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Harmonic Numbers: Insights, Approximations and Applications

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Abstract
The applications of spreadsheets to mathematics allow learners to analyze mathematical ideas in ways never before possible. In this paper harmonic numbers are defined, calculated, approximated, and applied. The concepts and techniques of recursion, approximation and combinatorial methods are illustrated. This paper also shows some properties of the harmonic series including divergence and its part in the determination of harmonic numbers. Analyzing the ideas of harmonic numbers presents to learners that one concept has many connections to others. Mathematical concepts and relationships between harmonic numbers, harmonic series, the Euler-Mascheroni constant and the harmonic mean are explored. Finding expressions in closed-form is not always possible and technology provides the capability to approximate the harmonic numbers.

Exploration and experimentation are discussed. With spreadsheets, complex calculations can be simplified, intuition can be developed and applications can be studied.

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Harmonic Series, Harmonic Numbers, Series Approximations, Euler-Mascheroni Constant and EXCEL Spreadsheets.

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1. Introduction

Instead of conducting rigorous proofs and performing tedious calculations this paper demonstrates accurate results through approximations with spreadsheets.

The concept that the harmonic series diverges is well known. Any calculus textbook has at least one proof [7]. This series diverges so slowly that relationships can be studied and various characteristics can be observed. Any person, student or professor that learns calculus or teaches calculus discovers the harmonic series. Figure 1 shows that the harmonic series diverges.

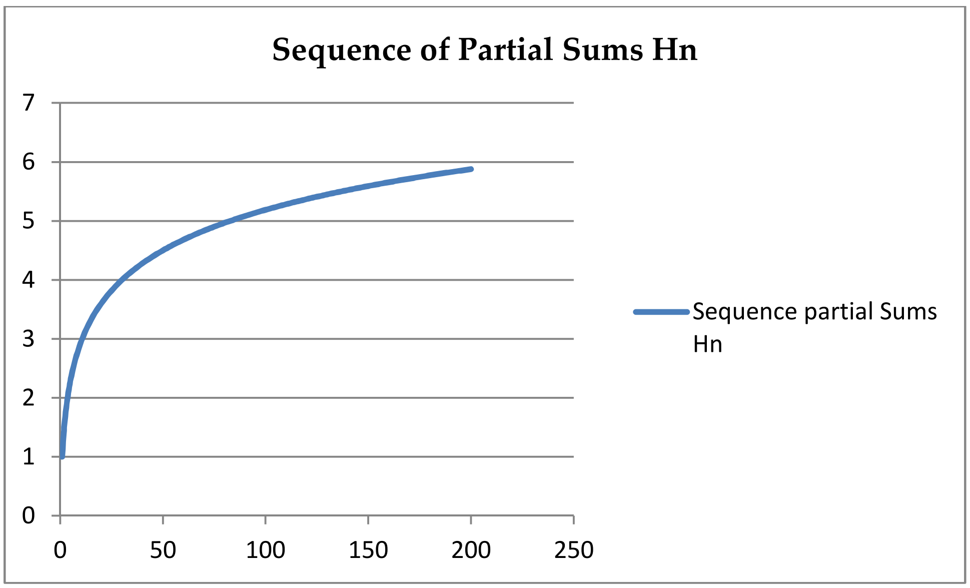


Figure 1: Sequence of Partial Sums Diverges

The definition for harmonic series is:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

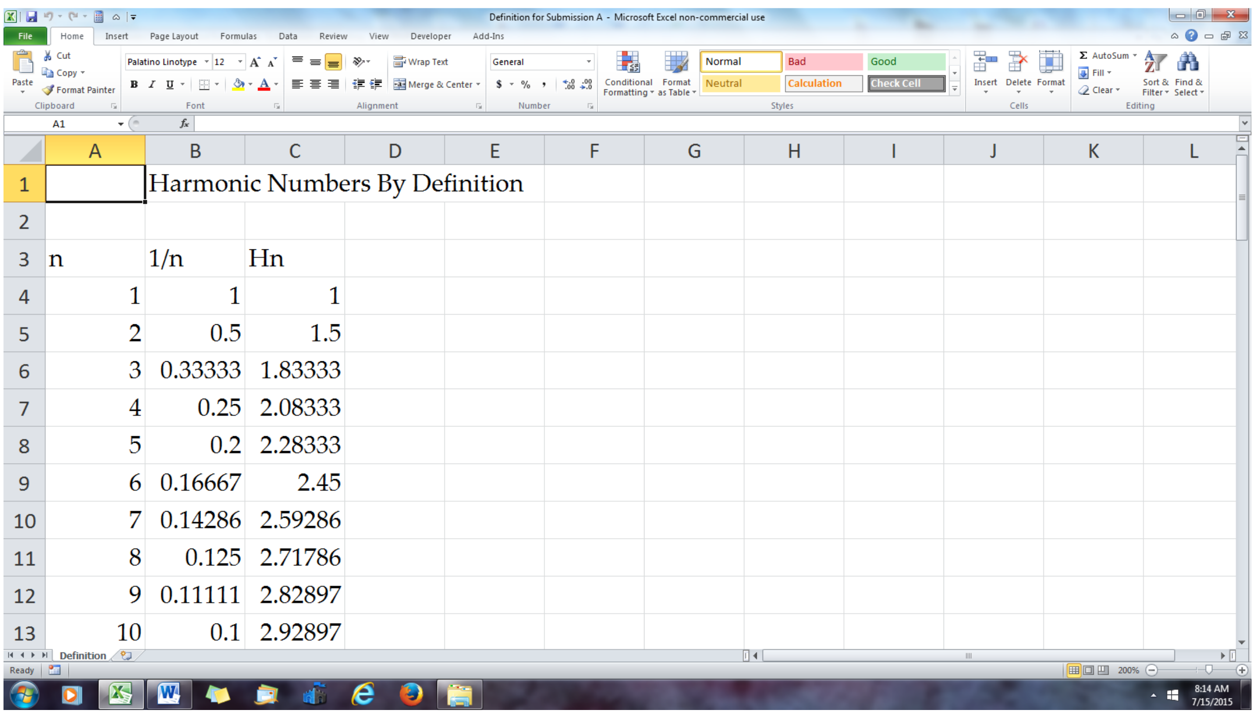
The focus of this paper is on showing that the harmonic numbers $H_n = \sum_1^n \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \frac{1}{n}$ can be calculated in a variety of different ways including direct calculations, harmonic means, recursion, combinations, and approximations. Analyzing concepts in different ways enlightens students to the understanding that mathematics is more than learning one idea. Harmonic numbers form the sequence: $H_1 = 1$; $H_2 = 1 + \frac{1}{2}$; $H_3 = 1 + \frac{1}{2} + \frac{1}{3}$; and so on. This sequence has many properties and relationships found in diverse areas of study including physics, music, business, and theoretical mathematics.

2. Harmonic Numbers by Definition

Harmonic numbers are sequences of partial sums of the harmonic series [7]. For example the ninth harmonic number

$$H_9 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \text{ is } 2.82897$$

The harmonic numbers can be calculated in EXCEL. The reciprocals are found first and then these terms are added. The ninth harmonic number H_9 for example is 2.82897. Due to the limitations of technology and specifically for EXCEL these sums become impossible for large n. The harmonic numbers are displayed in Figure 2.



Harmonic Numbers By Definition

n	1/n	Hn
1	1	1
2	0.5	1.5
3	0.33333	1.83333
4	0.25	2.08333
5	0.2	2.28333
6	0.16667	2.45
7	0.14286	2.59286
8	0.125	2.71786
9	0.11111	2.82897
10	0.1	2.92897

Figure 2: Harmonic Numbers by Definition

For the results displayed in Figure 2 enter the following: In column A enter 1 in cell A4 and =A4+1 in A5. In column B enter reciprocals of the values in column A starting with =1/A4. In column C sum the values by entering =SUM($B$4..B4) for the first harmonic number and copy>paste these values down to the desired number of harmonic numbers.

Technology can calculate directly and automatically and helps in understanding relationships and concepts.

3. Harmonic Numbers by Macro Programs

The calculations become impossible for finding any nth harmonic number for large n say 1000. The user would have to copy>paste down each of the columns until n is 1000. To find the nth harmonic number a macro can be written. A macro, or computer program using Visual Basic for Applications (VBA) automates the calculation process and allows the learner to find any harmonic number. VBA is an add-in for EXCEL. In order to use it the add-in has to be installed. (Details [3] for installing VBA and writing macros are available at https://msdn.microsoft.com/en-us/library/office/ee814737(v=office.14).aspx).)

The macro syntax for the calculation of the nth harmonic number is:

```
Sub harmonic()
Dim n As Variant
Dim t As Variant
Dim m As Variant
m = InputBox("Enter the number of terms desired")
For n = 1 to m
t = t + 1 / n
Next n
MsgBox ("For the first " & m & " terms of the harmonic series, the total is " & t)
End Sub
```

In developing a macro a subroutine (title) is written then the variables used must be dimensioned. The variables used are n (counter), t, and m (the number of the term desired). Then an InputBox is shown requesting the number of terms of the harmonic series the user wants. Entering a value and clicking ok reveals an MsgBox with the calculated nth harmonic number. The macro concludes with an End Sub statement.

The figures below show two interactive boxes one requesting the number of terms desired and the second box providing the nth harmonic number. The InputBox is displayed in Figure 3. The output from the MsgBox is displayed in Figure 4.

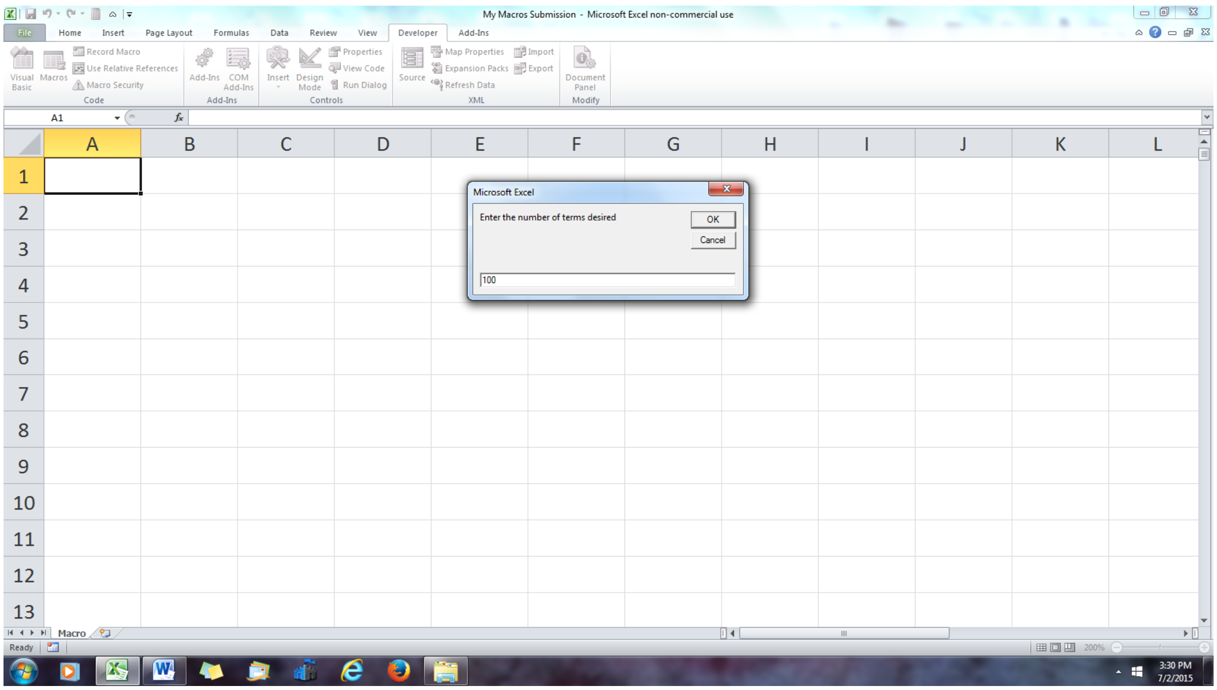


Figure 3: InputBox

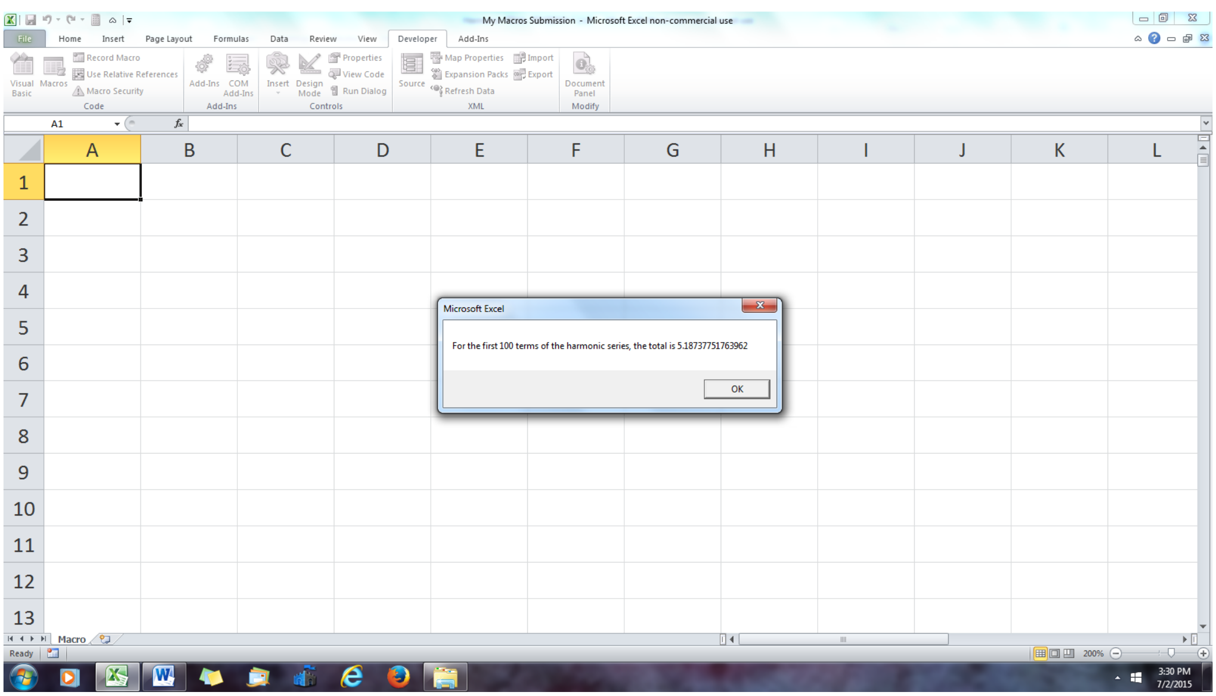


Figure 4: MsgBox

In Figure 3, the InputBox asks the user to enter the desired nth harmonic number. The number entered is 100. In Figure 4, the 100^{th} harmonic number is 5.18738.

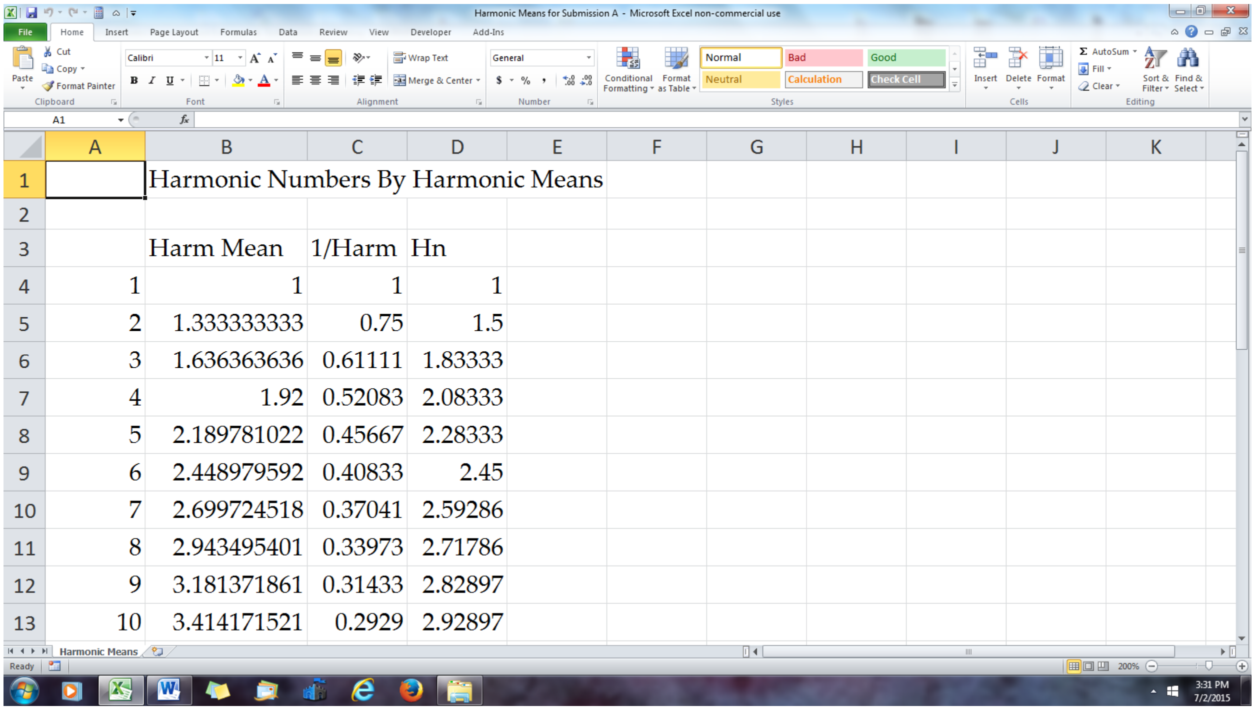
4. Harmonic Numbers by Harmonic Means

Another technique for calculating the harmonic numbers is based on applying harmonic means. The harmonic mean of a set of numbers is defined as the reciprocal of the arithmetic mean of the reciprocals. For a set of values x_i's the harmonic mean is

$$\frac{n}{1+\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}+\cdots.\frac{1}{x_n}}.$$

For example to find the harmonic mean of 1, 4, 6, 9, calculate: $\frac{4}{1+\frac{1}{4}+\frac{1}{6}+\frac{1}{9}}$. In EXCEL the syntax for the HARMEAN function is =HARMEAN(1,4,6,9). The result for the example is 0.2.

Any harmonic number is equal to n times the inverse or reciprocal of the harmonic mean of these natural numbers. Since the harmonic mean is the number of items n divided by the sum of the reciprocal numbers, the sum of the numbers is equal to n times 1/(harmonic mean). For example the ninth harmonic number is 9 times 1/harmonic mean of $(1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{7},\frac{1}{8},\frac{1}{9})$. The harmonic numbers using harmonic means are shown in Figure 5. The ninth harmonic number is 2.82897.



	A	B	C	D
1		Harmonic Numbers By Harmonic Means		
2				
3		Harm Mean	1/Harm	Hn
4	1	1	1	1
5	2	1.333333333	0.75	1.5
6	3	1.636363636	0.61111	1.83333
7	4	1.92	0.52083	2.08333
8	5	2.189781022	0.45667	2.28333
9	6	2.448979592	0.40833	2.45
10	7	2.699724518	0.37041	2.59286
11	8	2.943495401	0.33973	2.71786
12	9	3.181371861	0.31433	2.82897
13	10	3.414171521	0.2929	2.92897

Figure 5: Harmonic Numbers by Harmonic Means

The EXCEL formulas for finding the harmonic numbers by harmonic means are: In cell A4 enter 1 and then in A5 enter =A4+1. In cell B4 enter =HARMEAN($B$4..B4). In column C enter 1/the harmonic mean of the numbers in the previous column or starting with cell C4 =1/B4. In column D enter the product of n (column A) times 1/harmonic mean (column C). The harmonic numbers are shown in column D. The harmonic numbers calculated by harmonic means are the same as by definition (Figure 2).

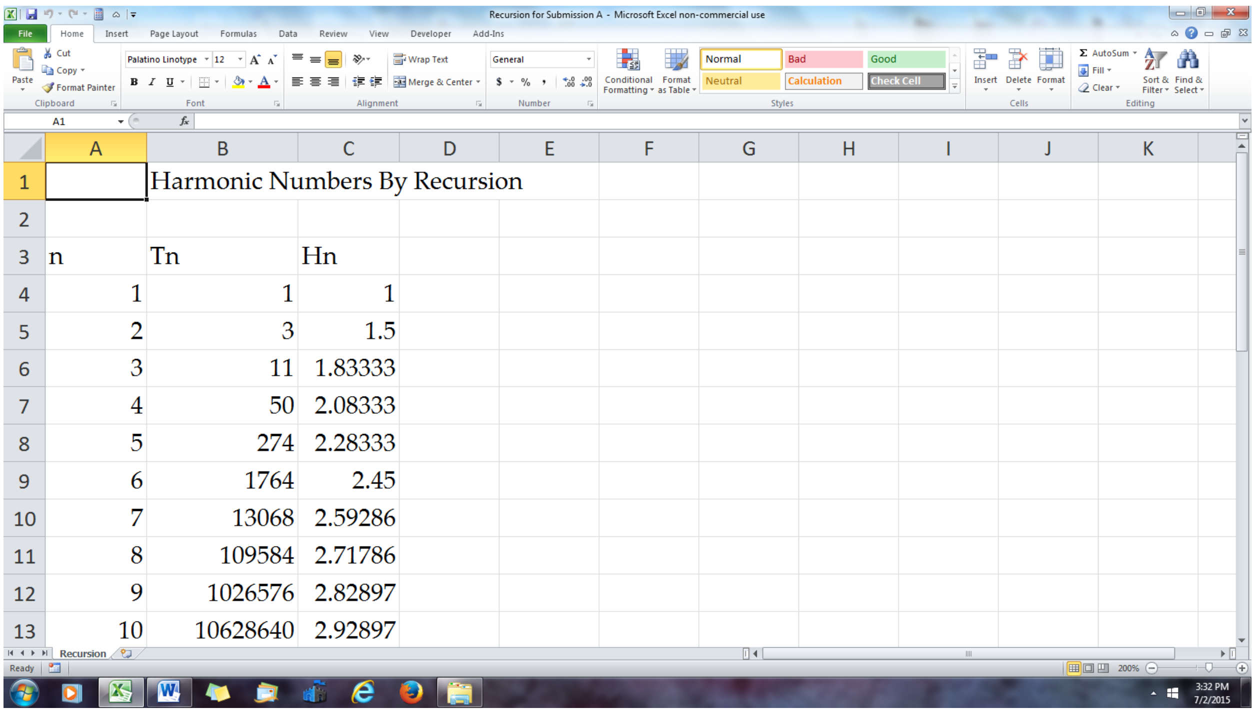
5. Harmonic Numbers by Recursion

There is no elementary equation or function for finding the nth harmonic number. Other ways such as recursion can be used to calculate the nth harmonic number.

Recursion is a technique where the next calculation or calculations are based on one two or more previous calculations. The Fibonacci numbers are an example. For the Fibonacci numbers recursion is done as follows: consider 1 and 1.The next term in the sequence is 2 and the next term is 3 and 5 and so forth. The terms in the sequence are found by adding the two previous terms.

The following steps are used for calculating harmonic numbers using recursion [6]. These steps are implemented in EXCEL and illustrated in Figure 6.

1) Define the harmonic numbers as: $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.
2) Calculate the ratio: $H_n = \frac{T_n}{n!}$, where $T_n = 1$ *when* $n = 1$ *and* $T_n = nT_{n-1} + (n-1)!$ *otherwise*.



	A	B	C
1		Harmonic Numbers By Recursion	
2			
3	n	Tn	Hn
4	1	1	1
5	2	3	1.5
6	3	11	1.83333
7	4	50	2.08333
8	5	274	2.28333
9	6	1764	2.45
10	7	13068	2.59286
11	8	109584	2.71786
12	9	1026576	2.82897
13	10	10628640	2.92897

Figure 6: Harmonic Numbers by Recursion

The EXCEL formulas are: In cell A4 enter 1 and in cell A5 enter =A4+1. In cell B4 enter 1 then in cell B5 enter =A5*B4+FACT(A5-1) for the T_n terms. In cell C4 enter =B4/FACT(A4) for the H_n terms and copy>paste down these values to the desired number of terms. The FACT function in EXCEL finds factorials for an integer value. Although the EXCEL function FACT has limitations on the largest factorial that can be calculated, this technique shows that harmonic numbers can be determined by using recursion.

6. Harmonic Numbers by Euler's Integral

Euler showed how to find harmonic numbers by using integral calculus [6]. He proved that the nth harmonic number is defined as

$$H_n = \int_0^1 \frac{1 - x^n}{1 - x}\, dx$$

For example, to find H_9, evaluate the integral

$$H_9 = \int_0^1 \frac{1 - x^9}{1 - x}\, dx$$

The expression $\frac{1-x^9}{1-x} = 1 + x + x^2 + x^3 + \cdots x^8$. The integration result is 2.82897. In general the fraction $\frac{1-x^n}{1-x}$ reduces to$1 + x + x^2 + x^3 + \cdots x^{n-1}$. The integral is evaluated using the fundamental theorem of calculus.

7. Harmonic Numbers by Combinations

Harmonic numbers can also be determined by applying [6]:

$$H_n = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k} \binom{n}{k}$$

where $\binom{n}{k}$ is the combination of n things taken k at a time and is found by calculating

$$C_k^n = \frac{n!}{k!\,(n-k)!}.$$

Applying this formula the results are shown in Figure 7.

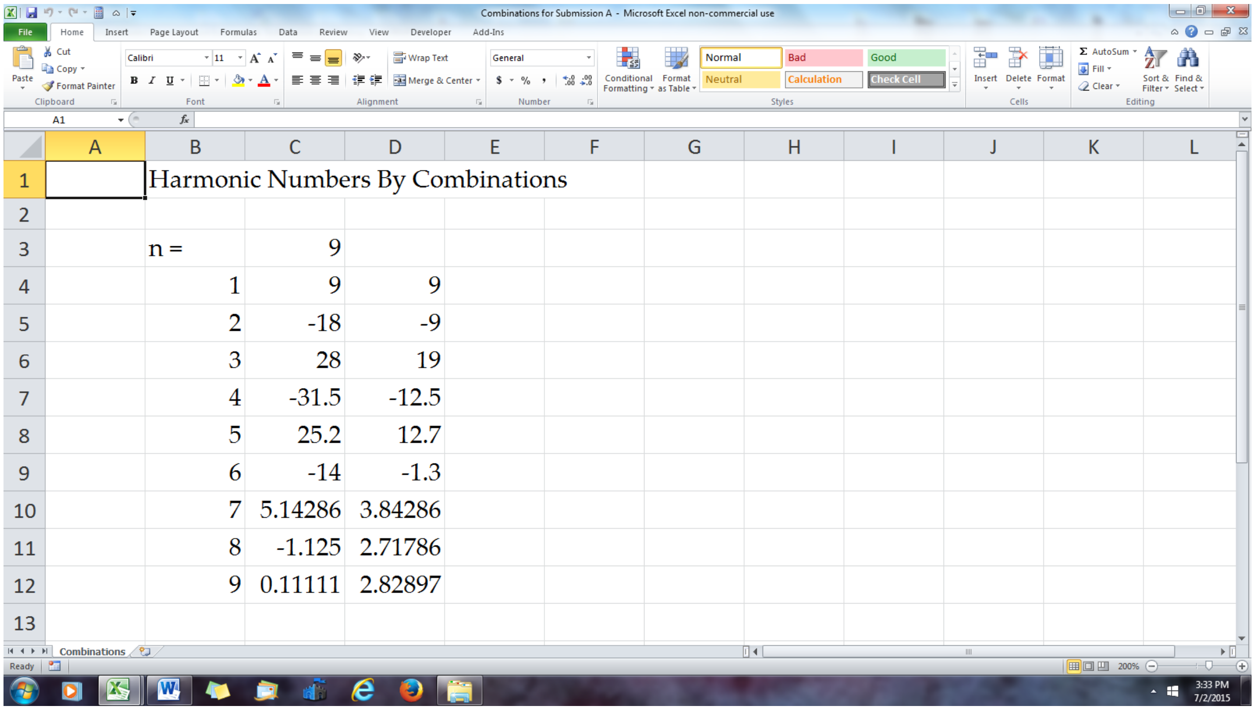


Figure 7: Harmonic Numbers by Combinations

In EXCEL the cell formulas for calculating the ninth harmonic number are: In column B enter 1 in B4 and then =B4+1 in B5. In cell C4 enter =(-1^(B4-1)/B4)*COMBIN($C$3,B4) for the $(-1)^{k-1}\frac{1}{k}\binom{n}{k}$ terms. In cell D4 enter=SUM($D4$...D4) to calculate the sums. Copy>paste these cells B4, C4 and D4 down to B12, C12 and D12. The results are shown in Figure 7. The COMBIN function calculates the combination of n things taken r at a time. Although EXCEL has limitations on calculations, the application of the COMBIN function shows that combinations can determine harmonic numbers.

The disadvantage of finding harmonic numbers using combinations is that to find one harmonic number for example, the ninth harmonic number, all the calculations displayed in Figure 7 are required.

8. Harmonic Numbers by Euler's Approximation

Euler [6] showed that the nth harmonic number H_n can be approximated by:

$$\ln(n) + \frac{1}{2n} + 0.57722$$

where ln(n) is the natural logarithm and 0.57722 is the gamma or the Euler-Mascheroni constant.

For n = 9, the calculation is ln(9) + $\frac{1}{18}$ + 0.57722= 2.82999. The value by definition is 2.82897 (Figure 2). Euler's approximation is displayed in Figure 8 for harmonic numbers 1 to 9.

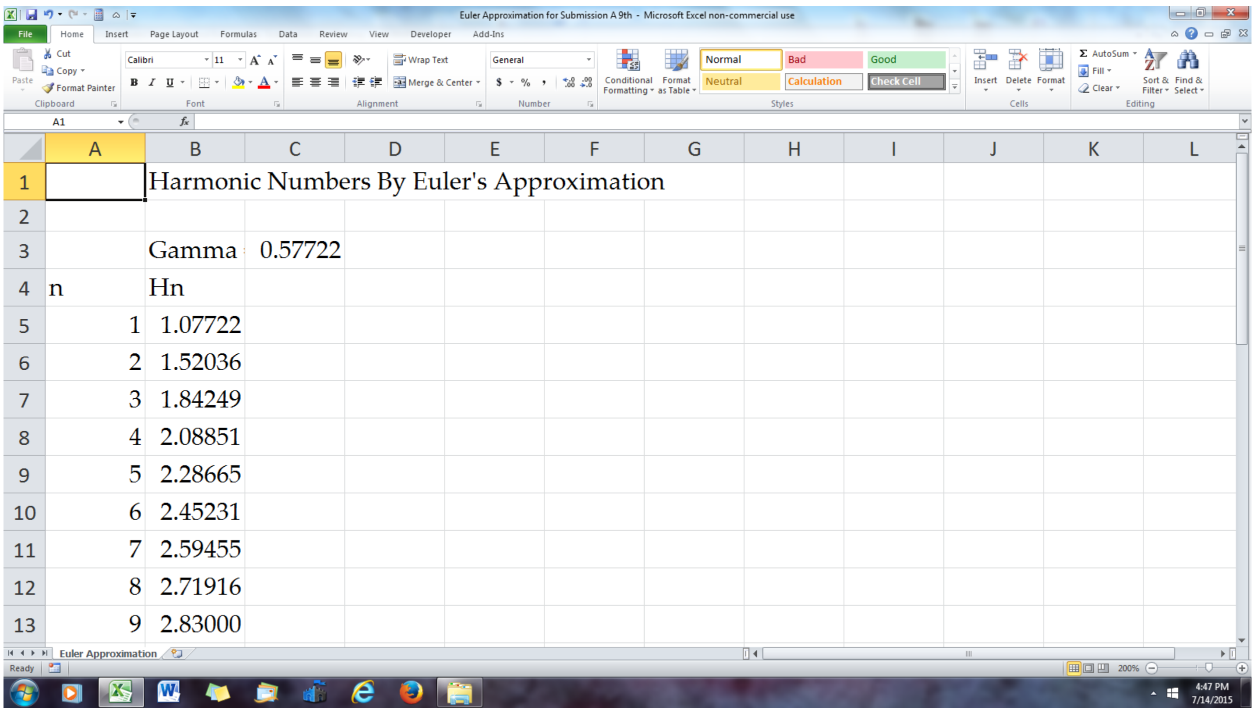


Figure 8: Harmonic Numbers by Euler's Approximation

In EXCEL the formulas are: In cell A5, enter 1 then in A6 enter =A5+1. In cell B5 enter =LN(A5)+1/(2*A5)+$C$3 and copy>paste these values down to the desired harmonic numbers. In cell C3 enter gamma 0.57722.

9. Harmonic Numbers by the Digamma Function

The harmonic numbers can also be found by applying the digamma or psi function. The digamma function is defined as

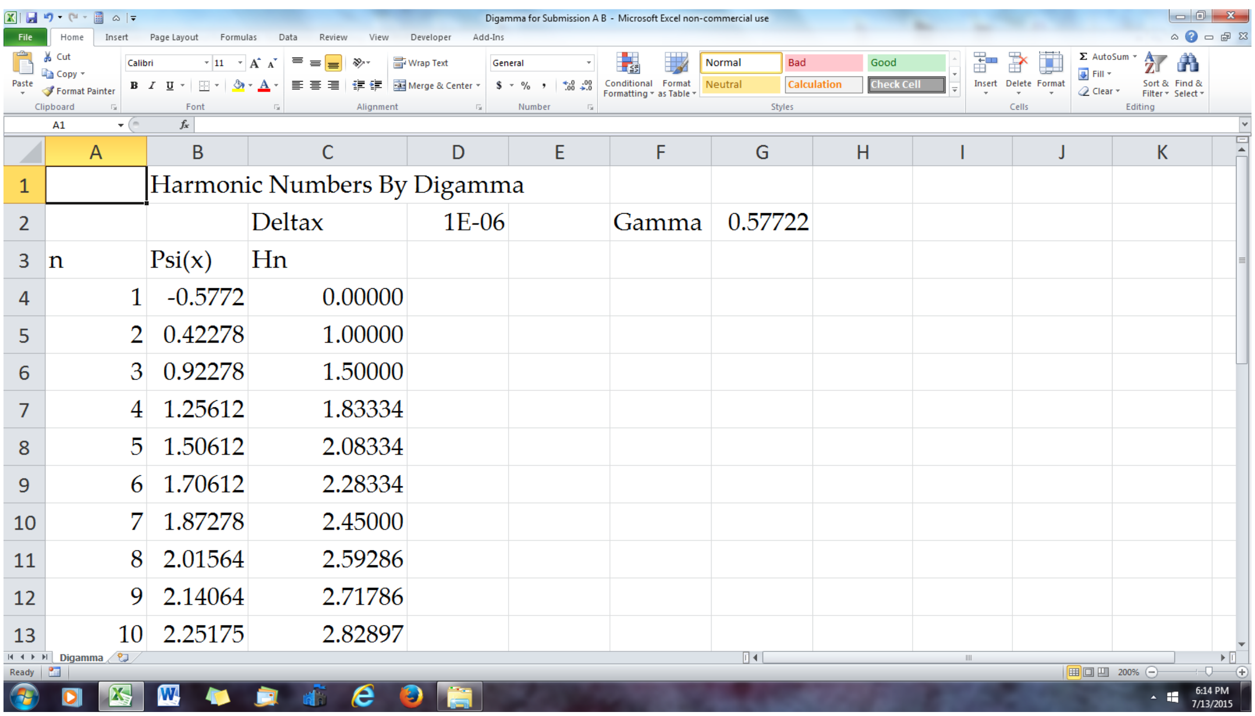
$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

where Γ is the gamma function and Γ′ is the derivative of the gamma function. There is no syntax in EXCEL for the digamma function. A numerical approximation can be made by using the definition of numerical differentiation where delta x is small. The following is a numerical differentiation approximation [11]:

$$\psi(x) = \frac{dln\big(\Gamma(x)\big)}{dx} = \frac{gammaln(x + delta\ x) - gammaln(x - delta\ x)}{2\ delta\ x}$$

The relationship between the harmonic numbers and the digamma function is defined as $\psi(n) = H_{n-1} - \gamma$ or $H_{n-1} = \psi(n) + \gamma$. That is the n-1st harmonic number is equal to the sum of the digamma function of n plus the Euler-Mascheroni constant. Figure 9 illustrates this approximation for the psi function and the n-1st harmonic number. The numerical derivative also can be approximated by:

$$\psi(x) = \frac{dln\big(\Gamma(x)\big)}{dx} = \frac{gammaln(x + delta\ x) - gammaln(x)}{delta\ x}$$



	A	B	C	D	E	F	G
1		Harmonic Numbers By Digamma					
2			Deltax	1E-06		Gamma	0.57722
3	n	Psi(x)	Hn				
4	1	-0.5772	0.00000				
5	2	0.42278	1.00000				
6	3	0.92278	1.50000				
7	4	1.25612	1.83334				
8	5	1.50612	2.08334				
9	6	1.70612	2.28334				
10	7	1.87278	2.45000				
11	8	2.01564	2.59286				
12	9	2.14064	2.71786				
13	10	2.25175	2.82897				

Figure 9: Harmonic Numbers by the Digamma Function

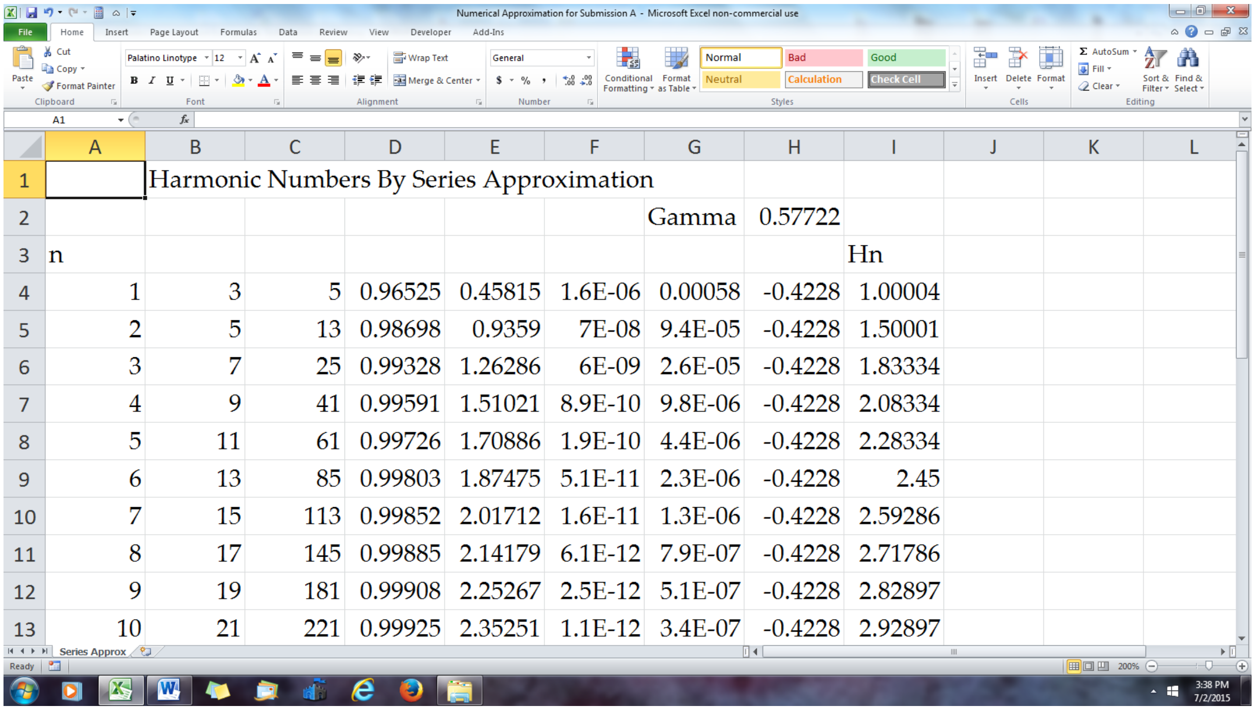
In cell A4 enter 1 and in A5 enter =A4+1. In cell B4, the numerical derivative is entered as =(GAMMALN(A4+$D$2)-GAMMALN(A4-$D$2))/(2*$D$2). In cell C4 the expression $H_{n-1} = \psi(n) + \gamma$ is calculated using =B4+$G$2. The EXCEL syntax GAMMALN(x) represents LN(Γ(x)). Copy>paste these cells down to the desired number of harmonic numbers.

10. Harmonic Numbers by Series Approximation

Since the harmonic numbers have no closed-form elementary expression and for large n the harmonic numbers cannot be found, approximations are possible. Consider the approximation [1]:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \approx (2n+1)tan^{-1}\left(\frac{1}{2n+1}\right) + \frac{1}{2}\ln\left(\frac{2n^2+2n+1}{2}\right) + \frac{56n^6+168n^5+140n^4-42n^2-14n-1}{2520(2n^2+2n+1)^7} - \frac{6n^2+6n+1}{180(2n^2+2n+1)^3} + \gamma - 1$$

γ is the Euler-Mascheroni constant, tan^{-1} is the inverse tangent function and ln is the natural logarithm function. Applying this expression in EXCEL is possible by breaking this expression into parts. Combining these parts then provides the results shown in Figure 10.



	A	B	C	D	E	F	G	H	I
1		Harmonic Numbers By Series Approximation							
2							Gamma	0.57722	
3	n								Hn
4	1	3	5	0.96525	0.45815	1.6E-06	0.00058	-0.4228	1.00004
5	2	5	13	0.98698	0.9359	7E-08	9.4E-05	-0.4228	1.50001
6	3	7	25	0.99328	1.26286	6E-09	2.6E-05	-0.4228	1.83334
7	4	9	41	0.99591	1.51021	8.9E-10	9.8E-06	-0.4228	2.08334
8	5	11	61	0.99726	1.70886	1.9E-10	4.4E-06	-0.4228	2.28334
9	6	13	85	0.99803	1.87475	5.1E-11	2.3E-06	-0.4228	2.45
10	7	15	113	0.99852	2.01712	1.6E-11	1.3E-06	-0.4228	2.59286
11	8	17	145	0.99885	2.14179	6.1E-12	7.9E-07	-0.4228	2.71786
12	9	19	181	0.99908	2.25267	2.5E-12	5.1E-07	-0.4228	2.82897
13	10	21	221	0.99925	2.35251	1.1E-12	3.4E-07	-0.4228	2.92897

Figure 10: Harmonic Numbers by Series Approximation

The EXCEL formulas in parts for this spreadsheet are:

In cell A4 enter 1
In cell B4 enter =2*A4+1
In cell C4 enter =2*A4^2+2*A4+1
In cell D4 enter =B4*ATAN(1/B4)
In cell E4 enter =0.5*LN(C4/2)
In cell F4 enter =(56*A4^6+168*A4^5+140*A4^4-42*A4^2-14*A4-1)/(2520*(C4^7))
In cell G4 enter =(6*A4^2+6*A4+1)/(180*(C4^3))
In cell H4 enter =$H$2-1 ($H$2 is gamma)
In cell I4 enter =D4+E4+F4-G4+H4
Then in cell A5 enter =A4+1.
Copy>paste these cell formulas down the spreadsheet to the desired number of harmonic numbers.

By definition the ninth harmonic number is 2.82897and by the series approximation the ninth harmonic number is 2.82897.

If only one harmonic number is desired start anywhere on a spreadsheet. Finding the 33rd harmonic number for example is determined by entering 33 anywhere on the spreadsheet. Start with cell A4 enter 33 then apply the details for one row above. The 33rd harmonic number H_{33} is 4.08879.

Some of the applications of the harmonic numbers are provided below.

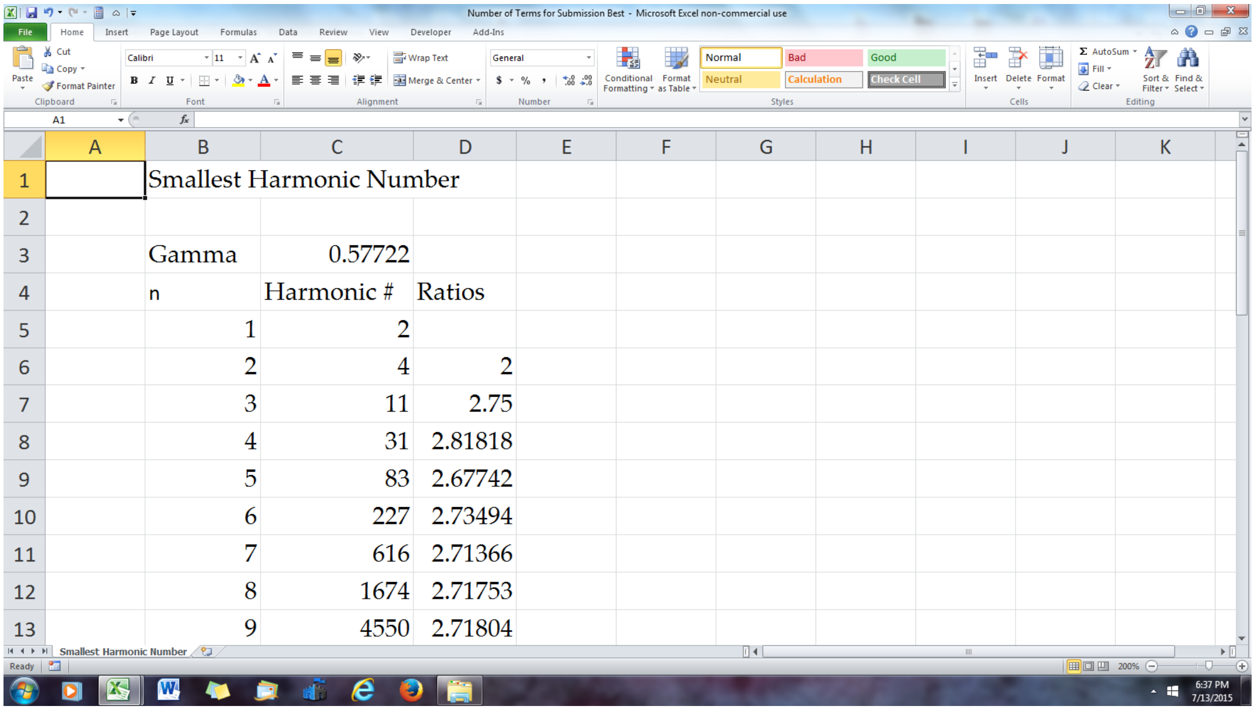
11. The Smallest Harmonic Number That Exceeds a Specific Integer

The harmonic series diverges as shown in Figure 1. Since the harmonic series diverges so slowly, finding the least number of terms of the harmonic series that exceeds a specific integer or the smallest harmonic number that exceeds that integer is a common application [2], [8].

An estimate for the number of terms of the harmonic series needed to exceed an integer or the smallest harmonic number that exceeds a specific integer is the greatest integer function of $[e^{a-\gamma} + 1/2]$ where a is the integer to be exceeded and γ is the Euler-Mascheroni constant [2], [8].

Calculating the number of terms needed to exceed a specific integer in the harmonic series arises from the Euler approximation. The formula provides the correct number of terms as shown in Figure 2. For example it takes 11 terms or H_{11} to exceed 3. The terms that are greater than a certain integer are: 1, 2, 4, 11, 31, 83, 227, 616, 1674, 4550 and so on. The ratios of the next term over the previous term approach e = 2.71828 (Figure 11). The number of terms needed to exceed a certain integer can be calculated by using the following EXCEL formulas.

In cell B5 enter 1 and in cell B6 enter =B5+1. In cell C5 enter =INT(EXP(B5-$C$3)+0.5). In cell D5 enter the ratios =C6/C5. Copy>paste these cells down to any desired number of calculations. Column C counts the number of terms of the harmonic series needed or the smallest harmonic number to exceed a specified integer. This application is displayed in Figure 11.



	A	B	C	D
1		Smallest Harmonic Number		
2				
3		Gamma	0.57722	
4		n	Harmonic #	Ratios
5		1	2	
6		2	4	2
7		3	11	2.75
8		4	31	2.81818
9		5	83	2.67742
10		6	227	2.73494
11		7	616	2.71366
12		8	1674	2.71753
13		9	4550	2.71804

Figure 11: The Smallest Harmonic Number

The number of terms of the harmonic series or the smallest harmonic number that exceeds a specific integer is calculated by entering a number in any cell and using =INT(EXP(number-gamma)+0.5). For example in B5 enter a number 4 then in C5 enter =INT(EXP(B5-0.577216)+0.5), the result is 31.The number of terms of the harmonic series needed is 31, H_{31} is the smallest harmonic number that exceeds 4.

12. The Coupon Collector's Problem

The Coupon Collector's Problem [6] is another application of the harmonic numbers. The coupon collector's or cards collector's problem involves determining the number of boxes of cereal or the number of packages of gum necessary to collect an entire set of coupons or sports cards. Consider the following example.

A student wants to collect a complete series of cards of famous mathematicians with their biographies. How many packs must he purchase to complete the entire set of 24 cards?

This example is an application of the geometric probability distribution [9] in which a person keeps purchasing cards until he gets the complete set.

The expected number of attempts to get the entire set of n items is nH_n [6]. If one needs a total set of 24 cards the number of purchases for each card is approximately H_{24} or 3.77 and the number of purchases needed for the total set of 24 is nH_n or 24*H_{24} or 24*3.77 or 91 purchases. The series approximation (Figure 10) calculates $H_{24}\, as\ 3.77 and\ 24 *\ H_{24}$ as 91.

13. Stacking Blocks

Another application of harmonic numbers is: How many wooden blocks of equal length and mass can be stacked in such a way that n blocks can be extended out from a table without falling to the ground?

This problem is often referred to as the Book Stacking Problem [10] or the Leaning Tower of Lire. The solution is based on measuring the length of overhang of blocks from a table. This distance or length is given as

$$s_n = \frac{1}{2}\sum_{i=1}^{n}\frac{1}{i} = \frac{1}{2}\,H_n$$

where H_n are the harmonic numbers. Using this formula and the series approximation (Figure 10), a stacking of 4 blocks has an overhang distance of $\frac{1}{2}\ (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4})$= 1.04167. More than one entire block length extends out from the table. And for stacking 31 blocks, more than 2 (2.01362) blocks extend beyond the table.

15. Record Snowfall

Another application of harmonic numbers pertains to the analysis of breaking records [4]. Suppose a meteorologist is interested in the number of record breaking years of snowfall amounts there will be in the next 50 years. The solution starts with the first year as being a record year for snowfall. Assuming for each year the amount of snowfall is independent of the other years and there is no pattern in the data for the amounts. In the second year the probability of a record snowfall will be $\frac{1}{2}$. At the end of 2 years the expected number of record snowfalls is $1 + \frac{1}{2} = 1.5$. In the nth year the expected number of record of snowfalls will be $\sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \frac{1}{n}$. This means in 50 years there will be on average $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{50} = 4.49921$(about 4.5) record snowfall years.

14. The Bug on the Rubber Tube

Suppose a bug is moving along a rubber tube that is 1 foot long and travels 1 inch in the first minute [5]. Assuming the rubber tube is being stretched uniformly and the bug is moving constantly along the tube, will the bug ever reach the end of the rubber tube? The bug travels $\frac{1}{12}$ of the rubber tube in the first minute. In the second minute, it travels 1 inch while the rubber tube is stretched 2 feet, so it travels $\frac{1}{2}$ times $\frac{1}{12}$ of the rubber tube. In the third minute, it travels $\frac{1}{3}$ times $\frac{1}{12}$ of the rubber tube. After n minutes, the bug travels $\frac{1}{12}(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n})$. The bug does reach the end of the tube because $\frac{1}{12}(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n})$ will be longer than 1 foot after n minutes since the harmonic series is divergent. With the approximation (Figure 11) for the least $H_n$that exceeds an integer n the value found is 91380. After 91380 minutes the bug travels 1foot and reaches the other end.

15. Conclusions

A closed-form expression (an expression that can be represented in elementary form) for computations and analysis is not always possible. Recursion techniques are useful. Approximations are as good as exact calculations and can be applied for large values of n. The concepts regarding harmonic numbers can be spanned over a variety of mathematical disciplines including algebra, approximation, and calculus.

Some of the relationships that exist for harmonic numbers include:

a) The difference between two harmonic numbers is $\frac{1}{n}$.
b) The quantity H_n-ln(n),the difference between the nth harmonic number and the natural logarithm of n approaches γ, Euler's Mascheroni constant $\gamma = 0.57722$.

c) Harmonic numbers are also related to the ln(n) and γ, the Euler-Mascheroni Constant 0.57722 by the following expression $H_n \approx \ln(n) + \frac{1}{2n} + \gamma$.

d) Harmonic numbers, the digamma function and the Euler-Mascheroni are related by the following: $\varphi(n) = H_{n-1} - \gamma$ or $H_{n-1} = \varphi(n) + \gamma$. That is, the n-1st harmonic number equals to the digamma function of n plus the Euler-Mascheroni constant.

e) The ratios of number of terms needed in the harmonic series to exceed a specific integer approach e.

f) The relationship $[e^{a-\gamma} + 1/2]$, where [] represents the greatest integer function, determines the number of terms of harmonic series needed to exceed a specific integer.

Harmonic numbers appear in many disciplines including business, statistics, and physics. Numerous relationships appear. When relationships cannot be defined in closed-form or elementary terms, ideas can be expressed through approximations and recursive techniques. From exact to approximate ways of finding these numbers each technique is helpful in showing how relationships exist. Visualizations and approximations convince the learner that all methods produce similar results. Approximations are close to the exact results.

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