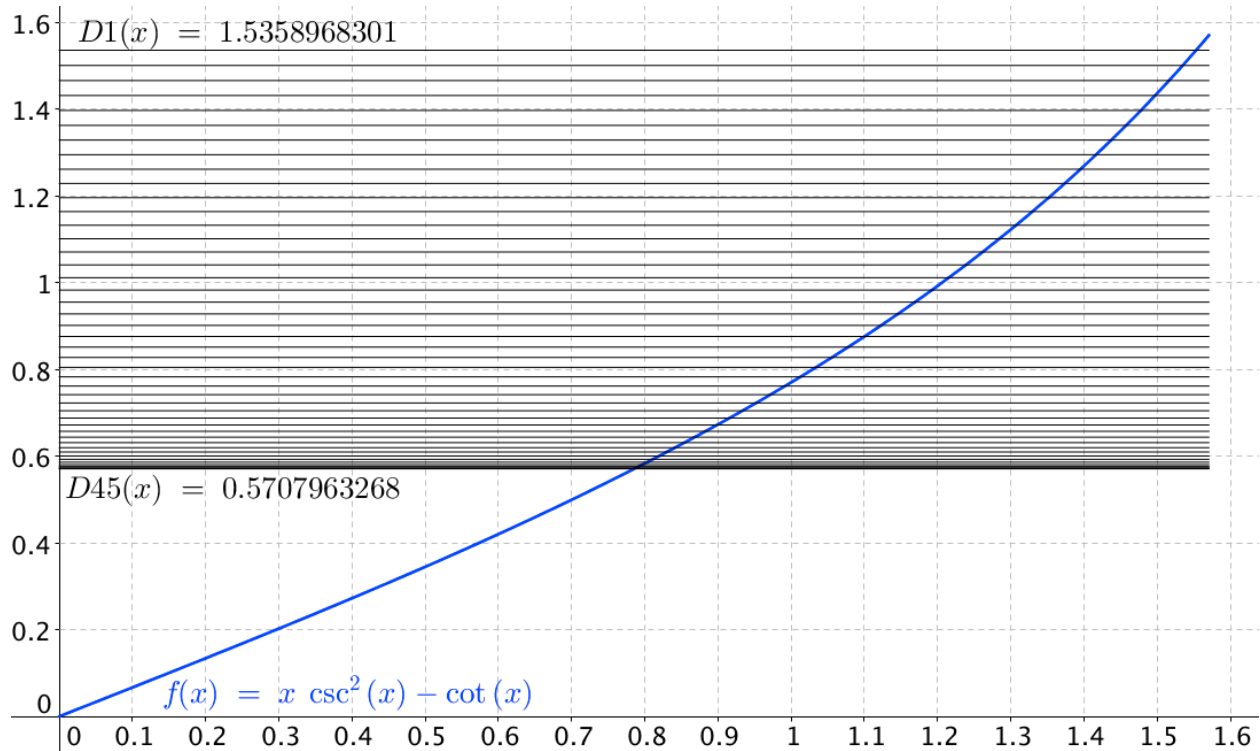


Figure 6: Horizontal lines from Column D

Column E will be used to define the abscissas (i.e., the values of angle α) of the intersections of the horizontal lines D1:D45 with the function f via the syntax **E1 = x(Intersect[f, D1,0,pi/2])** and dragging down to E45. That way, all α values (in radians) are obtained in E1:E45 making the equation $\alpha \csc^2 \alpha - \cot \alpha = \frac{\pi}{2} - \sin 2\beta$ true for each β defined in B1:B45. Column A will be used to enter the values of angle α (in degrees for comparison purposes) via the syntax **A1 = E1 (180/pi)** and dragging down to A45. Column F can be used to record any function of α and β , such as the sum $\alpha + \beta$ in degrees. Figure 7 illustrates all spreadsheet data on Columns A:F.

	A	B	C	D	E	F
1	88.986325686	0.0174532925	1.5358968301	1.5358968301	1.531043725	89.986325686
2	87.9454755908	0.034906585	1.5010398531	1.5010398531	1.5349381113	89.9454755908
3	86.8777800853	0.0523598776	1.4662678635	1.4662678635	1.5163033093	89.8777800853
4	85.7836844077	0.0698131701	1.4316232258	1.4316232258	1.4972077374	89.7836844077
5	84.6637621389	0.0872664626	1.3971481491	1.3971481491	1.4776614064	89.6637621389
6	83.5187131941	0.1047197551	1.362884636	1.362884636	1.4576765323	89.5187131941
7	82.3493913183	0.1221730476	1.3288744312	1.3288744312	1.4372680155	89.3493913183
8	81.1568119578	0.1396263402	1.295158971	1.295158971	1.4164535791	89.1568119578
9	79.9421440536	0.1570796327	1.2617793324	1.2617793324	1.3952536248	88.9421440536
10	78.7067511916	0.1745329252	1.2287761835	1.2287761835	1.3736919518	88.7067511916
11	77.452176863	0.1919862177	1.1961897334	1.1961897334	1.3517954991	88.452176863
12	76.1801648776	0.2094395102	1.1640596837	1.1640596837	1.3295947018	88.1801648776
13	74.892662495	0.2268928028	1.13242518	1.13242518	1.3071235461	87.892662495
14	73.5918250992	0.2443460953	1.101324764	1.101324764	1.2844196505	87.5918250992
15	72.2800243519	0.2617993878	1.0707963268	1.0707963268	1.2615244084	87.2800243519
16	70.9598415141	0.2792526803	1.0408770626	1.0408770626	1.2384828711	86.9598415141
17	69.6340713223	0.2967059728	1.0116034233	1.0116034233	1.2153438161	86.6340713223
18	68.3057160876	0.3141592654	0.9830110745	0.9830110745	1.1921596437	86.3057160876
19	66.9779782297	0.3316125579	0.9551348515	0.9551348515	1.1689862464	85.9779782297
20	65.654249676	0.3490658504	0.9280087171	0.9280087171	1.1458828248	85.654249676
21	64.3381100629	0.3665191429	0.9016657204	0.9016657204	1.1229118551	85.3381100629
22	63.032498167	0.3839724354	0.8761379563	0.8761379563	1.1001377475	85.032498167
23	61.7435702071	0.401425728	0.8514565265	0.8514565265	1.0776285921	84.7435702071

	A	B	C	D	E	F
24	60.4730404337	0.4188790205	0.8276515013	0.8276515013	1.0554536643	84.4730404337
25	59.2257317668	0.436332313	0.8047518837	0.8047518837	1.0336840212	84.2257317668
26	58.0057805912	0.4537856055	0.7827855732	0.7827855732	1.0123918565	84.0057805912
27	56.8173542156	0.471238898	0.7617793324	0.7617793324	0.9916499033	83.8173542156
28	55.6646248552	0.4886921906	0.7417587542	0.7417587542	0.9715309806	83.6646248552
29	54.5517379386	0.5061454831	0.7227482306	0.7227482306	0.9521074397	83.5517379386
30	53.4827779691	0.5235987756	0.704770923	0.704770923	0.9334505687	83.4827779691
31	52.4617408052	0.5410520681	0.6878487339	0.6878487339	0.9156301084	83.4617408052
32	51.4925034887	0.5585053606	0.6720022805	0.6720022805	0.898713726	83.4925034887
33	50.5787924515	0.5759586532	0.6572508692	0.6572508692	0.882766646	83.5787924515
34	49.7241605519	0.5934119457	0.6436124722	0.6436124722	0.8678503194	83.7241605519
35	48.9319599473	0.6108652382	0.631103706	0.631103706	0.8540238105	83.9319599473
36	48.2053208384	0.6283185307	0.6197398105	0.6197398105	0.8413415656	84.2053208384
37	47.5471291266	0.6457718232	0.6095346309	0.6095346309	0.8298539531	84.5471291266
38	46.9600117533	0.6632251158	0.6005006005	0.6005006005	0.8196068219	84.9600117533
39	46.4463177774	0.6806784083	0.5926487261	0.5926487261	0.8106411706	85.4463177774
40	46.0081080927	0.6981317008	0.5859885738	0.5859885738	0.8029929688	86.0081080927
41	45.6471432655	0.7155849933	0.5805282581	0.5805282581	0.7966929441	86.6471432655
42	45.3648748987	0.7330382858	0.5762744314	0.5762744314	0.7917664317	87.3648748987
43	45.1624391609	0.7504915784	0.5732322765	0.5732322765	0.7882332616	88.1624391609
44	45.0406507115	0.7679448709	0.5714054998	0.5714054998	0.7861076522	89.0406507115
45	45	0.7853981634	0.5707963268	0.5707963268	0.7853981634	90

Figure 7: Spreadsheet data in Columns A:F

Analysis of the Spreadsheet Data: It is worth noting that when $\beta = 45^\circ$, $\alpha = 45^\circ$ too. This particular case is prone to several observations.

- (1) This means that if the original right triangle ABC is isosceles, then points C and M would coincide; namely that triangles ABC and AMB would coincide too.
- (2) All three circular segments forming on the sides of triangle ABC would be similar. In particular, the segments AEC and BGC forming on the legs would be congruent (Fig.8a).
- (3) All three lunes forming on the sides of triangle ABC would be similar as well. In particular, the lunes $AFCE$ and $BHCG$ forming on the legs of triangle ABC would be congruent (Fig.8b).

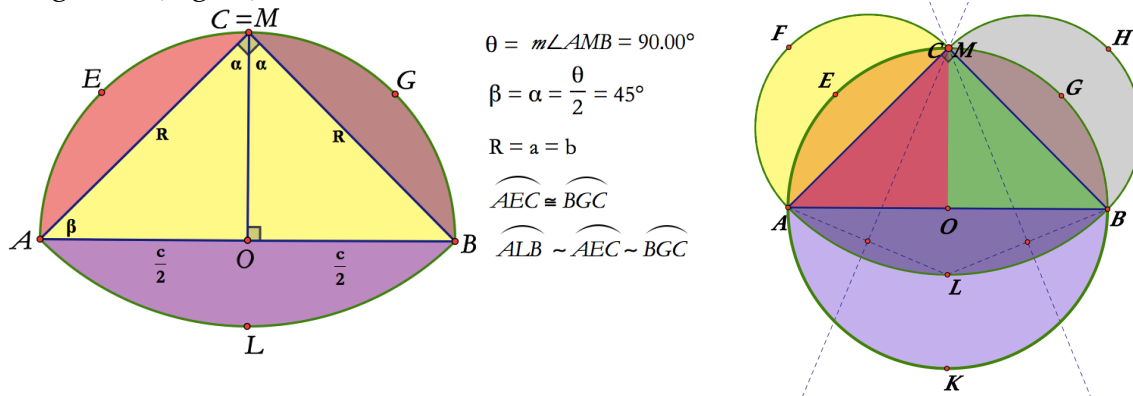


Figure 8: The particular case when ABC is a right isosceles triangle

Observations Regarding the Sum $\alpha + \beta$: For a given β , the maximum value of the sum $\alpha + \beta$ is 90° , which occurs for $\beta = 45^\circ$. The minimum value of the sum $\alpha + \beta$ occurs for $\beta \approx 31^\circ$. Is it possible to bound β so that the sum $\alpha + \beta$ is minimum? Redefining $B1 = 30.917 \left(\frac{\pi}{180}\right)$, $B2 = B1 + 0.00001 \left(\frac{\pi}{180}\right)$, etc., it is possible to regenerate the spreadsheet data with 2000 rows for $\beta \in [30.917^\circ, 30.919^\circ]$ to obtain the desired interval $\alpha + \beta \in [83.4615682475^\circ, 83.4615682477^\circ]$ for the sum (Fig.9).

	A	B	C	D	E	F
111	52.5434682479	0.5396226435	0.6891944887	0.6891944887	0.9170565213	83.4615682479
112	52.5434582478	0.539622818	0.689194324	0.689194324	0.9170563468	83.4615682478
113	52.5434482478	0.5396229925	0.6891941592	0.6891941592	0.9170561723	83.4615682478
114	52.5434382477	0.5396231671	0.6891939945	0.6891939945	0.9170559977	83.4615682477
115	52.5434282477	0.5396233416	0.6891938297	0.6891938297	0.9170558232	83.4615682477
116	52.5434182477	0.5396235161	0.689193665	0.689193665	0.9170556487	83.4615682477
117	52.5434082476	0.5396236907	0.6891935002	0.6891935002	0.9170554741	83.4615682476
118	52.5433982476	0.5396238652	0.6891933354	0.6891933354	0.9170552996	83.4615682476
119	52.5433882476	0.5396240397	0.6891931707	0.6891931707	0.9170551251	83.4615682476
120	52.5433782476	0.5396242143	0.6891930059	0.6891930059	0.9170549505	83.4615682476
121	52.5433682476	0.5396243888	0.6891928412	0.6891928412	0.917054776	83.4615682476
122	52.5433582476	0.5396245633	0.6891926764	0.6891926764	0.9170546015	83.4615682476
123	52.5433482476	0.5396247379	0.6891925117	0.6891925117	0.9170544269	83.4615682476
124	52.5433382476	0.5396249124	0.6891923469	0.6891923469	0.9170542524	83.4615682476
125	52.5433282476	0.5396250869	0.6891921821	0.6891921821	0.9170540779	83.4615682476
126	52.5433182477	0.5396252615	0.6891920174	0.6891920174	0.9170539033	83.4615682477
127	52.5433082477	0.539625436	0.6891918526	0.6891918526	0.9170537288	83.4615682477
128	52.5432982477	0.5396256105	0.6891916879	0.6891916879	0.9170535543	83.4615682477
129	52.5432882478	0.5396257851	0.6891915231	0.6891915231	0.9170533797	83.4615682478
130	52.5432782478	0.5396259596	0.6891913584	0.6891913584	0.9170532052	83.4615682478
131	52.5432682478	0.5396261341	0.6891911936	0.6891911936	0.9170530307	83.4615682478
132	52.5432582479	0.5396263087	0.6891910289	0.6891910289	0.9170528562	83.4615682479

Figure 9: Searching for a bound for the minimum value of the sum $\alpha + \beta$

Locus the Sum $+\beta$: Defining a slider for $\beta \in [0^\circ, 45^\circ]$, it is possible to obtain the locus of the sum $\alpha + \beta$ by introducing the locus point $P = (\beta, \alpha + \beta)$ where the values of α and β are obtained from the spreadsheets data (Fig.10). In Figure 10, the value of angle α (shown in radians) is obtained from the graph of f by the intersection of the horizontal line (obtained from Column D) and the graph of f . The locus curve indicates that the sum value is minimum for $\beta \approx 30.9^\circ$, in agreement with the spreadsheet data.

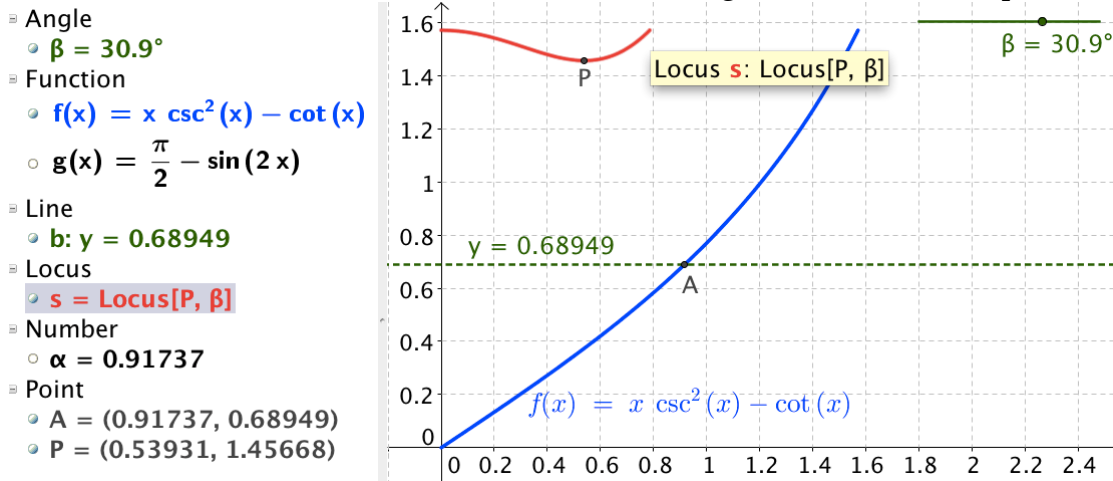


Figure 10: Graphing spreadsheet data and the locus of the sum

III. Conclusion

It therefore is possible to place a lune on the hypotenuse of a right triangle whose area is equal to the sum of the areas of the other two lunes placed on the legs of the triangle if and only if the equation $\alpha \csc^2 \alpha - \cot \alpha = \frac{\pi}{2} - \sin 2\beta$ holds (*the lune condition*). The

Pythagorean Theorem is indeed a relationship of areas, with or without the restriction that the lunes placed on the sides of a right triangle be similar.

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