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Spreadsheets: Laying a Foundation for Understanding Functions

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Spreadsheets: Laying a Foundation for Understanding Functions

Abstract
Linear, quadratic, and exponential functions, as well as polynomial functions, are the most basic mathematical expressions. Despite being among the most basic expressions in algebra, these functions are often used to approximate more complicated functions. The Common Core State Standards for Mathematics provide the framework for a discussion of the basic functions presented here. To that extent, the content of this paper consists of the use of spreadsheet technology in experimentation with linear transformations in the plane; experimentation with the way quadratic functions behave; constructing and comparing linear, quadratic, and exponential models of real-life data; and automation of the computation of real zeros of polynomials where such calculations require implementation of time consuming iterative procedures. An iterative procedure based on the Rational Zeros Theorem will be used to find exact values of rational zeros of a polynomial. A second iterative procedure based on the Bounds on Zeros theorem and the Intermediate Value Theorem will be used to approximate all real zeros of a polynomial.

Keywords
experimentation with functions, mathematical modelling, real zeros of polynomials

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Abstract

Linear, quadratic, and exponential functions, as well as polynomial functions, are the most basic mathematical expressions. Despite being among the most basic expressions in algebra, these functions are often used to approximate more complicated functions. The Common Core State Standards for Mathematics provide the framework for a discussion of the basic functions presented here. To that extent, the content of this paper consists of the use of spreadsheet technology in experimentation with linear transformations in the plane; experimentation with the way quadratic functions behave; constructing and comparing linear, quadratic, and exponential models of real-life data; and automation of the computation of real zeros of polynomials where such calculations require implementation of time consuming iterative procedures. An iterative procedure based on the Rational Zeros Theorem will be used to find exact values of rational zeros of a polynomial. A second iterative procedure based on the Bounds on Zeros theorem and the Intermediate Value Theorem will be used to approximate all real zeros of a polynomial.

Keywords: experimentation with functions, mathematical modelling, real zeros of polynomials

Introduction

Some eighty years ago, a University of Chicago mathematics professor by the name of Mayme I. Logsdon [6] wrote:

> *[e]arly in a course in algebra students are assigned the task of learning to substitute numbers for letters and perform indicated operations of arithmetic. It is amazing how inaccurate college students are in this elementary exercise*. (p. 50)

Logsdon's remark resonates with every algebra teacher even today. Students at all levels of schooling, from middle school to college, have difficulty working with algebraic variables. But this issue isn't exclusive to the learners of mathematics.

A study conducted by Clement, Lochhead, and Monk [3] found evidence that even freshman engineering students encounter difficulty when trying to translate information nested in word problems in order to recreate the context in symbolic form. Freshman engineering students were asked:

"*Write an equation using the variables C and S to represent the following statement: At Mindy's restaurant, for every four people who ordered cheesecake, there were five who ordered strudel*."

Many students gave the erroneous answer: *4C=5S.* Only 39% of the 497 students who took the test answered this question correctly as *5C=4S*. Overall, the fact that fewer than 50% of the students could solve the problems correctly indicated "the difficulty of translating into and out of algebraic notation" [3]. Moreover, the authors add that such "errors are also high for translations from pictures and data tables." Interestingly, more than 40% of the students failed the restaurant problem when it appeared once again on the final exam.

Clement et al. [3] concluded that student errors were caused because "They cannot translate reliably between algebra and other symbol systems, such as English, data tables, and pictures. We do not believe that this is a trivial problem." An in-depth study conducted by Abouchedid and Ramzi [1] corroborated those findings.

Figures 1 and 2 depict the use of the Excel spreadsheet in solving the restaurant problem for few specific cases. The data for the first few cases can be created easily. Cells A4 and B4 have the obvious entries which follow from the problem statement. Subsequently, the entries in cells A5:A6 and B5:B6 can easily be obtained by adding the appropriate numbers. Students can then be asked to fill in the column entitled "Strudel Column (formula)" with formula structures that would produce the results already placed in column B4:B6.

$$\frac{Strudel}{Cheesecake} = \frac{Strudel}{Cheesecake}$$

From cells A4 and B4 *or* the problem statement: $\frac{S}{C} = \frac{5}{4}$

$$5C = 4S$$

	A	B	C	D
1		**Mindy's**	**Restaurant**	
2			**Strudel Column**	**Strudel Column**
3	**Cheesecake**	**Strudel**	**(formula)**	**(alternative formula)**
4	4	5	=($B$4/$A$4)*A4	=1.25*A4
5	8	10	=($B$4/$A$4)*A5	=1.25*A5
6	12	15	=($B$4/$A$4)*A6	=1.25*A6
7	13	16.25	=($B$4/$A$4)*A7	=1.25*A7
8	14	17.5	=($B$4/$A$4)*A8	=1.25*A8
9	15	18.75	=($B$4/$A$4)*A9	=1.25*A9
10	16	20	=($B$4/$A$4)*A10	=1.25*A10

Figure 1: Two alternative solutions to Mindy's Restaurant problem in Excel.

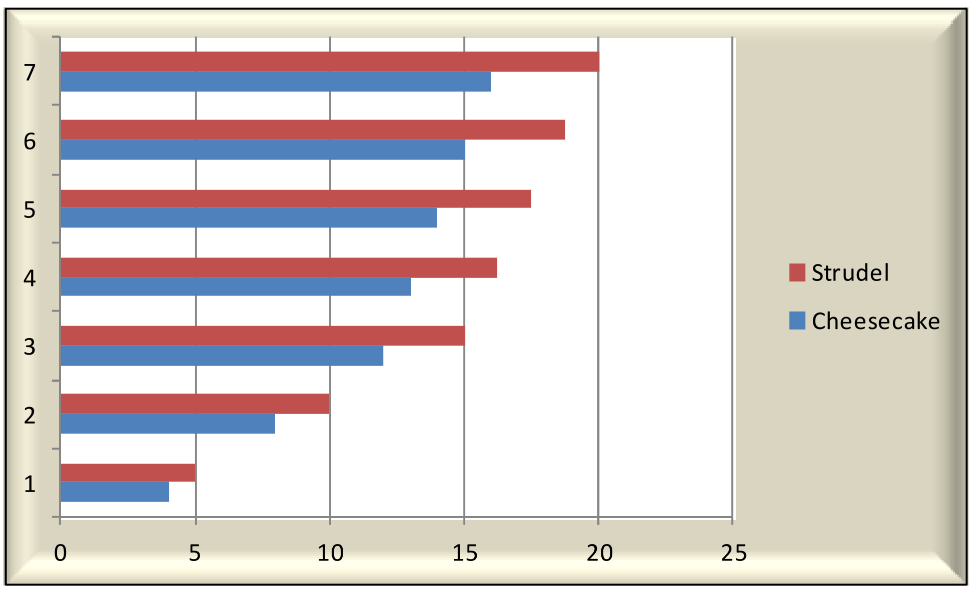


Figure 2: Excel Bar graph comparing strudel and cheesecake orders at Mindy's Restaurant.

Hence, a spreadsheet activity can be designed as intervention to provide practice in the very same areas of competency that the research found the students to be lacking: symbolic representation, data tables, and graphical representation.

Clement et al. [3] repeated the above study with another group of 150 calculus level science students while modifying the question and asking students to "Write an equation for the following statement:"

"There are six times as many students as professors at this university (use S for the number of students and P for the number of professors)."

It was reported that 37% of the students missed the problem. The same exact experiment was conducted with 47 non-science college students of whom 57% missed the problem. All of the erroneous answers were given as: $6S=P$.

A spreadsheet solution for this problem consisting of symbolic representation, a data table, and the associated linear graph are shown in Figures 3 and 4. It is clear that the 57% of the students who incorrectly answered $6S=P$ in the study would – with high likelihood – not make the same error when working with spreadsheets, because it simply would not make any sense at all for the university to employ, say, 300 professors for 50 students.

	A	B	C
16			**Students**
17	**Professors**	**Students**	**(formula)**
18	50	300	=(B18/A18)*A18
19	75	450	=(B19/A19)*A19
20	100	600	=(B20/A20)*A20
21	125	750	=(B21/A21)*A21
22	150	900	=(B22/A22)*A22
23	175	1050	=(B23/A23)*A23
24	200	1200	=(B24/A24)*A24

Figure 3: Excel Spreadsheet solution for the university problem.

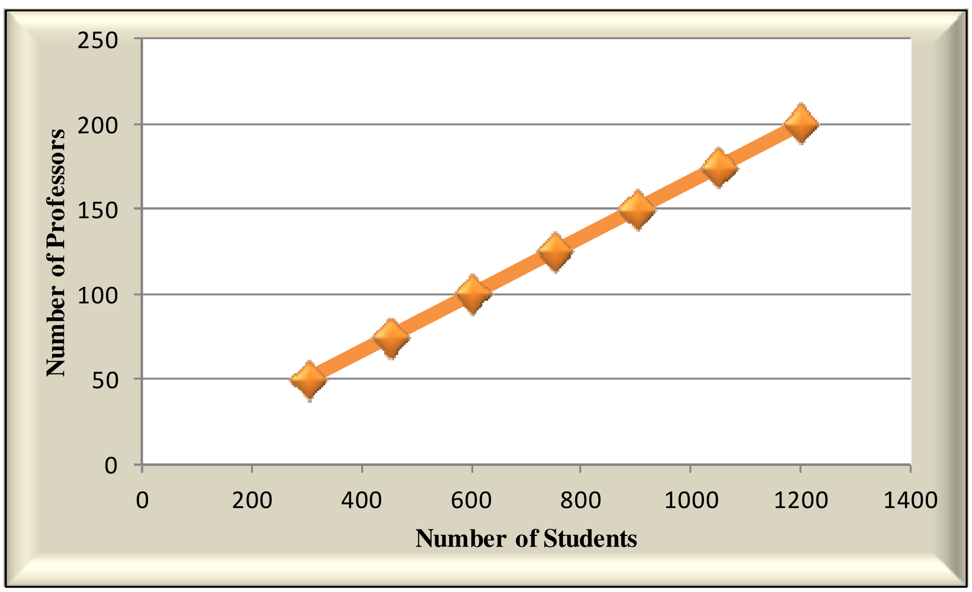


Figure 4: A linear graph of the number of Professors (as the dependent variable) vs. students (as the independent variable).

The Role that Spreadsheets Can Play to Help Teach and Learn About Mathematical Functions

Variable terms are indispensable to functions. Before the first spreadsheets like "Lotus 1-2-3" came around in the early 1980s, the most common ways of teaching students the meaning and use of variable symbols in algebra included hours of practice solving equations using pencil and paper, and exercises that built on students' generalization of some kind of numerical or geometrical pattern. On the other hand, science teachers often used real-world experiments like dropping a ball from different heights and measuring the height to which the ball bounced each time in order to demonstrate to their students the meaning and purpose of variables in a function and their connection to cause and effect phenomena in the real world.

Despite teachers' efforts, however, students continued to have difficulty bridging the gap between arithmetic problem solving and handling variable symbols in algebra. At the same time, the use of spreadsheet technology to aid in this transition was overlooked for the most part. Every college mathematics teacher in America today, for example, who has tried to use spreadsheets in his or her classroom knows about the large proportion of students who either have no prior experience with spreadsheet software or have worked with it so rarely that they are not proficient with it at all.

Kaput [5] discussed how computer programs like spreadsheets can be used to turn algebraic variables that seem static when used in a pencil and paper based setting that leads to a single solution into dynamic terms that can be used to experiment with and test multiple scenarios.

An initial series of examples will be used here to demonstrate how spreadsheets make this experimentation with multiple scenarios possible in a more efficient way than a pencil and paper format. Static cell contents entered directly by the user, as well as dynamic cell contents whose values change according to relative and absolute cell referencing will be used to highlight some of the most useful aspects of spreadsheets.

After these initial series of examples, real life data from the World Wide Web will be used to compare trend that is linear with both quadratic and exponential patterns. Linear, quadratic, and exponential functions are important because they are often used to approximate more complicated functions. Absolute percentage change will be used to quantify the difference between trends that match that of a linear, quadratic, or exponential function.

In the final portion of this paper, spreadsheets will be applied to iterative mathematical procedures for finding *real zeros* of polynomials. The procedures are based on the *Rational Zeros Theorem*, *Bounds on Zeros Theorem*, and the *Intermediate Value Theorem*. These methods would be extremely laborious and time consuming to implement in a pencil and paper setting.

Spreadsheet as a tool for experimentation with linear transformations in the plane

The Common Core State Standards for Mathematics [4] recommend that high school students should *experiment with transformations in the plane using software*. Furthermore, the transformations should involve systems of two linear equations in two variables such that the resulting straight lines either intersect at an angle at a unique point, or are parallel or perpendicular to each other.

Spreadsheet is an ideal tool of choice for this experimentation. For example, suppose two linear functions $F(x)$ and $G(x)$ have the standard forms:

$$F(x) = m_1 x + b_1 \qquad and \qquad G(x) = m_2 x + b_2 \tag{1}$$

Where:

m_1= slope of the linear function $F(x)$

b_1= y-intercept of the linear function $F(x)$

m_2= slope of the linear function $G(x)$

b_2= y-intercept of the linear function $G(x)$

The functions $F(x)$ and $G(x)$ are parallel if the condition in Equation 2 is true, and they are perpendicular if the condition in Equation 3 is true. If neither of these conditions is true, then the lines intersect at some other angle at a unique point.

$$m_1 = m_2 \tag{2}$$

$$m_1 \times m_2 = -1 \tag{3}$$

Figure 5 shows an Excel worksheet that has been designed to conduct experimentations with transformations of two linear equations in two variables in the plane. Figure 6 shows a portion of the Excel worksheet that contains the formula structure for functions $F(x)$ and $G(x)$. The functions $F(x) = -2x + 6$ and $G(x) = -2x + 30$ were used to demonstrate how the equal slope condition $m_1 = m_2 = -2$ creates the parallel lines shown in Figure 7. Students should be allowed to experiment with other pairs of equations with equal slopes on their own in order to understand that there are infinitely many possibilities for constructing parallel lines.

Figure 8 shows the same Excel worksheet set up shown in Figure 5 adjusted to experiment with the perpendicular set of functions $F(x) = -2x + 24$ and $G(x) = 0.5x - 5$. The slopes of the linear functions satisfy the condition in Equation 3 $(m_1 \times m_2 = (-2)(0.5) = -1)$ for perpendicular lines. A graph of the perpendicular linear system is shown in Figure 9.

There are infinitely many possibilities which students can experiment with. One other such example has been captured in the Excel segment shown in Figure 10 and its associated plot in Figure 11. The scroll bars can be used by students to experiment with various slopes and y-intercept values for both equations.

x	F(x)	G(x)
-20	46	70
-19.5	45	69
-19	44	68
-18.5	43	67
-18	42	66
-17.5	41	65
-17	40	64
-16.5	39	63
-16	38	62
-15.5	37	61
-15	36	60
-14.5	35	59
-14	34	58
-13.5	33	57
-13	32	56
-12.5	31	55
-12	30	54
-11.5	29	53
-11	28	52
-10.5	27	51
-10	26	50
-9.5	25	49

	Slope	Y-intercept
Initial_F(x)	0	0
Initial_G(x)	0	0

Slope of F(x)	Y-Intercept of F(x)	Slope of G(x)	Y-Intercept of G(x)
-2	6	-2	30

Slope of F(x)

Y-Intercept F(x)

Slope of G(x)

Y-Intercept G(x)

Figure 5. This is a partial Excel worksheet for experimenting with systems of linear equations in two variables within the plane. The column for X continues down to x=20 in this case. The scroll bars are set up to enable continuous incremental change of slope and y-intercept parameters for both equations F(x) and G(x), and to make experimentation with the graph of the system shown in Figure 7 effortless. The “Initial_F(x)” and “Initial_G(x)” cells for slope and y-intercept are set up to enable the user to use the scroll bars to choose through any continuous negative range of slope and intercept values if desired. Note that both current slopes of F(x) and G(x) are set to $m_1 = m_2 = -2$ in order to satisfy the condition in Equation 2 and create the graph of parallel lines shown in Figure 7.

	A	B	C
2	x	F(x)	G(x)
3	-20	=($F$2+$E$9)*A3+($G$2+$F$9)	=($F$3+$G$9)*A3+($G$3+$H$9)
4	-19.5	=($F$2+$E$9)*A4+($G$2+$F$9)	=($F$3+$G$9)*A4+($G$3+$H$9)
5	-19	=($F$2+$E$9)*A5+($G$2+$F$9)	=($F$3+$G$9)*A5+($G$3+$H$9)
6	-18.5	=($F$2+$E$9)*A6+($G$2+$F$9)	=($F$3+$G$9)*A6+($G$3+$H$9)
7	-18	=($F$2+$E$9)*A7+($G$2+$F$9)	=($F$3+$G$9)*A7+($G$3+$H$9)
8	-17.5	=($F$2+$E$9)*A8+($G$2+$F$9)	=($F$3+$G$9)*A8+($G$3+$H$9)
9	-17	=($F$2+$E$9)*A9+($G$2+$F$9)	=($F$3+$G$9)*A9+($G$3+$H$9)

Figure 6. A partial list of the only formulas entered for the Excel spreadsheet shown in Figure 5. The formulas follow the standard form of a linear equation $f(x) = mx + b$ and are used to create a link between different components of the worksheet. The ultimate objective is to create a dynamic graph in order to make continuous experimentation with different systems of linear equations in two variables possible.

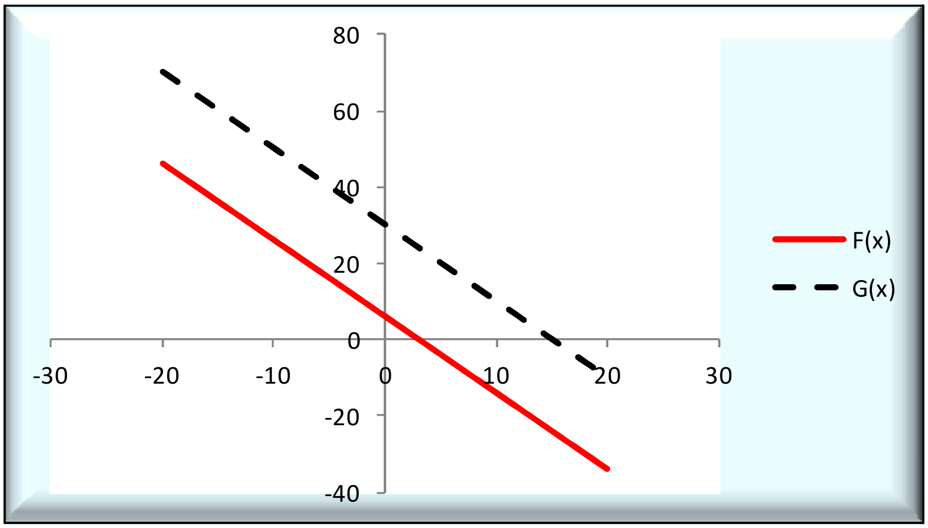
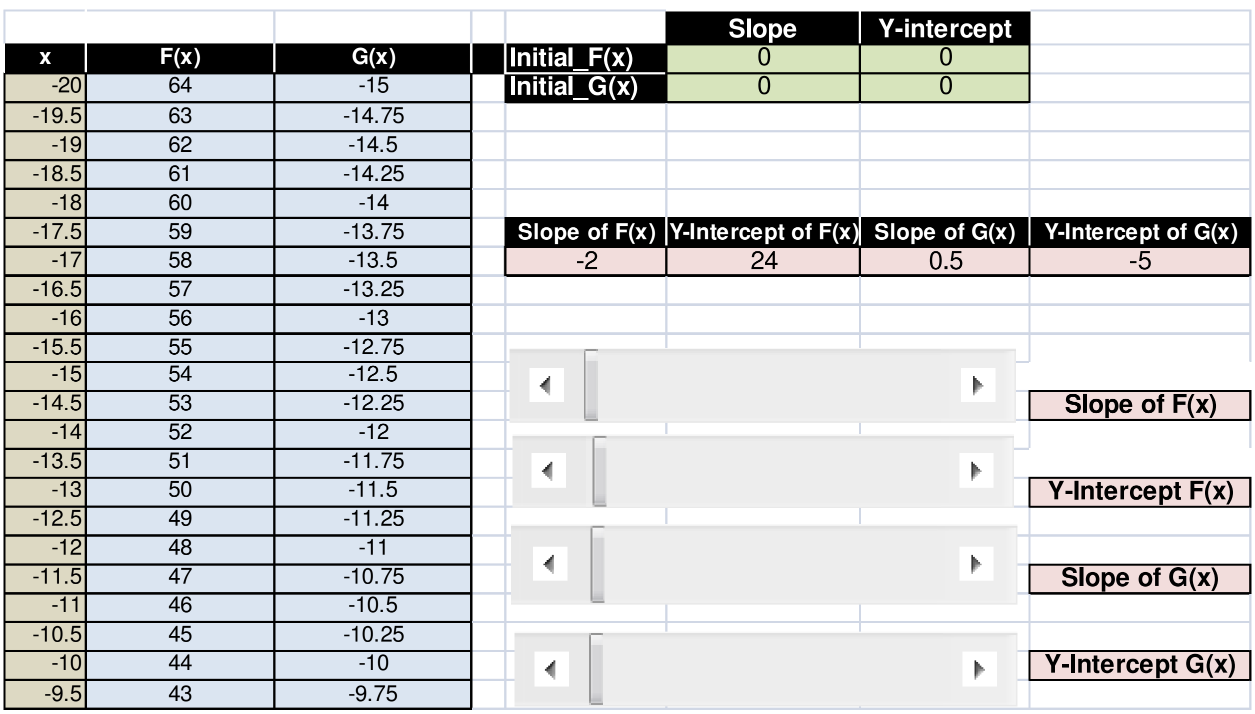


Figure 7. Graph of two parallel linear functions $F(x) = -2x + 6$ and $G(x) = -2x + 30$



x	F(x)	G(x)
-20	64	-15
-19.5	63	-14.75
-19	62	-14.5
-18.5	61	-14.25
-18	60	-14
-17.5	59	-13.75
-17	58	-13.5
-16.5	57	-13.25
-16	56	-13
-15.5	55	-12.75
-15	54	-12.5
-14.5	53	-12.25
-14	52	-12
-13.5	51	-11.75
-13	50	-11.5
-12.5	49	-11.25
-12	48	-11
-11.5	47	-10.75
-11	46	-10.5
-10.5	45	-10.25
-10	44	-10
-9.5	43	-9.75

	Slope	Y-intercept
Initial_F(x)	0	0
Initial_G(x)	0	0

Slope of F(x)	Y-Intercept of F(x)	Slope of G(x)	Y-Intercept of G(x)
-2	24	0.5	-5

Slope of F(x)

Y-Intercept F(x)

Slope of G(x)

Y-Intercept G(x)

Figure 8. A partial Excel worksheet for graphing the two perpendicular linear functions $F(x) = -2x + 24$ and $G(x) = 0.5x - 5$.

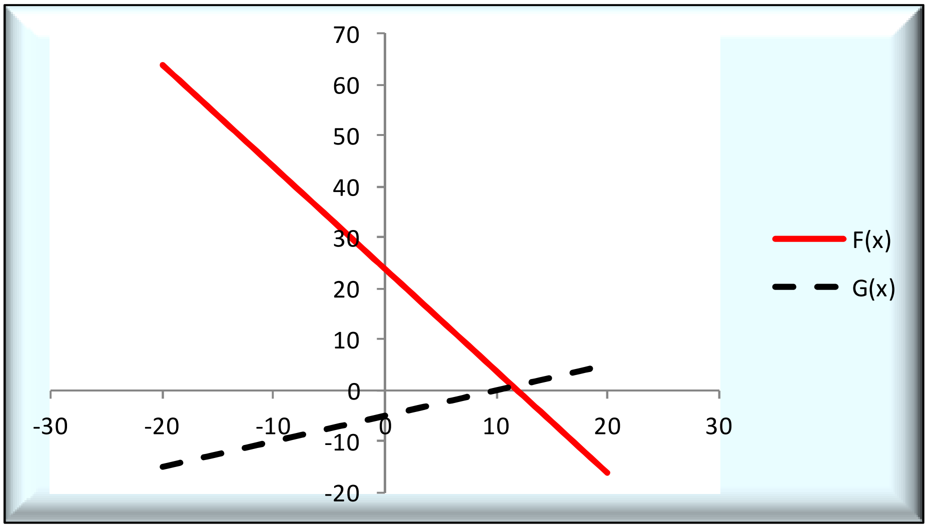
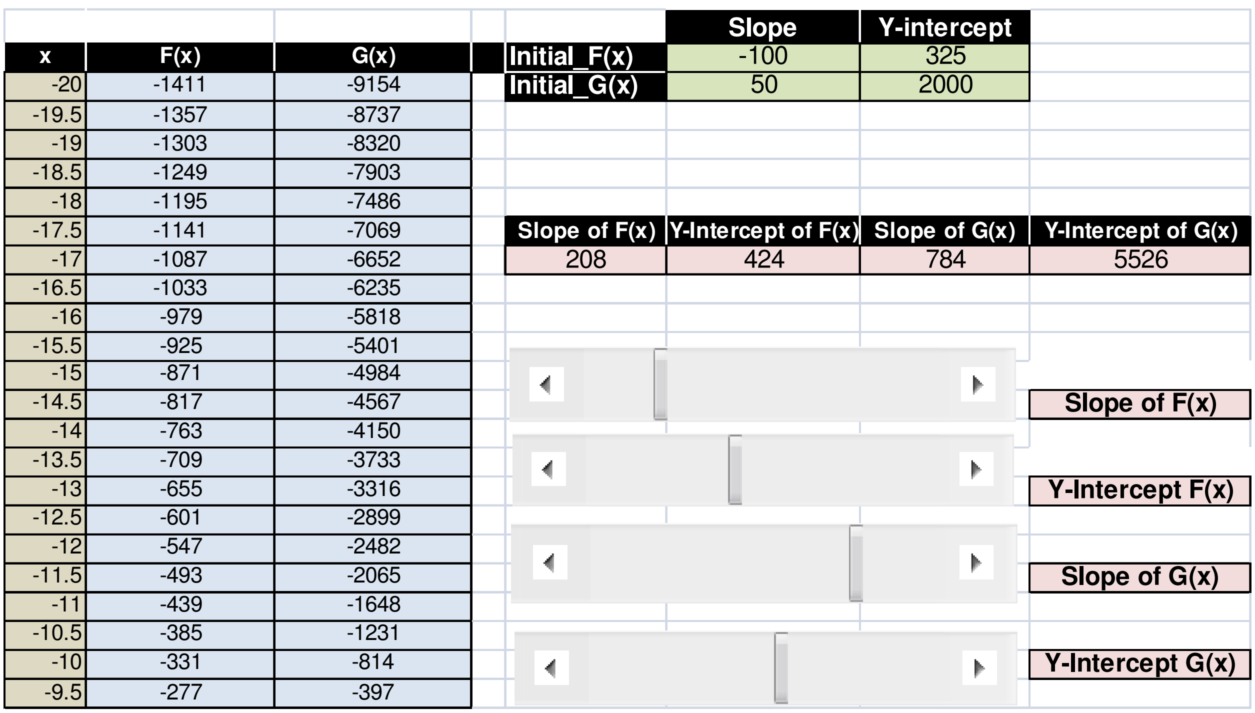


Figure 9. Graph of two perpendicular linear functions $F(x) = -2x + 24$ and $G(x) = 0.5x - 5$.



x	F(x)	G(x)
-20	-1411	-9154
-19.5	-1357	-8737
-19	-1303	-8320
-18.5	-1249	-7903
-18	-1195	-7486
-17.5	-1141	-7069
-17	-1087	-6652
-16.5	-1033	-6235
-16	-979	-5818
-15.5	-925	-5401
-15	-871	-4984
-14.5	-817	-4567
-14	-763	-4150
-13.5	-709	-3733
-13	-655	-3316
-12.5	-601	-2899
-12	-547	-2482
-11.5	-493	-2065
-11	-439	-1648
-10.5	-385	-1231
-10	-331	-814
-9.5	-277	-397

	Slope	Y-intercept
Initial_F(x)	-100	325
Initial_G(x)	50	2000

Slope of F(x)	Y-Intercept of F(x)	Slope of G(x)	Y-Intercept of G(x)
208	424	784	5526

Figure 10. A partial Excel worksheet for experimenting with a system of two linear functions in two variables. The functions in this particular snapshot are $F(x) = 208x + 424$ and $G(x) = 784x + 5526$.

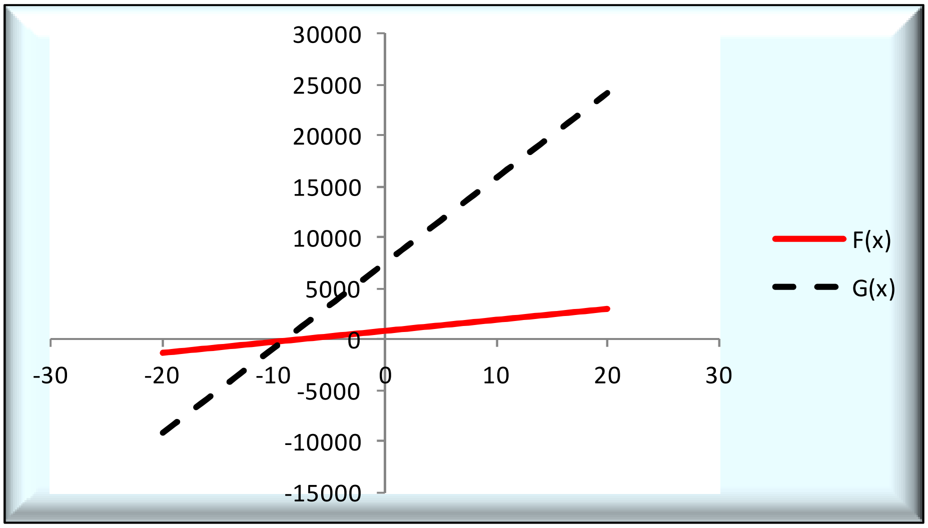


Figure 11. Graph of two linear functions $F(x) = 208x + 424$ and $G(x) = 784x + 5526$.

Spreadsheet as a tool for experimenting with quadratic functions

The Common Core State Standards for Mathematics [4] recommend spreadsheets as a tool for *experimenting* with how functions behave in high school mathematics classrooms. The quadratic formula is a common method for finding the solutions of a quadratic equation in high school algebra; especially if the equation is prime, which is to say it is not factorable [8]. More specifically:

$$\text{For:} \quad ax^2 + bx + c = 0 \quad \text{and} \quad a \neq 0 \tag{4}$$

$$\text{Quadratic formula:} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{5}$$

$$\text{Discriminant: } b^2 - 4ac = \begin{cases} a)\ if \quad b^2 - 4ac = perfect \quad square, \\ \quad then\ the\ quadratic\ equation\ is \\ \quad factorable \\ \\ b)\ if \quad b^2 - 4ac > 0,\ \ then\ two \\ \quad distinct\ real\ solutions\ exist \\ \\ c)\ if \quad b^2 - 4ac = 0,\ \ then\ thre\ is \\ \quad one\ repeated\ real\ solution \\ \\ d)\ if \ \ b^2 - 4ac < 0,\ \ then\ there\ is \\ \quad no\ real\ solution;\ solutions\ are \\ \quad complex\ conjugates \end{cases} \tag{6}$$

Before the invention of advanced programmable calculators and spreadsheets, in a traditional pencil and paper exercise, solving the quadratic equation $3x^2 - 5x + 1 = 0$ would require a basic calculator useful in computing the quadratic formula with coefficients $a = 3,\ \ b = -5,\ \ c = 1$. However, a mathematics instructor would have to assign many different problems of the same type in order to impress upon students the way solutions to the quadratic equation would vary depending on the value of the coefficients a, b, and c. Whereas, with spreadsheets, many experimentations with different sets of coefficient values may be carried out with the benefit of observing how the parabolic graph of the quadratic function changes instantaneously, given a change in the function's parameters.

For example, in Figure 12, the static cells B1, B2, and B3 contain the coefficients of a quadratic function whose values are determined by the user, while the values in cells E1, E2, E3, B5, and B6 are dynamic and change in response to input by the user.

	A	B	C	D	E
1	**a=**	3		**Discriminant=**	13
2	**b=**	-5		**x**	1.434258546
3	**c=**	1		**x**	0.232408121
4					
5	**X_minimum=**	0.833333			
6	**Y_minimum=**	-1.083333			

	A	B	C	D	E
1	**a=**	3		**Discriminant=**	=$B$2^2-4*$B$1*$B$3
2	**b=**	-5		**x**	=(-$B$2+SQRT($E$1))/(2*$B$1)
3	**c=**	1		**x**	=(-$B$2-SQRT($E$1))/(2*$B$1)
4					
5	**=IF($B$1<0,"X_maximum","X_minimum")**	=-$B$2/(2*$B$1)			
6	**=IF($B$1<0,"Y_maximum","Y_minimum")**	=$B$1*$B$5^2+$B$2*$B$5+$B$3			

Figure 12. The Excel segment on top shows the contents of user defined static cells (B1:B3) and formula-based dynamic cells (E1:E3 and B5:B6) for quadratic equation $3x^2 - 5x + 1 = 0$. The Excel segment on the bottom shows the formulas that compute the contents of the dynamic cells.

A comparison of the value of the discriminant in Figure 12 with the set of expressions in (6) indicates that the equation $3x^2 - 5x + 1 = 0$ is not factorable. Additionally, the function has two real solutions, i.e. two *real zeros,* at $x \cong 1.434$ and $x \cong 0.232$.

A calculation of the coordinates of the minimum or maximum point of a quadratic function at its vertex is based on Expressions in 7. The top Excel segment in Figure 12 also shows the coordinates of the minimum point at the vertex.

$$\left(x = -\frac{b}{2a},\ \ y = f\left(-\frac{b}{2a}\right)\right) \qquad (7)$$

The following calculations demonstrate a sample application of the Expressions in (7) to computing the coordinates of the vertex (a minimum point in this case) for the quadratic function: $f(x) = 3x^2 - 5x + 1$. Because the coefficient of the quadratic term, x^2, is positive, the parabola that represents the function, shown in Figure 13, opens upward and has a vertex at the point of the minimum.

$$a = 3 \qquad b = -5 \qquad c = 1$$

$$x_{min} = -\frac{-5}{2 * 3} = 0.83333$$

$$y_{min} = f(0.83333) = 3(0.83333)^2 - 5(0.8333) + 1 = -1.08333$$

$$(x_{min}, y_{min}) = (0.83333, -1.08333)$$

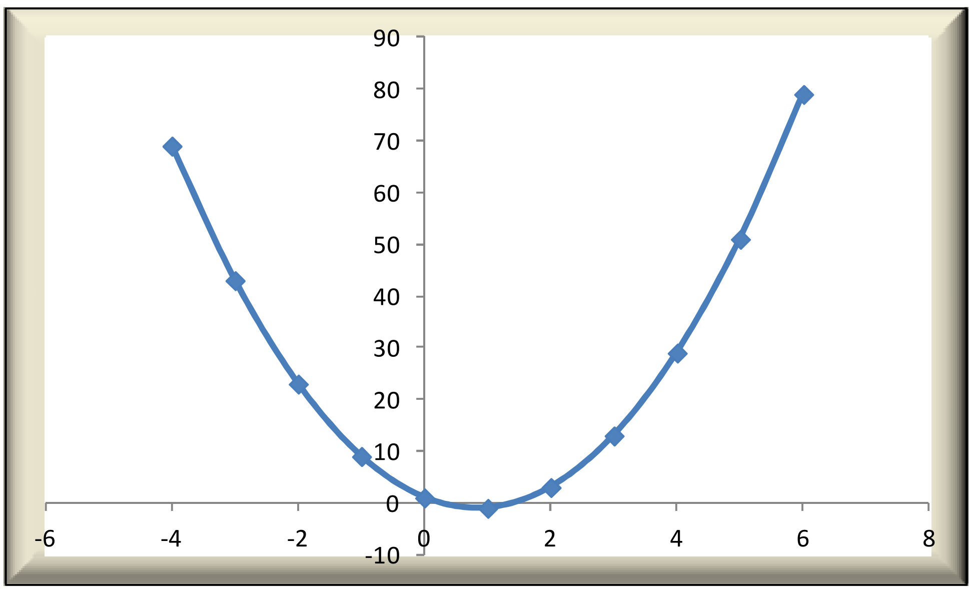


Figure 13. A plot of the quadratic function $3x^2 - 5x + 1 = 0$ with vertex at $(0.83333, -1.08333)$.

An instructor can ask students, just as easily, to *experiment* with different coefficient values, and observe the impact of this on the values of the discriminant, the *real zeros* or roots of the quadratic function, and the coordinates of the minimum or maximum point at the vertex. This has been demonstrated further in Figures 14, 15, and 16 using the quadratic function $x^2 - 0.1x + 6 = 0$. The coefficient values $a = 1$ and $b = -0.1$ and $c = 6$ are entered in the respective cells B1:B3. This will lead to the results shown in Figure 14.

	A	B	C	D	E
1	a=	1		Discriminant=	-23.99
2	b=	-0.1		x	#NUM!
3	c=	6		x	#NUM!
4					
5	X_minimum=	0.05			
6	Y_minimum=	5.9975			

	A	B	C	D	E
1	a=	1		Discriminant=	=$B$2^2-4*$B$1*$B$3
2	b=	-0.1		x	=(-$B$2+SQRT($E$1))/(2*$B$1)
3	c=	6		x	=(-$B$2-SQRT($E$1))/(2*$B$1)
4					
5	=IF($B$1<0,"X_maximum","X_minimum")	=-$B$2/(2*$B$1)			
6	=IF($B$1<0,"Y_maximum","Y_minimum")	=$B$1*$B$5^2+$B$2*$B$5+$B$3			

Figure 14. The results are for the quadratic function $f(x) = x^2 - 0.1x + 6$. The top Excel segment shows the coordinates of the minimum point and a negative discriminant value which explains why the function has no real zeros or roots. This outcome is indicated by the output "#NUM!" in cells E2:E3. The Excel segment on the bottom shows the formulas that generate the outcomes on top.

Figure 15 shows the data and the associated formula for a plot of the quadratic function $x^2 - 0.1x + 6 = 0$. The plot is shown in Figure 16. Note that the plot does not intersect the x-axis at any point. This is an indication that the function has complex conjugate solutions, and it is a reaffirmation of what we expected by comparing the negative discriminant in cell E1 of the top Excel segment shown in Figure 14 with Expressions in (6).

	G	H
1	x	f(x)
2	-5	31.5
3	-4	22.4
4	-3	15.3
5	-2	10.2
6	-1	7.1
7	0	6
8	1	6.9
9	2	9.8
10	3	14.7
11	4	21.6
12	5	30.5

	G	H
1	x	f(x)
2	-5	=$B$1*G2^2+$B$2*G2+$B$3
3	-4	=$B$1*G3^2+$B$2*G3+$B$3
4	-3	=$B$1*G4^2+$B$2*G4+$B$3
5	-2	=$B$1*G5^2+$B$2*G5+$B$3
6	-1	=$B$1*G6^2+$B$2*G6+$B$3
7	0	=$B$1*G7^2+$B$2*G7+$B$3
8	1	=$B$1*G8^2+$B$2*G8+$B$3
9	2	=$B$1*G9^2+$B$2*G9+$B$3
10	3	=$B$1*G10^2+$B$2*G10+$B$3
11	4	=$B$1*G11^2+$B$2*G11+$B$3
12	5	=$B$1*G12^2+$B$2*G12+$B$3

Figure 15. Function $f(x) = x^2 - 0.1x + 6$ evaluated at values of x ranging from -5 to 5. The data were used to plot the function.

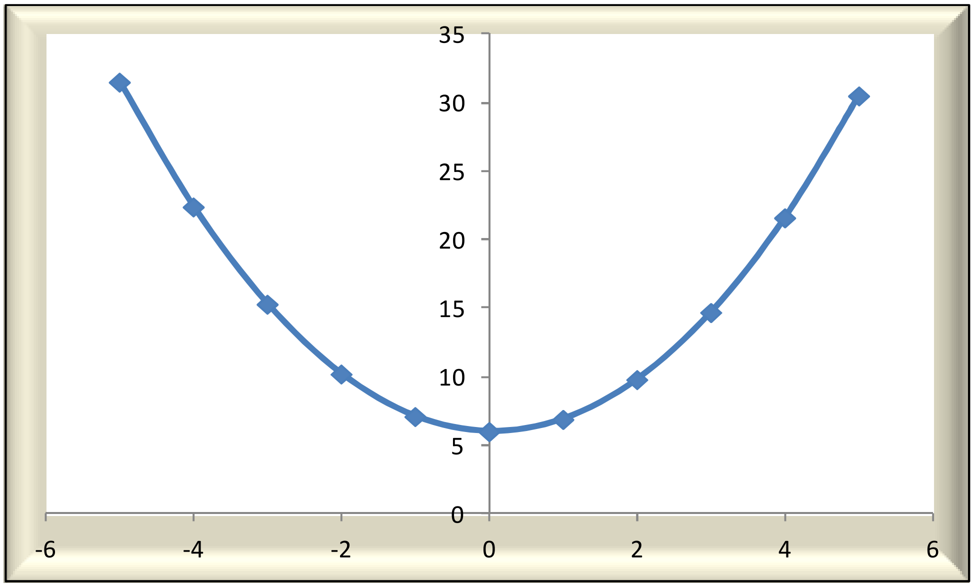


Figure 16. The plot of the function $f(x) = x^2 - 0.1x + 6$ with vertex at (0.05,5.9975). Note that the plot does not intersect the x-axis at any point, indicating that the solutions, or the *real zeros*, are complex conjugates.

Experimenting with the case of a quadratic function that has a maximum instead of a minimum point is possible by simply changing only one of the coefficients in the function. To accomplish this, the function $f(x) = x^2 - 0.1x + 6$ needs to be changed to $f(x) = -x^2 - 0.1x + 6$. On the other hand, all three coefficients may be changed to experiment with a new quadratic function altogether just as conveniently and efficiently.

For instance, the function $f(x) = -4x^2 - 12x + 10$ is one possible choice for a quadratic function that has its vertex at a maximum point. Thus, changing the coefficient values to $a = -4$ and $b = -12$ and $c = 10$ in the same worksheet produces the discriminant (304), both solutions or *real zeros* (-3.679449, 0) and (0.679449, 0), the data needed for the chart (cells G2:G9; H2:H9), and the coordinates of the maximum point (-1.5, 19). Figures 17, 18, and 19 contain the Excel segments for these results. Once again, sample calculations of the Expressions in (7) to computing the coordinates of the vertex are shown below:

$$a = -4 \qquad b = -12 \qquad c = 10$$

$$x_{max} = -\frac{-12}{2 * (-4)} = -1.5$$

$$y_{max} = f(-1.5) = -4(-1.5)^2 - 12(-1.5) + 10 = 19$$

$$(x_{max}, y_{max}) = (-1.5, 19)$$

	A	B	C	D	E
1	a=	-4		Discriminant=	304
2	b=	-12		x	-3.679449472
3	c=	10		x	0.679449472
4					
5	X_maximum=	-1.5			
6	Y_maximum=	19			

	A	B	C	D	E
1	a=	-4		Discriminant=	=$B$2^2-4*$B$1*$B$3
2	b=	-12		x	=(-$B$2+SQRT($E$1))/(2*$B$1)
3	c=	10		x	=(-$B$2-SQRT($E$1))/(2*$B$1)
4					
5	=IF($B$1<0,"X_maximum","X_minimum")	=-$B$2/(2*$B$1)			
6	=IF($B$1<0,"Y_maximum","Y_minimum")	=$B$1*$B$5^2+$B$2*$B$5+$B$3			

Figure 17. The results are for the quadratic function $f(x) = -4x^2 - 12x + 10$. The top Excel segment shows the coordinates of the maximum point (-1.5, 19) and a positive discriminant value (304) which indicates the function has two real roots at (-3.679449, 0) and (0.6794495, 0). The Excel segment on the bottom shows the Excel formulas for the outcomes shown on top.

	G	H
1	x	f(x)
2	-5	-30
3	-4	-6
4	-3	10
5	-2	18
6	-1	18
7	0	10
8	1	-6
9	2	-30

	G	H
1	x	f(x)
2	-5	=$B$1*G2^2+$B$2*G2+$B$3
3	-4	=$B$1*G3^2+$B$2*G3+$B$3
4	-3	=$B$1*G4^2+$B$2*G4+$B$3
5	-2	=$B$1*G5^2+$B$2*G5+$B$3
6	-1	=$B$1*G6^2+$B$2*G6+$B$3
7	0	=$B$1*G7^2+$B$2*G7+$B$3
8	1	=$B$1*G8^2+$B$2*G8+$B$3
9	2	=$B$1*G9^2+$B$2*G9+$B$3

Figure 18. Function $f(x) = -4x^2 - 12x + 10$ evaluated at values of x ranging from -5 to 2. This provides the data for a plot of the function.

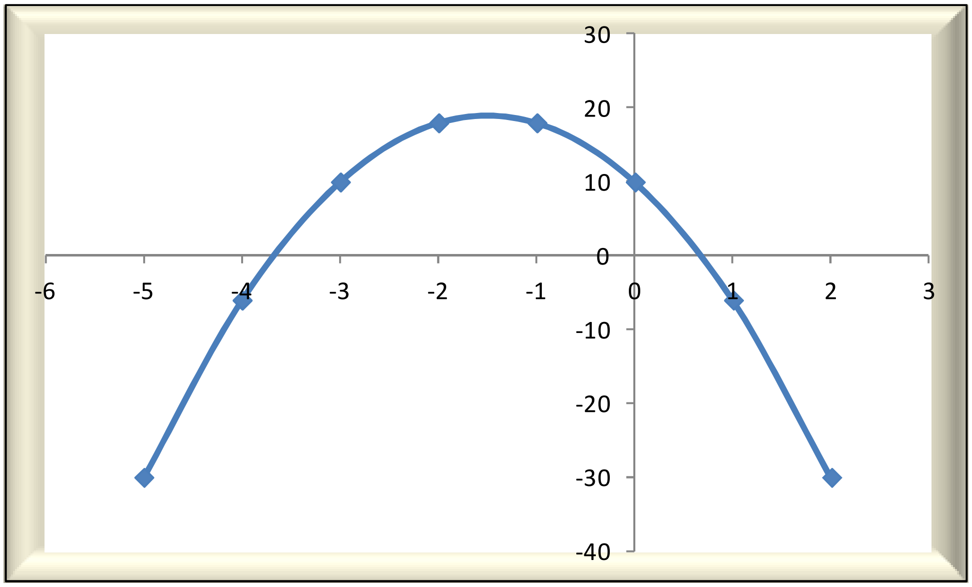


Figure 19. The plot of the function $f(x) = -4x^2 - 12x + 10$
The coordinates of the maximum point are $(-1.5,19)$.

Spreadsheet as a tool for modeling and comparing linear, quadratic, and exponential functions using real-life data

The Common Core State Standards for Mathematics [4] recommends that mathematics instructors teach their students to "construct and compare linear, quadratic, and exponential models." Equations 8, 9, and 10 show the standard forms of these types of functions.

$$\text{Linear functions:} \quad f(x) = mx + b \tag{8}$$

$$\text{Quadratic functions:} \quad f(x) = ax^2 + bx + c \tag{9}$$

$$\text{Exponential functions:} \quad f(x) = ca^x \tag{10}$$

Linear, quadratic, and exponential functions are covered in a typical high school algebra course. Although, some middle schools teach these topics in 8th grade, as well. But even college mathematics instructors are aware that the vast majority of their students do not have fluent understanding of these content areas. The large number of remedial algebra courses that are offered on nearly all of over 4000 college and university campuses in America is a clear indication of this fact.

Alagic and Palenz [2] used spreadsheets to examine the learning and teaching habits of a group of middle school mathematics teachers "about linear and exponential growth." Although the study's primary objective was to investigate how *the teachers'* "calculational and conceptual" orientations to mathematics would change in response to the spreadsheet-based pedagogy, the researchers believed that younger students would have “varying levels” of these same orientations.

Thus, from the outset, it was decided that successful implications of applying the approach to teach the teachers would be transferrable to the way the students of these teachers learned, as well. It was determined in the study that effective transfer of the spreadsheet-based pedagogy from teaching the teachers to teaching younger students would require that: 1) students should be provided with clear goals in the exercise; 2) when necessary, students should receive prompts, especially in relation to the use of spreadsheet technology; 3) students should receive feedback; and 4) should have opportunities for reflection. The only other criterion was that all participants, teachers and students alike, would have to come up with a real-life story prior to collecting of the data. With this blueprint in mind, real-life data were collected, graphed, and analyzed and interpreted with the aid of spreadsheets.

Although Alagic and Palenz [2] conducted their study seven years before the publishing of the Common Core State Standards for Mathematics in 2013, the four-step pedagogy used in their study is similar to the mathematical modeling process suggested by the Standards. Figure 20 displays an adaptation of the “basic modeling cycle” as discussed in the Common Core State Standards for Mathematics [4].

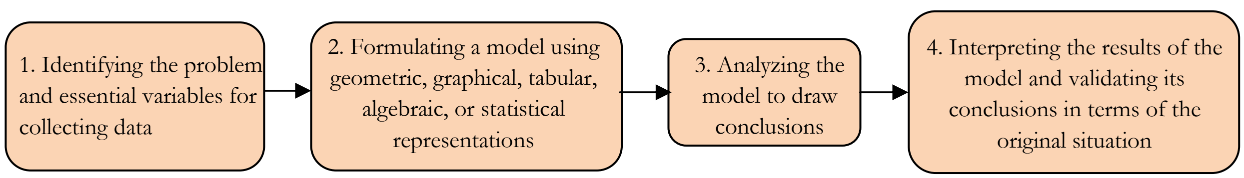


Figure 20: An adaptation of the basic modeling process as described in the Common Core State Standards for Mathematics.

The initial step in the modeling process is collecting data. Although, data may be measured and collected physically for an exercise in mathematical modeling, data from the World Wide Web were used for the example discussed here. For this reason, before discussing models of linear, quadratic, and exponential functions, two techniques for preparing electronic data for use in Excel will be discussed.

Importing electronic data directly from the Web

The “Data” menu in Excel 2007 or a later version of the software allows for importing electronic data directly from the Web. Under this menu, in the “Get External Data” category, one of the available options is “From Web.” Selecting this option opens a window entitled “New Web Query,” where the URL for where the data is located on the Web may be entered. The *Query* tool will then bring up the website that contains the data and a yellow arrow will allow the user to select and import tabulated data that are in a format other than that created by Excel.

Importing electronic data from a compact disc

Data in the form of portable space-delimited, comma-delimited, or tab-delimited text files are still a common form of creating very large data files. Oftentimes, these data files are over 10 megabytes in size, contain hundreds of thousands of rows of data, and are commonly stored on compact discs. As spreadsheets like Excel have expanded their data storage and manipulation capability by several fold, the potential for importing data from CD-ROMs into Excel has increased. The following steps outline the procedure for importing a space or character delimited data file into Excel. The procedure is mainly intended to be used with formatted data files located on a CD-ROM. However, in order to avoid the need for a CD-ROM in this exercise, a relatively small space-delimited data file will be imported from the Web. Nevertheless, the procedure to import larger data files of this type from a CD-ROM is the same for the most part.

Exercise: The data will be imported from the website: http://lib.stat.cmu.edu/DASL.

1. Once on the website, click on the rectangular block that reads "List all methods"
2. Select "Correlation" from the list of methods
3. Select "US Crime Story" from the list of datasets
4. On the screen that comes up, there's a line that reads "**Datafile Name:** US Crime". Click on the hyperlink to go to the data page. It would be preferable to have students spend time on this page and familiarize themselves with the different variables and also to get a visual sense of what the dataset looks like.
5. Use the mouse to click and select all of the data from this page and paste it into a blank WORD document. Because there are too many columns of data that can fit on a "portrait" page layout, before pasting the data, go to "Page Layout" menu in WORD and change the page layout to "Landscape."
6. Save the WORD file on your DESKTOP with these settings:
   File name: US_Crime_data and Save as Type: plain text
7. After you select the settings mentioned above and as you try to save the file, another window may open up which is called "File Conversion – US_Crime_data.txt". If this happens, make sure that the radio dial that says "Other encoding" is checked and then click OK to finish saving the file on your DESKTOP.
8. Go to your DESKTOP and open the text file US_Crime_data file that you have just saved. Visually inspect the data and make sure that everything looks good. Close and exit the text file.
9. Launch the Excel spreadsheet and from your DESKTOP open the text file US_Crime_data which you have just created. When doing this, make sure that "Text Files" is shown as the type of file you wish to open. Otherwise, Excel will not be able to see the file on your desktop. Next, click OK.
10. At this point, a window will open up that is titled "Text import Wizard – Step 1 of 3." In this window, make sure that the radio dial next to "Delimited" has been checked. Then click "Next" to go to the next window.
11. This window has the title "Text import Wizard – Step 2 of 3" and is very important because towards the bottom it shows you how your data will be imported into Excel. Make sure the data in all the columns are nicely aligned. If they are not, experiment with several of the radio dials that are on the left side of the window to see which one of the selections will align the data in the right space. For example, note that if you uncheck the radio dial next to "Tab" your data will look scrambled. You do not want this. Check the

radio dial next to the "Tab" option back on again and hit the "Next" button to go to the next window.

12. This window has the title "Text import Wizard – Step 3 of 3" and it is also very important in that it is here where you must choose the type of data stored in each column. If the type of data stored in a column is numeric, the heading for that column should be set to "GENERAL." Otherwise, you must use the cursor to select the column that is non-numeric and set its type to one of the other available options of "Text", "Date", or "Do not import column (skip)" by clicking the radio dial next to the option.
13. Once the correct types have been set for all columns of data, simply click the "Finish" button. At this point, you should see all the data imported into Excel neatly and in the right place.

Although the modeling exercise discussed hereafter is based on real-life data obtained from the Web, neither of the two methods that have been explained for downloading of electronic data could be used because the data were not located in a single file or location. Therefore, the data for the following series of activities were obtained from the Web and entered into Excel, manually.

Data about a real-life story were gathered in order to follow the pedagogy used by Alagic and Palenz [2] for providing a meaningful context for the exercise. The real-life story is one that made the national news in the United States in December 2013.

Activity 1: Find A Story With Connection To Real-life

In December 2013, New York City made national news for having unseasonably warm temperatures. Global warming has been the subject of controversy for at least two decades, and students might be experiencing strange weather patterns where they live, too. Hence, this example can easily be adapted to studying the historical temperature at other locations that may be of interest to individual students.

Activity 2: Identify the Essential Variables for Collecting the Data and Graph the Data

The variables that should be investigated depend on the goal of the exercise. The goal here is to develop a model in order to explain: *How different the December 2013 weather in New York City was from any other year?*

To answer this question, we would need the daily temperatures for New York City area for December 2013 and at least one other year. Hence, we will create two variables for our study: the day in December (D) about which data from an officially operated thermometer was collected, and the temperature (T) for that day.

The data used for this exercise was obtained from http://www.wunderground.com/history for New York City. Students may visit the website to obtain historical temperature data for where they live. Data is available at the website from 1945 to present. Minimum and maximum daily air temperatures were obtained for December 1990 and December 2013. Simple scatterplots for the "High" and "Low" temperatures for the two time periods were compared separately and inspected visually to see if there were any significant differences within each category for the selected periods. Figures 21 and 22 provide the full data as they were entered into an Excel worksheet.

	A	B	C
1	Day	December 1990 (High)	December 2013 (High)
2	1	48	49
3	2	48	49
4	3	47	53
5	4	47	52
6	5	47	60
7	6	46	62
8	7	46	41
9	8	45	33
10	9	45	39
11	10	45	37
12	11	44	33
13	12	44	30
14	13	44	35
15	14	43	34
16	15	43	40
17	16	43	33
18	17	42	32
19	18	42	37
20	19	42	47
21	20	42	53
22	21	41	65
23	22	41	71
24	23	41	64
25	24	41	42
26	25	40	31
27	26	40	36
28	27	40	40
29	28	40	55
30	29	39	48
31	30	39	45
32	31	39	32

Figure 21: Daily "High" December temperatures for New York City from 1990 and 2013.

	A	B	C
54	Day	December 1990 (Low)	December 2013 (Low)
55	1	37	36
56	2	37	41
57	3	36	38
58	4	36	41
59	5	35	48
60	6	35	37
61	7	35	32
62	8	34	29
63	9	34	31
64	10	34	30
65	11	33	27
66	12	33	23
67	13	33	23
68	14	32	22
69	15	32	30
70	16	32	25
71	17	32	24
72	18	31	23
73	19	31	30
74	20	31	41
75	21	30	51
76	22	30	61
77	23	30	42
78	24	30	26
79	25	29	19
80	26	29	30
81	27	29	31
82	28	29	36
83	29	28	41
84	30	28	23
85	31	28	21

Figure 22: Daily "Low" December temperatures For New York City from 1990 and 2013.

Figures 23 and 24 show the scatter plots for each pair of high and low temperatures. According to the average "high" temperatures, December 2013 (44.45 ◎F) was warmer than December 1990 (43.03 ◎F) by less than two degrees Fahrenheit. At the same time, according to the "low" temperature averages, December 2013 (32.65 ◎F) was warmer than December 1990 (32.03 ◎F) by less than one degree Fahrenheit.

Clearly, the average values do not reveal much as we look at the erratic behavior of the weather in New York City during December 2013 evident in both Figures 23 and 24.

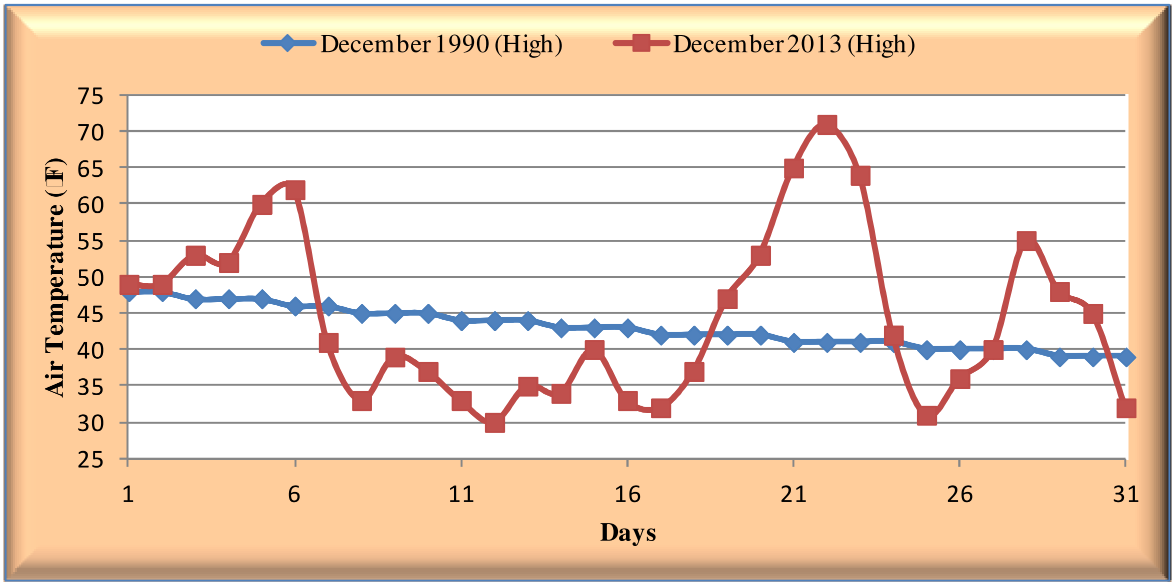


Figure 23: Comparison between December 1990 and 2013 "High" temperatures for New York City.

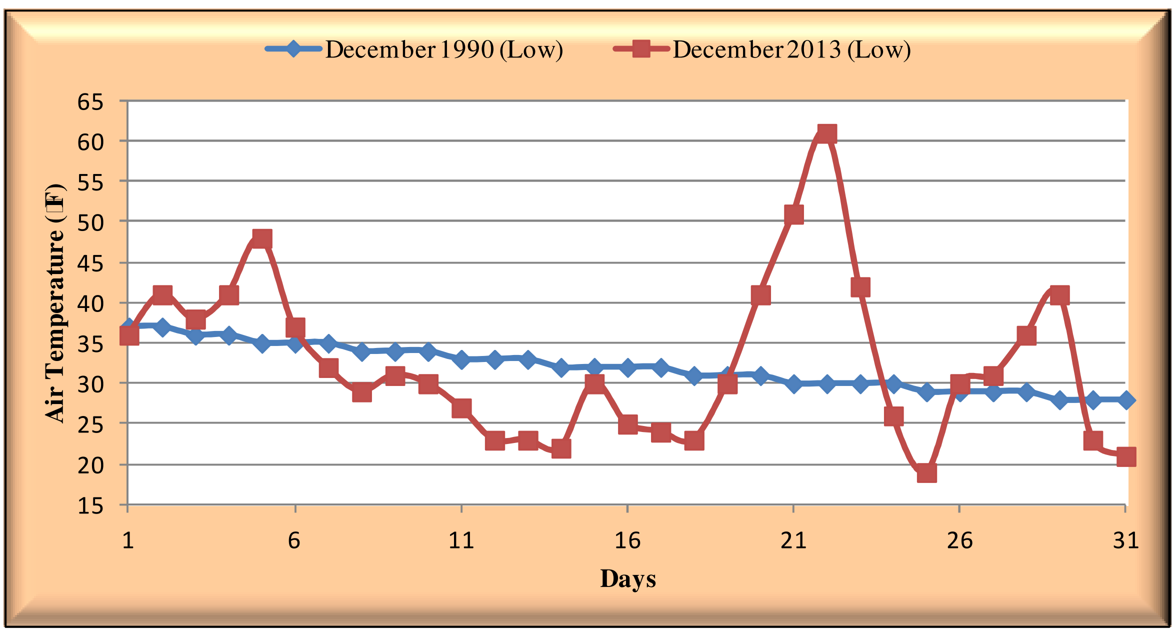


Figure 24: Comparison between December 1990 and 2013 "Low" temperatures for New York City

Activity 3: Create Mathematical Models for the Most Salient Temperature Patterns in December 1990 and 2013

Although it seems feasible to use a single model to quantify the salient linear pattern apparent in the December 1990 data, it would be too complicated to do the same for 2013. However, linear, quadratic, and exponential models may be used to create *piecewise* mathematical models for any specific period of the data that is of interest.

When comparing the December high temperatures (Figure 23), the most salient contrast is observed from December 17 to December 25. We will first focus on this range of days for our mathematical modeling exercise. For these nine days, the daily high temperatures from December 1990 and 2013 were separated from the rest of the data and graphed in Figure 25. For December 1990, as expected, the graph is very nearly similar to the graph of a linear function, whereas the data for the same period from 2013 matches the graph of a quadratic function. Next, we will generate a quantitative mathematical model for each data segment.

To do this we will make use of the “Trendline” feature in the Excel spreadsheet. First, the graph for which we wish to generate a model must be selected. This will bring up the “Chart Tools” at the top of the spreadsheet worksheet screen. There are three toolboxes available here: 1. “Design,” 2. “Layout,” and 3. “Format.” Clicking on the “Layout” toolbox will open it. This toolbox has several tool bins. In the “Analysis” bin, there is a tool called “Trendline.” Selecting the “Trendline” tool will give access to a window from where we can then choose “More trendline options.” In the window that opens next, an appropriate radio dial must be selected. The “Linear” dial in combination with “Display Equation on chart” option was used to generate the piecewise linear mathematical model for the data segment chosen from 1990. Separately, the radio dial “Polynomial” was selected to generate the piecewise quadratic mathematical model for the data segment from 2013. Both mathematical models are also shown in Figure 25.

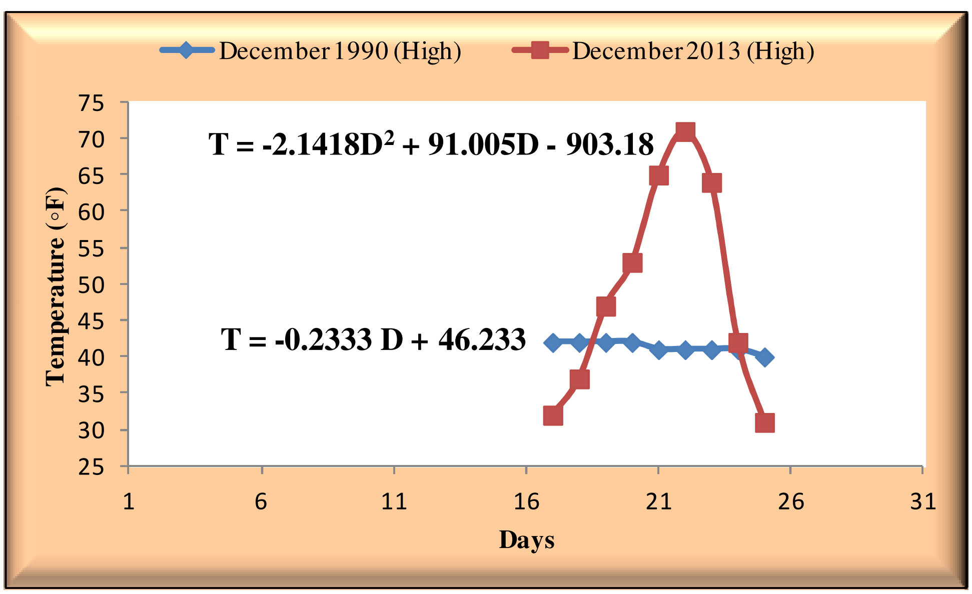


Figure 25: Daily High temperature graphs and piecewise mathematical models from December 17 through December 25 for the years 1990 and 2013.

The same steps were followed to generate the graphs and the piecewise mathematical models for the low daily temperatures of interest shown in Figure 26. Except, in this case, the chosen dates ranged from December 15 through 22. A "Linear" trendline and an "Exponential" trendline were used to generate the piecewise mathematical models for these data segments.

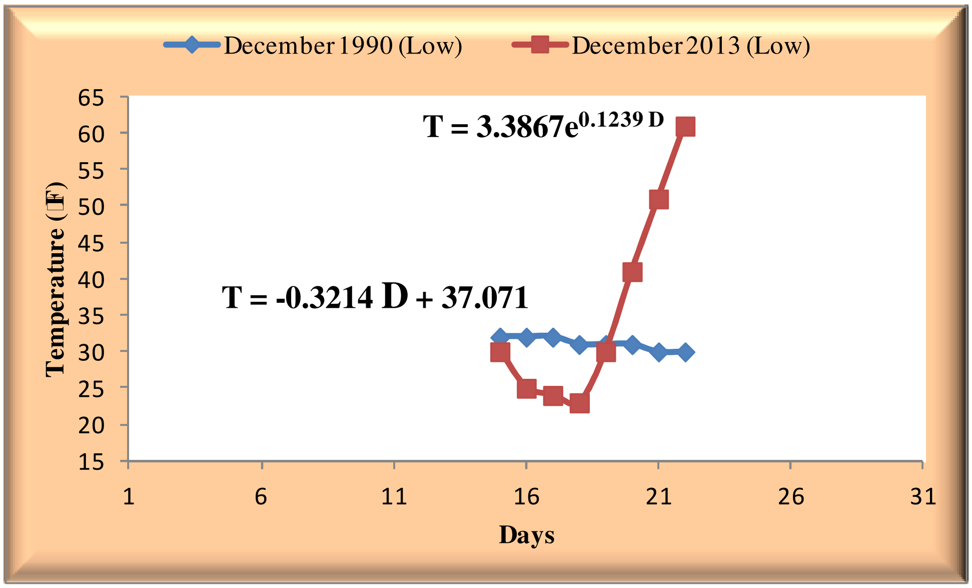


Figure 26: Daily low temperature graphs and piecewise mathematical models from December 15 through December 22 for the years 1990 and 2013.

Activity 4: Analyze and Interpret the Results of the Piecewise Models in Relation to the Original Question

In Figure 25, we have the following piecewise linear model for the data segment of high temperatures from 1990:

$$T = -0.2333\, D + 46.233$$

The validity of this model may be tested by substituting a value for "D" within the range 17 – 25 (e.g. D=21) and calculating the expected temperature for that day using the model. This yields the following result:

$$T = -0.2333(21) + 46.233 \cong 41.3\ °\mathrm{F}$$

From the plot in Figure 25, as well as from the actual data we have recorded in Figure 21, it can be verified that the outcome predicted by the model is very close to the actual temperature value of T=41 °F. The absolute percentage difference with respect to the actual field data is:

$$\left|\frac{41.3-41}{41}\right| \times 100 = 0.73\%$$

Other values within the range of the data segment shown in Figure 25 may be used to analyze the validity of the piecewise model for that period during the month.

Next, we want to analyze the validity of the piecewise model we have for high temperatures during the same December days from 2013. The model we have from Figure 25 is:

$$T = -2.1418D^2 + 91.005D - 903.18$$

Because this piecewise model is quadratic (i.e. non-linear), we will investigate its validity by calculating its value at several points, where D=17, D=22, and D=25. The actual temperatures for these days are shown in Figure 21, and they were T=32, 71, and 31 degrees Fahrenheit, respectively. The predicted results from our piecewise model are:

$$T = -2.1418(17)^2 + 91.005(17) - 903.18 \cong 24.9\ °\text{F}$$
$$T = -2.1418(22)^2 + 91.005(22) - 903.18 \cong 62.3\ °\text{F}$$
$$T = -2.1418(25)^2 + 91.005(25) - 903.18 \cong 33.3\ °\text{F}$$

Therefore, the absolute percentage differences with respect to the actual field values are:

$$\left|\frac{24.9-32}{32}\right| \times 100 = 22.2\%$$
$$\left|\frac{62.3-71}{71}\right| \times 100 = 12.3\%$$
$$\left|\frac{33.3-31}{31}\right| \times 100 = 7.4\%$$

The greatest deviation from the actual value is 22.2%, where *the expected temperature* for December 17 was calculated. If desired, one may choose to improve the accuracy of the piecewise quadratic model by excluding the December 17 data from the analysis and recalculating both the linear and quadratic models for December 18 through December 25 time period, instead. This would yield more accurate piecewise models but, of course, would narrow the range of days when the models could be applied. We will attempt trimming and improving a mathematical model in this way in our next analysis of the daily low temperatures, where the selected data segment gives a better demonstration of the degree to which a model can be improved using this method.

As shown in Figure 26, slightly different dates were chosen to create the piecewise models for the daily low temperatures category. Once again, a piecewise linear model could best quantify the pattern in the data segment from December 1990.

$$T = -0.3214D + 37.071$$

From the plot, it appears that the data point at D=18 has the least alignment with the other data points. Therefore, this would be an ideal data point to choose to get an idea about the measure of the validity of our piecewise linear model, since the variation between the expected value from the model and the actual data collected in the field is expected to be the greatest at this point; that is, the variation would be less for other data points within the same time period. We have:

$$T = -0.3214(18) + 37.071 \cong 31.3\ °\mathrm{F}$$

On the other hand, from Figure 22, we can see that the actual low daily temperature for December 18, 1990 was T=31 °F. Thus, the absolute percentage difference with respect to the actual temperature is:

$$\left|\frac{31.3 - 31}{31}\right| \times 100 = 0.97\%$$

This means that the actual temperature value deviates from the temperature predicted by the model for that day by less than 1%.

We now turn to the same data segment for daily low temperatures, but from the year 2013, for which we have the following piecewise exponential model:

$$T = 3.3867e^{0.1239\,D}$$

Because this is not a linear model, we choose the three days of D=15, D=18, and D=22 to investigate its validity. Figure 22 shows that the actual field low temperature measurements on these three days in 2013 were T=30, 23, and 61 degrees Fahrenheit, respectively.

$$T = 3.3867e^{0.1239(15)} \cong 21.7\ °\mathrm{F}$$
$$T = 3.3867e^{0.1239(18)} \cong 31.5\ °\mathrm{F}$$
$$T = 3.3867e^{0.1239(22)} \cong 51.7\ °\mathrm{F}$$

Accordingly, the absolute percentage differences between the expected and actual measurements are:

$$\left|\frac{21.7-30}{30}\right| \times 100 = 27.7\%$$
$$\left|\frac{31.5-23}{23}\right| \times 100 = 37\%$$
$$\left|\frac{51.7-61}{61}\right| \times 100 = 15.2\%$$

These percentage differences seem rather large. Let us see if we can improve the model's validity by omitting the data for December 15 and recalculating both piecewise models. Figure 27 depicts the impact on the graph and piecewise models of removing December 15 from the analysis.

First, using D=18 in the new piecewise linear model from 1990 data for low temperatures , we have the result:

$$T = -0.3571(18) + 37.786 \cong 31.4\ °\mathrm{F}$$

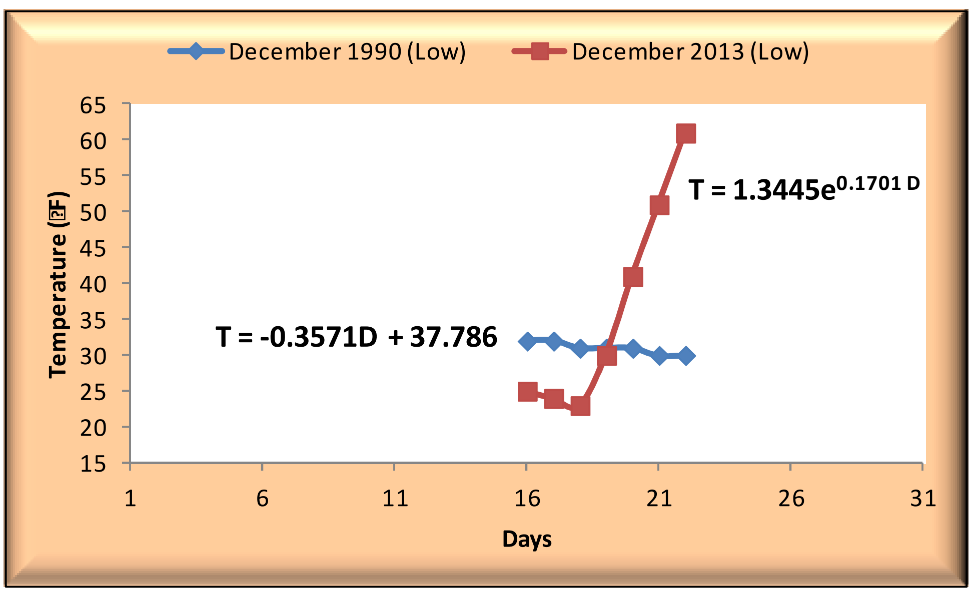


Figure 27: Daily low temperature graphs and piecewise mathematical models from December 16 through December 22 for the years 1990 and 2013.

This will give an absolute percentage difference of 1.3% for the piecewise linear model. Our action to remove December 15 from the data segment seems to have increased the absolute percentage difference for the linear model somewhat from 0.97%, previously. However, this increase is not much and it may be a tradeoff worth having if it helps improve the exponential model for the same period.

We will now look into calculating the absolute percentage differences for the low temperature data from 2013, using the new piecewise exponential model from the period. Choosing D=16, 18, and 22, we have:

$$T = 1.3445e^{0.1701(16)} \cong 20.4\ °\mathrm{F}$$
$$T = 1.3445e^{0.1701(18)} \cong 28.7\ °\mathrm{F}$$
$$T = 1.3445e^{0.1701(22)} \cong 56.7\ °\mathrm{F}$$

From these predicted results, the absolute percentage differences with respect to the actual temperatures are calculated as 18.4%, 24.5%, and 7.05%, respectively. It is apparent from comparing the percentage differences for December 18 and 22 temperatures with those from the previous model that the accuracy of the piecewise exponential model has been improved while the data for December 15 was removed from the analysis.

The question that guided this modeling exercise was: *How different the December 2013 weather in New York City was from any other year?* The spreadsheet graphs and mathematical models of the data for December 1990 and 2013 suggest that linear models are a good fit for the 1990 data, whereas piecewise exponential and quadratic models were a better match for the 2013 data. These models indicate a stable weather pattern for December 1990, but a volatile and unstable pattern of changing temperatures for the same period in 2013. The 1990 data was chosen at random to compare with and shed light on how different daily temperatures in 2013 were from any given year in the city's past.

The above analyses and interpretations provide only one way of approaching a mathematical modeling exercise. It is preferable to allow students to use the spreadsheet to explore with the graphs and the mathematical models on their own in order to develop their own analytical and interpretive skills. Albeit, guidance and prompting provided by the instructor are indispensable to the successful implementation of the modeling process.

Spreadsheet as a tool for finding the real zeros of polynomial functions of higher degree

A polynomial is a function of one variable with a power that can only be a positive integer. Polynomials have the following standard form [8]:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0 \qquad (11)$$

Where a_1, a_{n-2}, a_{n-1}, and a_n are coefficients of the polynomial. The coefficient of the x term with the highest power, i.e. the coefficient a_n, is called the ***leading coefficient*** of the polynomial.

The ***degree of a polynomial*** in one variable, x, is the largest power of x that appears in the function. A few examples of polynomials are:

$P(x) = 16$ (leading coefficient = 0 ; the degree of the polynomial = 0)

$P(x) = 2x + 5$ (leading coefficient = 2; the degree of the polynomial = 1)

$P(x) = 3x^4 - 7$ (leading coefficient = 3; the degree of the polynomial = 4)

$P(x) = 5x^6 - x^5 + 0.5x^4 + 3x^2 + 2x + 9$ (leading coefficient = 5; the degree of the polynomial = 6)

Like linear, quadratic, and exponential models, polynomials are among the simplest functions in algebra. Because of this, they are useful in modeling many natural and social phenomena ranging from climate to market conditions.

The Common Core State Standards for Mathematics [4] recommend the identification of *zeros of a polynomial* an essential part of a course in high school algebra. The *real* root, or *real zero*, of a polynomial $P(x)$ is any number, β_1, such that $P(\beta_1) = 0$; that is, the polynomial intersects the $x - axis$ at its *real zero*.

The *zeros* of a polynomial function, *real* or *complex*, may be of interest for various reasons. According to the *Factor Theorem* [8], for instance, knowledge of *zeros* of a polynomial makes it possible to rewrite a polynomial in terms of simpler functions. That is, given a polynomial function, $P(x)$, if $P(\beta_1) = 0$, then $(x - \beta_1)$ is a factor of polynomial $P(x)$. The factor $(x - \beta_1)$ of a polynomial always is a simpler function than the original polynomial it factorizes. A polynomial $P(x)$ of degree n has exactly n *zeros*.

For example, the quadratic polynomial function $P(x) = x^2 - 1$ may be solved for its *real zeros* and written in terms of its easier to understand linear factors as follows:

$$P(x) = x^2 - 1$$

$$x^2 - 1 = 0 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1 \ \textit{are two real zeros}$$

Therefore: $x^2 - 1 = (x - 1)(x + 1)$

By the same token, suppose that the polynomial described in Equation 11 has n *real zeros* $\beta_1, \beta_2, \dots, \beta_r$. Thus, this means:

$$P(\beta_1) = P(\beta_2) = \cdots = P(\beta_r) = 0$$

Where, r, is an index such that $r = 1,2,3, \dots, n$.

If the *real zeros* $\beta_1, \beta_2, \dots, \beta_r$ are all distinct (i.e. non-repeating), then we can write Equation 11 in terms of its factors (Riley et al., 2008):

$$P(x) = A(x - \beta_1)(x - \beta_2) \dots (x - \beta_r) \qquad (12)$$

Where A must be some non-zero constant.

Recall from our discussion of quadratic polynomial functions, however, that *zeros* of a polynomial may repeat; that is, *zeros* may have ***multiplicity*** greater than one. In such cases, when there are *real zeros* that repeat, factorized form of Equation 12 takes the form shown in Equation 13, where powers $m_1, m_2, \dots, m_r$ are the *multiplicity* of the *real zeros* $\beta_1, \beta_2, \dots, \beta_r$, respectively.

$$P(x) = A(x - \beta_1)^{m_1}(x - \beta_2)^{m_2} \times \dots \times (x - \beta_r)^{m_r} \qquad (13)$$

Because a polynomial has exactly n *zeros*, the number of *real zeros* of a polynomial cannot exceed, n, the *degree* of the polynomial. Therefore, the following condition must be satisfied if all *zeros* of a polynomial are *real*; i.e. there are no *complex zeros*.

$$m_1 + m_2 + m_3 + \cdots + m_r = n$$

When a polynomial function is of the linear form $P(x) = mx + b$, its only *real zero* may be found easily by setting the equation equal to zero and solving for the value of x. When a polynomial is of quadratic form $P(x) = ax^2 + bx + c$, its *real* and *complex zeros* may be found either with factoring or using the quadratic formula. But these methods are not sufficient, or even applicable, in finding the *zeros* of the vast majority of polynomials that do not fall in these two categories.

Special software is often used when mathematicians need to find approximations of all or few particular *zeros* of a polynomial, where other simpler methods are not available. However, effective use of this kind of software often requires an understanding of mathematics that is not yet attained by a pupil who is just beginning to learn about functions.

Nevertheless, it is possible to find the *real zeros* of a polynomial of higher degree with a spreadsheet, because the methods to accomplish this are based on iterative procedures that would be possible to simulate within the software. It would be laborious and very time consuming to apply the iterative procedures using a pencil and paper, however.

Exact rational zeros of polynomials

The *Rational Zeros Theorem* [8] provides an iterative procedure that helps find all the rational *real zeros* of a polynomial. The method is based on the value of the *leading coefficient*, a_n, and the value of the constant term, a_0, in Equation 11, where these coefficients must be non-zero. That is, if:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 \qquad where: \quad a_n \neq 0 \quad , \quad a_0 \neq 0 \tag{14}$$

The theorem states that if the coefficients a_n and a_0 are non-zero integers, then let p be all possible factors of a_0 and q be all possible factors of a_n. Then, all *rational zeros* of the polynomial $P(x)$ may be found among one or more combinations of the ratio, $\frac{p}{q}$.

Because there could be many combinations of $\frac{p}{q}$, and not all combinations will be a *zero* of the polynomial, a spreadsheet program that sifts through all possible iterations in a timely manner is desirable. The following example is a case in point:

Example: We would like to identify all the *rational zeros* of the polynomial:

$$f(x) = 2x^3 + 11x^2 - 7x - 6$$

Therefore:

$$p: \quad \pm 1, \pm 2, \pm 3, \pm 6$$
$$q: \quad \pm 1, \pm 2$$
$$\frac{p}{q}: \quad \pm 1, \pm 2, \pm 3, \pm 6, \pm \tfrac{1}{2}, \pm \tfrac{3}{2}$$

Hence, we must go through 12 iterations of $f\left(\frac{p}{q}\right)$ to find which one or ones lead to the result $f\left(\frac{p}{q}\right) = 0$.

The spreadsheet shown in Figure 28 is designed to run iterations for up to 22 values. If a polynomial requires more iterations, the "click-and-drag" technique can copy the formula down to checking as many numbers as it is needed. The sample Excel formula shown below comes from cell B3 of the worksheet. Basically, it is a single "IF" statement that is checking each value entered in column "A" as a potential *rational zero*.

```
=IF($C$3+$C$4*A3^$D$4+$C$5*A3^$D$5+$C$6*A3^$D$6+$C$7*A3^$D$7+$C$8*A3^$D$8+
$C$9*A3^$D$9+$C$10*A3^$D$10+$C$11*A3^$D$11+$C$12*A3^$D$12+$C$13*A3^$D$13+$
C$14*A3^$D$14+$C$15*A3^$D$15+$C$16*A3^$D$16+$C$17*A3^$D$17+$C$18*A3^$D$18+
$C$19*A3^$D$19+$C$20*A3^$D$20+$C$21*A3^$D$21+$C$22*A3^$D$22+$C$23*A3^$D$23
+$C$24*A3^$D$24=0,"Rational Root!","N/A")
```

It is clear from the output shown in column B of Figure 28 that the only *rational zeros* of the polynomial are: $1, -\frac{1}{2}$, and -6. Because a polynomial can only have as many total *zeros* of any kind as the degree of the polynomial, and because the polynomial in this example has degree 3, therefore, the three *rational zeros* that have been identified are all of the *zeros* of the polynomial.

	A	B	C	D
1	Possible Rational Zeros			
2	(Enter as X value)	Is it?	Coeff_Value	Power
3	1	Rational Root!	-6	N/A
4	-1	N/A	2	3
5	2	N/A	11	2
6	-2	N/A	-7	1
7	3	N/A		
8	-3	N/A		
9	6	N/A		
10	-6	Rational Root!		
11	0.5	N/A		
12	-0.5	Rational Root!		
13	1.5	N/A		
14	-1.5	N/A		
15		#NUM!		
16		#NUM!		
17		#NUM!		
18		#NUM!		
19		#NUM!		
20		#NUM!		
21		#NUM!		
22		#NUM!		
23		#NUM!		
24		#NUM!		

Figure 28. Excel spreadsheet designed to find the *rational zeros* of a polynomial. In this case, the output in column B is for the polynomial $f(x) = 2x^3 + 11x^2 - 7x - 6$. The coefficients and the power of the terms in the polynomial are shown in columns C and D, respectively.

The *Rational Zeros Theorem* leads to *exact zeros* of a polynomial as long as they are rational numbers. Thus, the method is limited because it does not find *irrational zeros*.

The *Bounds on Zeros* theorem and the *Intermediate Value Theorem*, however, may be used in conjunction to find *approximations of all real zeros* of a polynomial function. This approach is useful when *complex zeros* of a polynomial are not a concern.

Approximate real zeros of polynomials

The *Bounds on Zeros* theorem is an elegant method for determining the interval within which the *real zeros* of a polynomial may be found [8]. The method is based on using the coefficients of a polynomial while the leading coefficient is set to 1. Thus, in case the leading coefficient is not 1, coefficients must be factored such that the leading coefficient becomes 1.

To understand the method, suppose there exists a polynomial $f(x)$ whose leading coefficient is 1.

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad (15)$$

Then, taking the notation $Max\{\quad\}$ to mean "choose the largest entry in { },” consider:

$$Max\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\} \qquad (16)$$

$$and$$

$$1 + Max\{|a_0|, |a_1|, \ldots, |a_{n-1}|\} \qquad (17)$$

Then, $\pm$ of the smaller of the results obtained from Expressions 16 and 17 set the upper and lower boundaries for the interval of real numbers that contains the *real zeros* of the polynomial.

At the same time, the *Intermediate Value Theorem* states that for a polynomial $f(x)$, if two numbers a and b exist such that $a < b$ and $f(a)$ and $f(b)$ have opposite algebraic signs, then there is at least one real zero of the polynomial between a and b [8]. This means that sign changes of a polynomial evaluated over the interval found using the *Bounds on Zeros* theorem indicate the approximate locations of all the *real zeros* of the polynomial. Subsequent iterations over subintervals of decreasing size will be necessary to find reasonably accurate approximations of the *real zeros*. The following two examples demonstrate this.

Example #1: We would like to find all *real zeros* of the polynomial $f(x) = x^5 - 2x^2 - 3$.

Solution: Note that the leading coefficient of the polynomial is already 1. Thus, no factorization of the coefficient terms is necessary. Expressions 16 and 17 may be calculated using the coefficients:

$$a_0 = -3, \quad a_1 = 0, \quad a_2 = -2, \quad a_3 = 0, \quad a_4 = 0, \quad a_5 = 1$$

$$Max\{1, |-3| + |0| + |-2| + |0| + |0|\} = Max\{1,5\} = 5$$

$$1 + Max\{|-3|, |0|, |-2|, |0|, |0|\} = 1 + 3 = 4$$

The number 4 is the smaller of the two values calculated from applying Expressions 16 and 17. Therefore, all *real zeros* of the polynomial must be within the interval $[-4,\ 4]$.

The Excel segment in Figure 29 shows that in the first iteration, the value of the polynomial (cells B8:B18) changes only once over the interval $[-4,\ 4]$ of x values (cells A18:A18). This means that the polynomial has only one *real zero* and the other four zeros are all complex conjugate numbers. The additional iterations 2, 3, 4, and 5 were carried out to obtain a close approximation of the *real zero*. We can see that at $x = 1.49512$ (cell I17), the polynomial has value $f(1.49512) = 0.000258$, which is nearly zero. This result matches that obtained by Riley, Hobson, and Bence [7] using the *Rearrangement of Equation* method.

	A	B	C	D	E	F	G	H	I	J
1		1st_Iteration	2nd_Iteration	3rd_Iteration	4th_Iteration	5th_Iteration				
2	Lower_bound	-4	0.8	1.44	1.488	1.4944				
3	Upper_bound	4	1.6	1.52	1.496	1.4952				
4	Parsing Increment	0.8	0.08	0.008	0.0008	0.00008				
5										
6										
7	1st_ Iteration	f(x)_1	2nd_Iteration	f(x)_2	3rd_Iteration	f(x)_3	4th_Iteration	f(x)_4	5th_Iteration	f(x)_5
8	-4	-1059	0.8	-3.95232	1.44	-0.955463578	1.488	-0.1334667	1.4944	-0.01341
9	-3.2	-359.02432	0.88	-4.021068083	1.448	-0.827757223	1.4888	-0.1185988	1.49448	-0.01189
10	-2.4	-94.14624	0.96	-4.027827302	1.456	-0.696420694	1.4896	-0.1036911	1.49456	-0.01037
11	-1.6	-18.60576	1.04	-3.946547098	1.464	-0.561389223	1.4904	-0.0887438	1.49464	-0.00886
12	-0.8	-4.60768	1.12	-3.746458317	1.472	-0.422597327	1.4912	-0.0737566	1.49472	-0.00734
13	0	-3	1.2	-3.39168	1.48	-0.279978803	1.492	-0.0587295	1.4948	-0.00582
14	0.8	-3.95232	1.28	-2.840826163	1.488	-0.133466725	1.4928	-0.0436625	1.49488	-0.0043
15	1.6	2.36576	1.36	-2.046612582	1.496	0.017006562	1.4936	-0.0285554	1.49496	-0.00278
16	2.4	65.10624	1.44	-0.955463578	1.504	0.171509442	1.4944	-0.0134083	1.49504	-0.00126
17	3.2	312.06432	1.52	0.492881203	1.512	0.330111036	1.4952	0.00177904	**1.49512**	**0.000258**
18	4	989	1.6	2.36576	1.52	0.492881203	1.496	0.01700656	1.4952	0.001779

Figure 29. Excel spreadsheet showing the iterations needed for finding the *real zeros* of the polynomial $f(x) = x^5 - 2x^2 - 3$. Five iterations were carried out to approximate the *real zero* with reasonable accuracy. The lower and upper bounds for the first iteration were determined using the *Bounds on Zeros* theorem. The lower and upper bounds for the second through fifth iterations were selected based on where in the previous iteration the polynomial changed sign.

Figures 30 and 31 give details of the sample Excel formulas that were used to run the five iterations shown in Figure 29.

	A	B
1		1st_Iteration
2	Lower_bound	-4
3	Upper_bound	4
4	Parsing Increment	=IF(AND(B2<0,B3<0),(B3-B2)/10,IF(B2<0,(ABS(B2)+ABS(B3))/10,(B3-B2)/10))

Figure 30. Sample Excel segment showing the calculation of the subinterval within each *Bounds on Zeros* interval. Iterations for finding the *real zero*s of the polynomial $f(x) = x^5 - 2x^2 - 3$ were carried out on the intervals: [-4, 4], [0.8, 1.6], [1.44, 1.52], [1.488, 1.496], [1.4944, 1.4952] as shown in Figure 29. Each interval was split into 10 equal subintervals. The formula in cell B4 shown in this segment was then simply copied over to cells C4, D4, E4, F4 in order to establish the width of each of the subintervals.

	A	B
7	**1st_ Iteration**	**f(x)_1**
8	=$B$2	=A8^5-2*A8^2-3
9	=IF(A8<0,IF(A8+$B$4>-0.0001,0,A8+$B$4),A8+$B$4)	=A9^5-2*A9^2-3
10	=IF(A9<0,IF(A9+$B$4>-0.0001,0,A9+$B$4),A9+$B$4)	=A10^5-2*A10^2-3
11	=IF(A10<0,IF(A10+$B$4>-0.0001,0,A10+$B$4),A10+$B$4)	=A11^5-2*A11^2-3
12	=IF(A11<0,IF(A11+$B$4>-0.0001,0,A11+$B$4),A11+$B$4)	=A12^5-2*A12^2-3
13	=IF(A12<0,IF(A12+$B$4>-0.0001,0,A12+$B$4),A12+$B$4)	=A13^5-2*A13^2-3
14	=IF(A13<0,IF(A13+$B$4>-0.0001,0,A13+$B$4),A13+$B$4)	=A14^5-2*A14^2-3
15	=IF(A14<0,IF(A14+$B$4>-0.0001,0,A14+$B$4),A14+$B$4)	=A15^5-2*A15^2-3
16	=IF(A15<0,IF(A15+$B$4>-0.0001,0,A15+$B$4),A15+$B$4)	=A16^5-2*A16^2-3
17	=IF(A16<0,IF(A16+$B$4>-0.0001,0,A16+$B$4),A16+$B$4)	=A17^5-2*A17^2-3
18	=IF(A17<0,IF(A17+$B$4>-0.0001,0,A17+$B$4),A17+$B$4)	=A18^5-2*A18^2-3

Figure 31. Sample Excel formulas for the first iteration to find the *real zero* of the polynomial $f(x) = x^5 - 2x^2 - 3$. Column "1st_Iteration" is calculating the 10 subintervals based on the provided lower and upper limits for the main interval. In this case, that would be the interval [-4, 4]. Column "f(x)_1" evaluates the polynomial at values from column "1st_iteration." It is the sign change in column "f(x)_1" that determines the lower and upper limits of the interval over which the next set of iterations of the polynomial will be run.

Example #2: We would like to find all *real zeros* of the polynomial $f(x) = 4x^5 - 2x^3 + 2x^2 + 1$.

Solution: Note that the leading coefficient of the polynomial is 4. This must be converted to 1 before *Bounds on Zeros* and the *Intermediate Value Theorem* can be applied. Factoring the coefficients can achieve this.

$$f(x) = 4x^5 - 2x^3 + 2x^2 + 1 = 4\left(x^5 - \frac{1}{2}x^3 + \frac{1}{2}x^2 + \frac{1}{4}\right)$$

The leading coefficient of the factorized polynomial is now equal to 1, and it is in the correct format for the application of the *Bounds on Zeros* theorem.

$$a_0 = \frac{1}{4}, \quad a_1 = 0, \quad a_2 = \frac{1}{2}, \quad a_3 = -\frac{1}{2}, \quad a_4 = 0, \quad a_5 = 1$$

$$Max\left\{1, \left|\frac{1}{4}\right| + |0| + \left|\frac{1}{2}\right| + \left|-\frac{1}{2}\right| + |0|\right\} = Max\left\{1, \frac{5}{4}\right\} = \frac{5}{4}$$

$$1 + Max\left\{\left|\frac{1}{4}\right|, |0|, \left|\frac{1}{2}\right|, \left|-\frac{1}{2}\right|, |0|\right\} = 1 + \frac{1}{2} = \frac{3}{2}$$

The number $\frac{5}{4}$ is the smaller of the two values calculated from Expressions 16 and 17. Then, all *real zeros* of the polynomial must be within the interval $\left[-\frac{5}{4}, \frac{5}{4}\right]$ or $[-1.25, 1.25]$.

The Excel segment in Figure 32 shows that in the first iteration, the algebraic sign of the polynomial changes only once within the interval $[-1.25, 1.25]$ of x values. This means that the polynomial has only one *real zero* and the other four zeros must be complex conjugates. The additional iterations 2, 3, 4, and 5 were carried out to obtain a close approximation of *the real zero*. We can see that at $x = -1.078425$, the polynomial has value $f(-1.078425) = -0.00016$, which is nearly zero. This yields $(-1.078425, -0.00016)$ a satisfactory *approximation* of the *real zero* of the polynomial $f(x) = 4x^5 - 2x^3 + 2x^2 + 1$. Higher accuracy of the result may be obtained by increasing the number of iterations.

	A	B	C	D	E	F	G	H	I	J
1		1st_Iteration	2nd_Iteration	3rd_Iteration	4th_Iteration	5th_Iteration				
2	Lower_bound	-1.25	-1.25	-1.1	-1.08	-1.0785				
3	Upper_bound	1.25	-1	-1.075	-1.0775	-1.07825				
4	Parsing Increment	0.25	0.025	0.0025	0.00025	0.000025				
5										
6										
7	1st_ Iteration	f(x)_1	2nd_Iteration	f(x)_2	3rd_Iteration	f(x)_3	4th_Iteration	f(x)_4	5th_Iteration	f(x)_5
8	-1.25	-4.17578125	-1.25	-4.17578125	-1.1	-0.36004	-1.08	-0.0250883	-1.0785	-0.00135
9	-1	1	-1.225	-3.356408164	-1.0975	-0.316263276	-1.07975	-0.0211181	-1.078475	-0.00095
10	-0.75	2.01953125	-1.2	-2.61728	-1.095	-0.273040214	-1.0795	-0.0171531	-1.07845	-0.00056
11	-0.5	1.625	-1.175	-1.953070586	-1.0925	-0.230366495	-1.07925	-0.0131933	**-1.078425**	**-0.00016**
12	-0.25	1.15234375	-1.15	-1.35867875	-1.09	-0.18823782	-1.079	-0.0092388	-1.0784	0.000231
13	0	1	-1.125	-0.829223633	-1.0875	-0.146649911	-1.07875	-0.0052895	-1.078375	0.000625
14	0.25	1.09765625	-1.1	-0.36004	-1.085	-0.105598511	-1.0785	-0.0013454	-1.07835	0.001019
15	0.5	1.375	-1.075	0.053326445	-1.0825	-0.065079382	-1.07825	0.0025935	-1.078325	0.001412
16	0.75	2.23046875	-1.05	0.41512375	-1.08	-0.025088307	-1.078	0.00652717	-1.0783	0.001806
17	1	5	-1.025	0.729398398	-1.0775	0.01437891	-1.07775	0.01045564	-1.078275	0.0022
18	1.25	12.42578125	-1	1	-1.075	0.053326445	-1.0775	0.01437891	-1.07825	0.002594

Figure 32. Excel spreadsheet showing the iterations needed for finding the *real zeros* of the polynomial $f(x) = 4x^5 - 2x^3 + 2x^2 + 1$. Five iterations were carried out to approximate the *real zero* with reasonable near zero accuracy.

Both polynomials discussed had only one *real zero*. In case more *real zeros* are identified within the initial interval, a separate set of iterations will need to be carried out for each instance where the polynomial changes sign over a particular subinterval.

Conclusion

Functions have been part of the school mathematics in K-12 and college mathematics curricula for over a century. For at least as long, mathematics teachers have looked for better ways of teaching the content so that all of their students understand it. Recently, the Common Core Sate Standards for Mathematics [4] have renewed the call for improvement of mathematics education. A large portion of the Standards focuses on how functions ought to be taught for better understanding of all students. The Common Core State Standards for Mathematics urge the use of spreadsheets in various capacities ranging from mathematical modeling to experimentation with functions and numerical methods. Indeed, the properties of spreadsheets make it possible to engage students with the kind of abstract mathematical thinking that makes understanding functions inevitable. Not to mention the strategic benefits that students will reap from gathering such skills later in life as professionals.

This paper intended to offer practical spreadsheet-based ideas for implementing a wide range of recommendations that have been outlined and discussed with respect to teaching functions in the Common Core State Standards for Mathematics.

References

[1] Abouchedid, K., & Nasser, R. (2000). *The Role of Presentation and Response Format in Understanding, Preconceptions and Alternative Concepts in Algebra Problems*. Zouk Mosbeh, Lebanon: Notre Dame University, Lebanon.

[2] Alagic, M., & Palenz, D. (2006). Teachers explore linear and exponential growth: Spreadsheets as cognitive tools. *Journal of Technology and Teacher Education*, 14(3), 633 – 649.

[3] Clement, J., Lochhead, J., & Monk, G. S. (1981). Translation Difficulties in Learning Mathematics. *American Mathematical Monthly*, 88(4), 287 – 290.

[4] Common Core State Standards for Mathematics (2013). Retrieved March 12, 2015 from http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

[5] Kaput, J. (1992). Technology and mathematics education. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning (pp. 515-556)*. Reston, VA, USA: National Council of Teachers of Mathematics.

[6] Logsdon, M. I. (1935). *A mathematician explains*. Chicago, IL: University of Chicago Press.

[7] Riley, K.F.; Hobson, M.P. & Bence, S.J. (2008). *Mathematical methods for physics and engineering (3rd edition)*. New York: Cambridge University Press.

[8] Sullivan, M. (2008). *Algebra and trigonometry (8th edition)*. Upper Saddle River, New Jersey: Pearson Prentice Hall.