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Conditional Formatting Revisited: a Companion for Teachers and Others

Steve Sugden
Bond University, ssugden@bond.edu.au

John E. Baker
Natural Maths, john@naturalmaths.com.au

Sergei Abramovich
State University of New York at Potsdam, abramovs@potsdam.edu

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Abstract

Conditional formatting is a truly remarkable feature of the modern spreadsheet program, exemplified by Microsoft Excel. In the decade since the first publication on conditional formatting in this journal, and with the singular exception of articles in *Spreadsheets in Education*, we note that this capability of Excel is still not cited much in the mathematics education literature. The present paper is written to review and summarize published accounts over the past decade and then to showcase basic and more advanced uses of conditional formatting in the hope that more mathematics educators will take the time to master this powerful tool for the benefit of their students.

Keywords

conditional formatting, mathematics education

1 Background

One of the earliest papers in this journal was devoted to conditional formatting (Abramovich & Sugden, 2005), ten years ago. In that paper, the following observations were made.

1. *Excel 97 and its successors allow dynamic, automatic formatting of any cell, based on its current value, i.e., conditional formatting.*
2. *With the introduction of conditional formatting into Excel, the basic human ability of pattern recognition, including colour perception, algorithms for which are so difficult for conventional binary computers, may be exploited for educational gain.*
3. *References to spreadsheet conditional formatting in the general literature, at least for any didactic use, are sparse indeed.*
4. *Conditional formatting not only enhances problem solving through a visual representation of the concepts involved, but it can also create an environment from which, quite unexpectedly, challenging problems can emerge.*
5. *Conditional formatting certainly appears to be a very much under-used feature of Excel; certainly for educational purposes. In a 2003 survey of the use of spreadsheets in education (Baker & Sugden, 2003), it was found that almost no published examples of this potentially very powerful and useful facility exist.*
6. *The possibilities for educational applications of conditional formatting would seem to be very great indeed.*

Ten years is a long time in the rapidly-changing world of computers in mathematics education. It is therefore somewhat surprising to note that not many authors have documented their use of conditional formatting as a powerful learning tool.

In the earlier paper on conditional formatting cited above (Abramovich & Sugden, 2005), some examples of conditional formatting and how it may be used in the mathematics classroom were suggested. We (and other contributors to the present journal *Spreadsheets in Education*) have also published further examples of conditional formatting and to a very limited extent, elsewhere, in the interim. However, looking back on all of this, and considering our own continued use of conditional formatting, the main point to note is that, outside of *Spreadsheets in Education*, not much at all has been published, at any time, on the use of conditional formatting in mathematics education.

Reflecting on possible reasons, we conjecture that:

1. For those using technology in the classroom, some are simply not aware of the existence of conditional formatting or perhaps do not perceive its vast potential for mathematical learning.
2. Those aware of it may have dismissed it as “another Microsoft gimmick”, designed to sell more copies of Excel/Office, but otherwise just a curiosity. This may even be

the case for some of those who might otherwise profitably use spreadsheets in their classes.

3. Conditional formatting can be tricky to set up in Excel, even for some relatively "simple" examples.
4. Conditional formatting is not only tricky but may be mathematically challenging for a classroom teacher.
5. Conditional formatting, by its nature, represents the second order (or meta-) processing of information within a spreadsheet; that is, building a formula within a formula, thereby, analyzing data generated through another process. It is a reflection on data resulting from its visual modification to allow for a distinction between elements of the data without it being replicated.

In the present work, our intention is to motivate further the use of conditional formatting in the mathematics classroom and beyond and to give teachers, and others, some examples of its use. The examples are downloadable with this paper from the *Spreadsheets in Education* journal website.

It is worth noting that conditional formatting may be used to *hide information* as well as to highlight. There are valid reasons for doing this, and our present example of the modular multiplication table (or *Cayley table*, from a group-theoretic perspective), is a classic case. This is described in Section 3, see also the eBook (Miller & Sugden, 2011) and Section 7 of (Lovászová & Hvorecký, 2003).

We are interested in illustrating examples of conditional formatting which are directly useful for teachers, in general, and mathematics teachers, in particular. Conditional formatting may also be of interest to educational researchers who want to use a spreadsheet to deal with large samples of data.

2 Published accounts of conditional formatting for educational use since 2003

Since the inaugural paper (Baker & Sugden 2003), *Spreadsheets in Education* has published several papers featuring or highlighting the use of conditional formatting in educational settings. We give a brief outline of these below. However apart from a few trivial cases, we have seen no conditional formatting examples published elsewhere in that period (2005-2015). A search (12 May 2015) on Google Scholar produced very little of relevance, apart from articles in *Spreadsheets in Education* noted below, where we briefly summarize significant published uses of conditional formatting from 2003 to 2015. These are almost exclusively from the present journal, *Spreadsheets in Education*.

1. **Brief overview of conditional formatting and its application to constructivist education**. Survey paper, inaugural issue of *Spreadsheets in Education* (Baker & Sugden, 2003).

2. **Computer programming.** Earliest known published instance of conditional formatting in education to hide information (Lovászová and Hvorecký, 2003).
3. **Mathematics education**. In the paper (Abramovich & Sugden, 2005) the following examples were considered: limits and convergence of a sequence, graphical solution of non-linear equations, modular arithmetic, simultaneous linear congruences and patterns in the multiplication table, Ramanujan's taxicab numbers (based on the famous story of Hardy about the number 1729 recognized by Ramanujan as the smallest positive integer expressible as the sum of two cubes in more than one way). The authors then went on to consider broader mathematics education contexts in which conditional formatting may profitably be used. These were:
    - Designing environments for younger learners
    - Developing meta-context through conditional formatting
    - Conditional formatting as a window on unsolved problems
4. **Demonstration of period-doubling bifurcation**. In discrete phase-locked loop control systems, conditional formatting was applied to the numerical representation of orbits allowing one to observe the phenomenon of period-doubling bifurcations that are otherwise difficult to recognize visually (Abramovich, Kudryashova, Leonov & Sugden, 2007).
5. **Recursion and Spreadsheets.** Conditional formatting rates a brief mention here; again, to highlight values in tables (Baker, Hvorecký, & Sugden, 2006).
6. **Developing definitions**. Abramovich (2007) describes the use of conditional formatting in the context of the explorations within the multiplication table and to illustrate definition of various classes of numbers, such as square and triangular numbers, and also inductive transfer.
7. **Aid for interpretation of error residuals**. Sinex (2006) makes use of conditional formatting to highlight errors which may be positive, negative, or on rare occasions, zero. The author states that "*This approach works to strengthen the "rule of four," where mathematics educators are trying to emphasize and relate graphical, numerical, symbolic, and verbal descriptions*".
8. **Differentiate between primes and zeroes in the Goldbach Comet & Goldbach Glacier**. Baker (2007) makes use of conditional formatting to pinpoint primes satisfying certain conditions, and graphically make up what he terms the *Goldbach Glacier*.
9. **Automatic warning to students when answers are incorrect or out of range** (Wetzel & Whicker, 2007). Answers seen by a student using conditional formatting parameters serve to clearly contrast three possibilities: green cells contain correct answers, while pink and blue cells contain incorrect answers that are too large and too small, respectively.

10. **Sudoku**. In (Sugden, 2008), conditional formatting is used to highlight "definite singletons" and "possible pairs" in the game of Sudoku. The Excel model described there is intended to take some of the drudgery of computing the sets of "possibles" for each currently vacant cell.

11. **Display of Fibonacci Sieve.** In (Abramovich & Leonov, 2009), conditional formatting is used to display subsequences of Fibonacci numbers that survive the so-called *Fibonacci sieves* of orders one through five.

12. **Developing conditional formatting skill using a mathematical definition.** In (Abramovich, Nikitina & Romanenko, 2010), it is shown how basic mathematical skills, once being used as professional skills, become advanced spreadsheet programming skills enabling conditional formatting.

13. **Potential difference in electromagnetic boundary-value problems**. Lau & Kuruganty (2010) make excellent use of conditional formatting to vividly show change of potential in boundary-value problems included in a mathematically challenging electrical engineering curriculum.

14. **Modular arithmetic**. In (Miller & Sugden, 2010), the authors use conditional formatting to highlight instances of a user-selected value in a modular multiplication table (or *Cayley* table). Since the modulus and also the value to be so highlighted are both selectable via sliders, one needs to limit table display to inputs less than the modulus. Again, this is easily accomplished via conditional formatting, and is an example of creative use of conditional formatting to *hide information*; a topic which, in itself, is well-worth exploring further.

15. **Change of sign of function and zero of function and derivatives; root finding** (Sugden, 2010). Here, conditional formatting is used to solve $f(x) = 0$ by highlighting ordinates which are negative, zero or positive in three different colours. Change of sign of the function is easily seen and simple zoom buttons are supplied to allow the user to zero in on the solution, or to get the "big picture" with just a few mouse clicks. The model also estimates first and second derivatives and highlights change of sign in the same manner, so extrema and inflections also may be seen and their location estimated reasonably accurately, at least for functions which are not too wild.

16. **Fundamental theorem of arithmetic.** Here, conditional formatting is used for suppression of the display of essentially redundant information in tables of primes, (Sugden & Miller, 2010).

17. **Detection of cycles (recurrent herd configurations)**. In (Sugden, 2011), a spreadsheet is described where a sequence of evolving partitions for a herd of goats (into sub-herds) is modelled. One of the functions of the model is to automatically recognize and clearly display recurrences of an earlier partition. The well-known computer-scientific technique of *hashing*, combined with conditional formatting to highlight each occurrence, is used to achieve this recognition.

18. **Solution highlight in a long financial schedule/table.** In (Sugden & Miller, 2011), conditional formatting is employed to highlight which adjacent periods in a long sequence of periods give rise to a change of sign of a debt, thus indicating the final payment of a loan. This may be regarded as a trivial use, however the intention is to save the reader's eyes some unnecessary extra work, and is a general benefit of even the simplest application of conditional formatting.

19. **Highlight of solution pairs and also of improper values.** In (Sugden, 2011b), conditional formatting is used to highlight solution ordered pairs and also incorrect values in tables.

20. **Visualization of random sequences.** In (Juster, 2013), conditional formatting is used to highlight outcomes of a sequence of events with binary random outcomes (e.g., tossing a coin). It is also used in a very clever manner to simulate the rolling of a die (spots are turned on or off using conditional formatting).

21. **Patterns in binomial modular tables.** In (Karakirik, 2015), the author uses conditional formatting to highlight striking recursive *Sierpinski-like* patterns in binomial tables.

22. **Suggestion or motivation of proof of theorem**. In (Baker & Sugden, 2014, 2015) conditional formatting is employed to generate patterns in a binary truth-table, thus suggesting a line of proof for an important theorem concerning representation of integers by Fibonacci numbers (the *Zeckendorf* expansion).

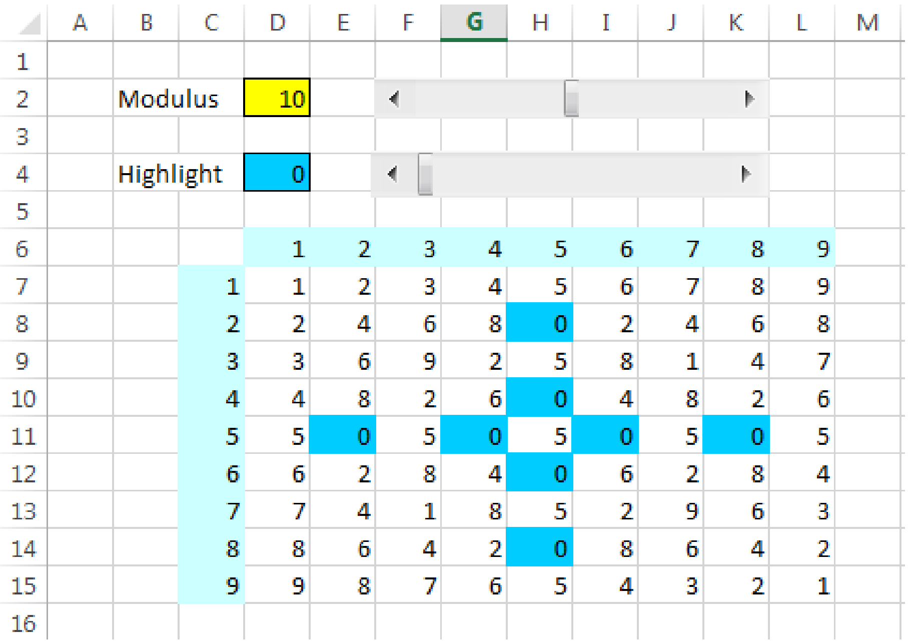
23. **Identify matching columns in a table.** In (Gomez, Oppenheim & Yurekli, 2015), conditional formatting is used in support of teaching topics in ethnomathematics.

3 Why use conditional formatting?

Conditional formatting is described as an "essential feature" of Excel (Baker, J. & Baker, A. 2011). Rapid identification of numbers or strings with special properties within an array of values is made. For example, to identify a square number in a matrix without conditional formatting, one needs another table; as in the case of Pythagorean triples (Abramovich, 1999). Conditional formatting also permits economy of representation: it is defined in terms of a single cell but then may be applied to an entire highlighted array. A form of highlighting information would be to individually format each cell of an array in which information is presented. However, conditional formatting enables the application of colour and other formatting features to a whole array of numbers. It is a second level of 'computer intelligence' and can be called meta-data, that is, data about data. This is similar to how a picture illustrates a verbally described context/situation. For students who are colour-blind, a teacher can replace colour by a pattern style.

In a nutshell, the ability to automatically draw attention to certain patterns or values, or perhaps to hide unwanted content in cells, is the essence of conditional formatting. True to the nature of an electronic spreadsheet, this all happens in a dynamic way. So, as parameters change, various patterns may appear or disappear, drawing attention to important mathematical principles. An excellent example of this is the modular multiplication table of Figure 1, where fundamental results about prime moduli are instantly shown as students are allowed to vary either modulus or residue to be highlighted.

By highlighting zeros we emphasize a hidden pattern that pairs of numbers (2, 8) and (4, 6) are symmetrical about the number 5 on the number line. From the point of view of the modulo 10 system, this is because 2, 8, 4, and 6 are the only numbers in the range 1 through 9 that when multiplied by 5 yield a multiple of 10. In fact, a reason for the teacher to highlight zeros would be to ask students to come up with an interpretation of the pattern such as that given. That is, by highlighting zeros (or any other number for that matter) we



Modulus 10

Highlight 0

	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	0	2	4	6	8
3	3	6	9	2	5	8	1	4	7
4	4	8	2	6	0	4	8	2	6
5	5	0	5	0	5	0	5	0	5
6	6	2	8	4	0	6	2	8	4
7	7	4	1	8	5	2	9	6	3
8	8	6	4	2	0	8	6	4	2
9	9	8	7	6	5	4	3	2	1

Figure 1: Highlighting values in a modular multiplication table

encourage students to think about hidden patterns and make sense of them.

Note also that here, highlighting zeroes does not require any programming sophistication. We do see zero, but through conditional formatting we emphasize the pattern made by the position of the zeroes in the table and this enhances visualization. Here, one can say that the use of conditional formatting goes back to the famous "method of dual stimulation" by Vygotsky (cited in Wertsch, 1991, p. 32) when formatting features of Excel serve as stimuli that mediate learner's response to the first set of stimuli which in the case of numeric modelling include symbolic recognition of special types of numbers such as zeroes. A different case is when we highlight numbers with special properties, e.g., prime numbers. This is done in Section 4.3 where we use conditional formatting to display the sieve of Eratosthenes.

In this manner, live mathematics can be seen by all and important theorems motivated. Even *lines of proof* can be suggested by this powerful, visual approach to mathematics. A large component of the teacher's role is then to guide this discovery process and perhaps to suggest new lines of investigation. It is even better if students are, at some point, able to offer their own ideas for further exploration and discovery. When this happens in the classroom, the potential for learning is great.

4 Classroom-ready examples and how to construct them

The examples discussed below are included in the spreadsheet that accompanies this paper.

4.1 General use of conditional formatting

Some general ways in which conditional formatting may be employed include the following.

1. Identifying numbers with special properties through conditional formatting as an agency for learning mathematics.
2. Students inventing their own problems related to conditional formatting.
3. The use of conditional formatting when conducting quantitative research in education (managing large samples based on a given threshold).
4. Digital fabrication (making rectangle, square, triangle, circle, etc. via conditional formatting).

4.2 Use of conditional formatting to *hide* information

The previous section discussed the modular multiplication table. This example also ably illustrates the use of conditional formatting to usefully hide information. Why do we want to hide things? The simplest reason is that certain cells may not be relevant for the current state of the model. For example, if a slider has been used to limit the size of a table (as in the modular table example), then material outside of the table is a distraction. The simplest way to hide cells using conditional formatting is to employ white font on white ("No Color") background. The conditional formatting dialog for the modular multiplication table is as shown in Figure 2.

4.3 Example: sieve of Eratosthenes and Dirichlet's theorem

In this example, we use conditional formatting to reveal rather than highlight a special set of numbers, the primes between 1 and 500. In the accompanying spreadsheet, the cells C3:AA22 were first filled with the numbers 1 – 500 and the background of that range was set to black. In that way, the numbers in the cells were completely hidden. Then two conditional formatting rules were applied:

1) Select C3:AA3 and apply the conditional formatting rule
=(C3 - 2)*(C3 - 3)*(C3 - 5)*(C3 - 7)*(C3 - 11)*(C3 - 13)*(C3 - 17)*(C3 - 19)*(C3 - 23) = 0
Set the format to a white background and this will reveal the primes up to 23. Note that only the cells that contain one of the primes in the range [2, 23] will satisfy the

conditional formatting rule.

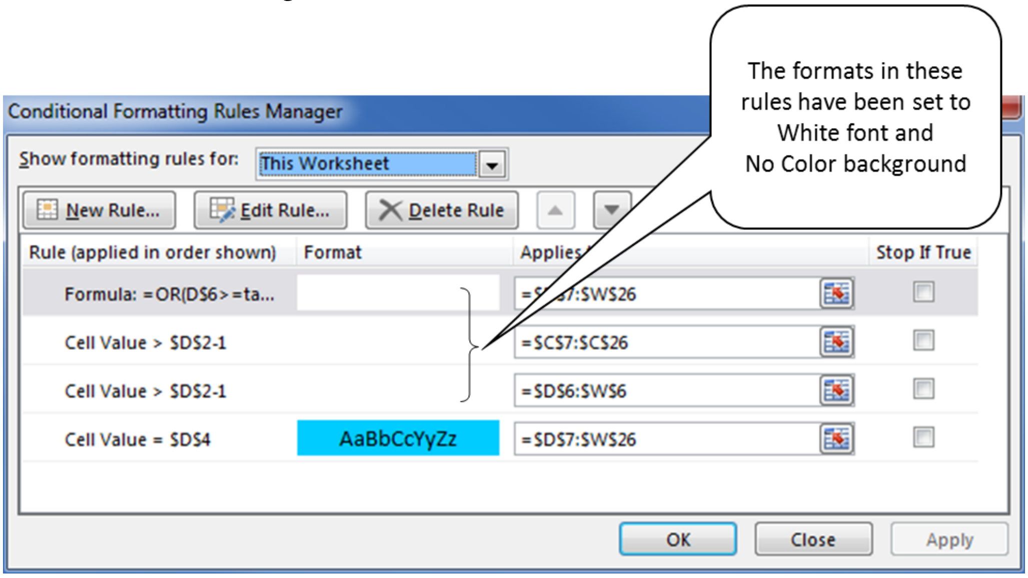


Figure 2: Conditional Formatting Dialog

Since $23^2 = 529$, we only need to go as far as 23 in order to show the primes up to 500.

2) Select C4:AA22 and apply the conditional formatting rule
=GCD(C4, 2*3*5*7*11*13*17*19*23) = 1
Again set the format to a white background and the primes from 25 – 500 appear as shown in Figure 3. Note that only the cells that contain the primes from this range will not be divisible by any of the primes from the range [2, 23].

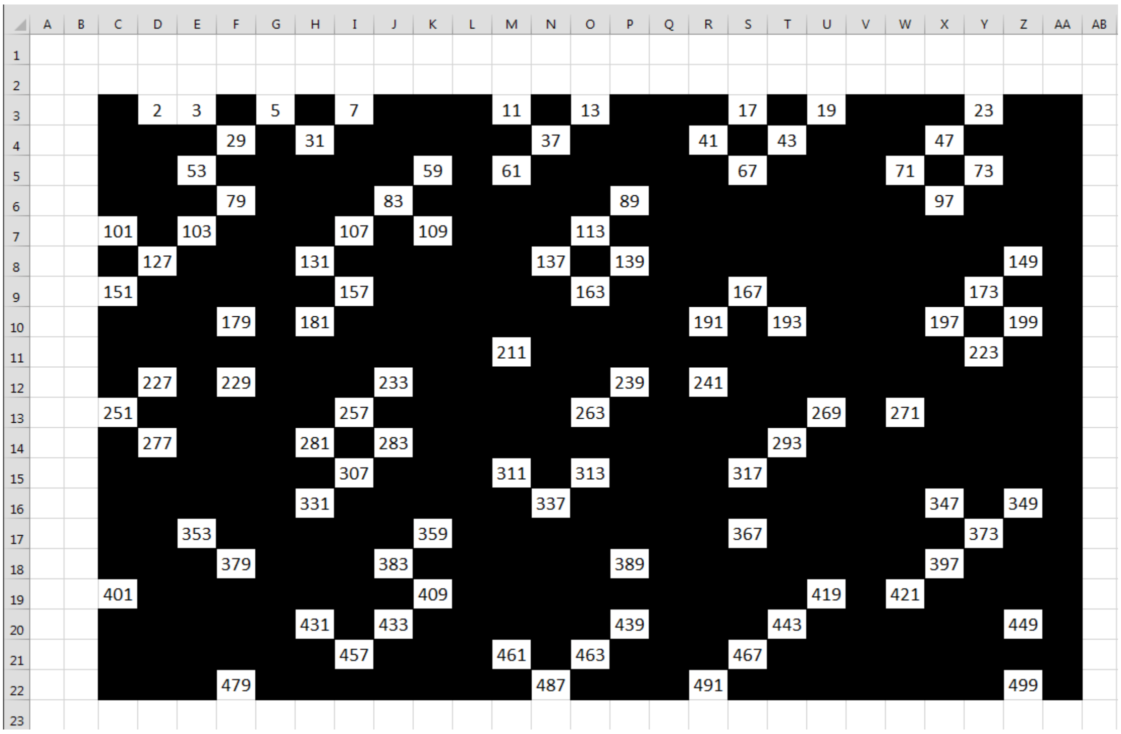


Figure 3: Conditional formatting illustrates the Sieve of Eratosthenes

Figure 3 can also be used to illustrate Dirichlet's theorem about integer arithmetic sequences with the first term a and difference d: If $d \geq 2$ and $a \neq 0$ are relatively prime numbers then there are infinitely many primes among the terms of the sequence $x_n = a + dn$. Here, $d = 25$ and for $a = 5, 10, 15, 20, 25$ (row 3) only, conditional formatting shows no prime numbers generated through the corresponding sequences. That is, the pairs (25, 5), (25, 10), (25, 15), (25, 20) and (25, 25) comprise not relatively prime numbers. In other words, the only columns which are totally dark because they do not display prime numbers (except cell G3 in column G) are those that correspond to the multiples of 5 not greater than 25. Without conditional formatting such a phenomenon of number theory would be difficult to recognize.

4.4 Example: A grade book

A simple, but very useful, application of conditional formatting is in a spreadsheet gradebook. Here, the teacher has recorded marks for each assessment item and the spreadsheet is used to assemble and finalize grades. It is useful to automatically highlight those students who fall into certain categories: failure, or perhaps 1% or 2% short of a passing grade. Such students are then easily identified by conditional formatting and perhaps reconsidered for a pass.

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2		Student	Mark out of	Grade	Final %			Breakpoint	Grade	Frequency	Percentage	
3			100					0	F	7	29%	
4		A	87	HD	87%			50	P	6	25%	
5		B	66	C	66%			65	C	3	13%	
6		C	52	P	52%			75	D	5	21%	
7		D	57	P	57%			85	HD	3	13%	
8		E	50	P	50%				Total	24		
9		F	73	C	73%							
10		G	48	F	48%							
11		H	85	HD	85%							
12		I	41	F	41%							
13		J	79	D	79%							

Figure 4: Basic, but useful application of conditional formatting for a teacher's gradebook.

4.5 Example: inductive transfer in a mathematical induction proof

In this section, we consider an example of an induction proof where conditional formatting assists with the inductive step.

PROPOSITION. The sum $S(n)$ of all products in the $n \times n$ multiplication table is given by the formula:

$$S(n) = \left(\frac{n(n+1)}{2}\right)^2 \quad (1)$$

PROOF. The right-hand side of Formula (1) depends on an integer variable n and thus it is a candidate for proof by the method of mathematical induction. To this end, note that a spreadsheet allows one to computationally enhance the inductive development of Formula (1). In addition, the conditional formatting feature of a spreadsheet allows for the demonstrative (inductive) phase of the proof to be visually enhanced.

The first step is to show that Formula (1) is true for n = 1. Indeed, when n = 1, the multiplication table consists of unity only, whence $S(1) = 1$. The right-hand side of Formula (1) assumes the same value when n = 1.

The second step is to test the transition from n to n+1; that is, assuming that Formula (1) holds true for n (making the inductive assumption), one has to show that after replacing n by $n+1$, the structure of the formula does not change, that is

$$S(n+1) = \left(\frac{(n+1)(n+2)}{2}\right)^2 \qquad (2)$$

To visually support this step, that is, the inductive phase of the proof, we can use conditional formatting. This clearly shows what augments the table when it acquires a new row and a new column. This feature can highlight any augmentation of the table from its current state by a *gnomon*. As shown in Figure 5, conditional formatting highlights such a gnomon consisting of the numbers that belong to $S(11)$ but do not belong to $S(10)$. More specifically, the transition from the table of size 10 to that of 11 can be described as follows.

$$S(11) - S(10) = 2 \times 11 \times (1 + 2 + \cdots + 11) - 11^2$$

In general, the transition from n to n+1 yields the relationship

$$S(n+1) = S(n) + 2(n+1)\big(1 + 2 + 3 + \cdots + (n+1)\big) - (n+1)^2$$

Next, by making use of the formula: $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$, we can show that $S(n+1) = S(n) + (n+1)^3$. Indeed,

$$2(n+1)\big(1 + 2 + 3 + \cdots + (n+1)\big) - (n+1)^2 = 2\frac{(n+1)^2(n+2)}{2} - (n+1)^2 = (n+1)^3$$

That is, the sum of all numbers in the $(n+1)$-gnomon is equal to $(n+1)^3$. Finally, using the inductive assumption yields the following chain of equalities:

$$S(n+1) = S(n) + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4}$$
$$= \frac{(n+1)^2}{4}(n^2 + 4n + 4) = \frac{(n+1)^2}{4}(n+2)^2 = \left(\frac{(n+1)(n+2)}{2}\right)^2$$

This concludes the proof of Formula (1) by the method of mathematical induction. The role of conditional formatting was to show what constitutes the core of mathematical induction proof: the transition from n to $n+1$ always adds a gnomon of rank $n+1$ which is equal to $(n+1)^3$. The rest is for algebra skill. In particular, by adding numbers that belong to each of the n gnomons one obtains the formula $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

The following programming steps are presented with reference to Figure 5.

1. Create a multiplication table using Row 4 as the x-values and Column B as the y-values; that is, name the values in Row 4 as x and those in Column B as y. The table can extend to, say, 25 cells in both directions.

2. Add a scrollbar in Row 2 and link it to D2 and set the range as 1 – 25. Give D2 the name *size*.

We are now ready to add the conditional formatting that will change the visible table as the scrollbar changes the value of n in cell D2 and highlight the $S(n+1) - S(n)$ part of the table, that is, the table is augmented by a gnomon of rank $n + 1$ as the scroll bar attached to cell D2 replaces n by $n + 1$. It is important to create the rules in the right order, as the last rule to be created is the last one to be applied.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1														
2			Size	11										
3														
4			1	2	3	4	5	6	7	8	9	10	11	
5		1	1	2	3	4	5	6	7	8	9	10	11	
6		2	2	4	6	8	10	12	14	16	18	20	22	
7		3	3	6	9	12	15	18	21	24	27	30	33	
8		4	4	8	12	16	20	24	28	32	36	40	44	
9		5	5	10	15	20	25	30	35	40	45	50	55	
10		6	6	12	18	24	30	36	42	48	54	60	66	
11		7	7	14	21	28	35	42	49	56	63	70	77	
12		8	8	16	24	32	40	48	56	64	72	80	88	
13		9	9	18	27	36	45	54	63	72	81	90	99	
14		10	10	20	30	40	50	60	70	80	90	100	110	
15		11	11	22	33	44	55	66	77	88	99	110	121	
16														

Figure 5: Transition from size 10 to size 11 adds a gnomon of rank 11.

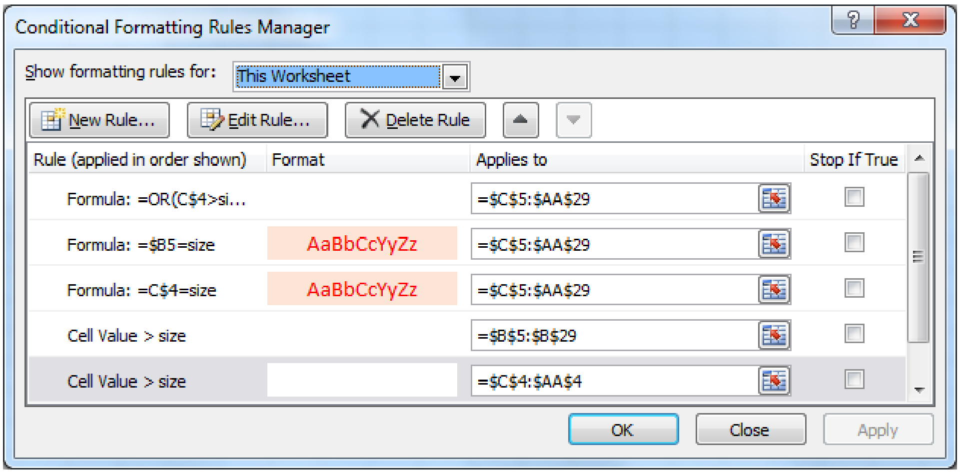


Figure 6: Using Formulas in Conditional Formatting

3. Select C4:AA4 which is the x-value range and select **Conditional Formatting: Highlight Cells Rules: Greater Than**. Type *size* in the value box and choose a custom format in which the Font Color is white. When you apply this formula you should see that all the x-values greater than *size* have been made invisible.

4. Select B4:B29 and repeat step 1 with this range, i.e. the y-values.

We will now highlight the column that contains the $x \times y$ values for the current value of x, which is *size*.

5. Select the whole of the multiplication table C5:AA29. Choose **Conditional Formatting: New Rule …** and in the dialog, select the last option for **Rule Type**, which is **Use a formula ….** When typing a formula for a range, you need to start with = and create a formula that works for the top left corner of the range. The formula needed here is

   = C$4 = size

   and the formatting should be for both Font and Fill.

6. Repeat the previous step for the y-values by adding the formula rule

   = $B5 = size

   and the formatting should be for the same Font and Fill.

We will now make invisible the part of the table that contains no $x \times y$ values.

7. With the table C5:AA29 still selected, as before, choose **Conditional Formatting: New Rule …** and in the dialog, select the last option for **Rule Type**, which is **Use a formula ….** The following formula makes invisible those cells that are either beyond where $x = size$ or beyond where $y = size$.

   = OR(C$4 > size, $B5 > size)

   and the formatting should be a **White** font and a **No Color** background.

4.6 Example: an investigation of the postage stamp problem

In this section, we explore one of the famous unsolved problems of mathematics and, with the help of conditional formatting, discover a new feature of the solution. The problem to be addressed is well-known as the *Postage Stamp Problem* or as the *Frobenius Coin Problem*, which dates back to the 19th century. The standard version is described at (Stackoverflow, 2015) as:

> *The postage stamp problem is a mathematical riddle that asks what is the smallest postage value which cannot be placed on an envelope, if the letter can hold only a limited number of stamps, and these may only have certain specified face values.*

The problem can be rephrased into a more general form and the limit on the number of stamps allowed can be removed, to give:

> Given two coprime positive integers a and b, what is the largest number that cannot be expressed as $ma + nb$, where m and n are non-negative integers?

The general form of the problem is given in (Weisstein, 2015) and word has it that a solution to the general form would be worth a large amount of money. However, it is not our intention to go for the unsolved form. Rather we would like to show how useful a spreadsheet can be in undertaking an investigation of aspects of a famous mathematical problem.

To set up a spreadsheet that finds the totals that can be made with two stamps requires no more than a simple table.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y
1																									
2			Stamp A		5	7	Stamp B		8																
3																									
4																									
5		A+B	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90		Num	Count	
6		0	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90		1	0	
7		8	8	13	18	23	28	33	38	43	48	53	58	63	68	73	78	83	88	93	98		2	0	
8		16	16	21	26	31	36	41	46	51	56	61	66	71	76	81	86	91	96	101	106		3	0	
9		24	24	29	34	39	44	49	54	59	64	69	74	79	84	89	94	99	104	109	114		4	0	
10		32	32	37	42	47	52	57	62	67	72	77	82	87	92	97	102	107	112	117	122		5	1	
11		40	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120	125	130		6	0	
12		48	48	53	58	63	68	73	78	83	88	93	98	103	108	113	118	123	128	133	138		7	0	
13		56	56	61	66	71	76	81	86	91	96	101	106	111	116	121	126	131	136	141	146		8	1	
14		64	64	69	74	79	84	89	94	99	104	109	114	119	124	129	134	139	144	149	154		9	0	
15		72	72	77	82	87	92	97	102	107	112	117	122	127	132	137	142	147	152	157	162		10	1	
16		80	80	85	90	95	100	105	110	115	120	125	130	135	140	145	150	155	160	165	170		11	0	
17		88	88	93	98	103	108	113	118	123	128	133	138	143	148	153	158	163	168	173	178		12	0	
18		96	96	101	106	111	116	121	126	131	136	141	146	151	156	161	166	171	176	181	186		13	1	
19		104	104	109	114	119	124	129	134	139	144	149	154	159	164	169	174	179	184	189	194		14	0	
20		112	112	117	122	127	132	137	142	147	152	157	162	167	172	177	182	187	192	197	202		15	1	
21		120	120	125	130	135	140	145	150	155	160	165	170	175	180	185	190	195	200	205	210		16	1	
22		128	128	133	138	143	148	153	158	163	168	173	178	183	188	193	198	203	208	213	218		17	0	
23		136	136	141	146	151	156	161	166	171	176	181	186	191	196	201	206	211	216	221	226		18	1	
24		144	144	149	154	159	164	169	174	179	184	189	194	199	204	209	214	219	224	229	234		19	0	
25		152	152	157	162	167	172	177	182	187	192	197	202	207	212	217	222	227	232	237	242		20	1	
26		160	160	165	170	175	180	185	190	195	200	205	210	215	220	225	230	235	240	245	250		21	1	
27		168	168	173	178	183	188	193	198	203	208	213	218	223	228	233	238	243	248	253	258		22	0	
28		176	176	181	186	191	196	201	206	211	216	221	226	231	236	241	246	251	256	261	266		23	1	
29		184	184	189	194	199	204	209	214	219	224	229	234	239	244	249	254	259	264	269	274		24	1	
30																							25	1	
31																							26	1	
32																							27	0	
33																							28	1	

Figure 7: A table of values for the Two Stamp problem

However, the table of Figure 7 does not make it clear which totals can be made and which cannot. To find out, we need to make a list of the numbers 1 to n and check which of these are in the table. This is done in Columns W and X.

After naming the table of Figure 7 as *stamps*, the formula in cell X6 is

(X6) = COUNTIF(stamps, W6)

This formula is filled down and cells that have non-zero entries are highlighted with conditional formatting.

Let us denote the largest value which cannot be expressed as $ma + nb$ by $M(a, b)$. Then it is known that

$$M(a, b) = (a - 1)(b - 1) - 1 = ab - a - b$$

Within the literature, for example (Turner, 2008), we can find proofs that this formula is correct and move on to a feature that the conditional formatting makes visible.

A	3	A	3	A	4	A	5
B	7	B	8	B	7	B	7

Num	Count	Num	Count	Num	Count	Num	Count
1	0	1	0	1	0	1	0
2	0	2	0	2	0	2	0
3	1	3	1	3	0	3	0
4	0	4	0	4	1	4	0
5	0	5	0	5	0	5	1
6	1	6	1	6	0	6	0
7	1	7	0	7	1	7	1
8	0	8	1	8	1	8	0
9	1	9	1	9	0	9	0
10	1	10	0	10	0	10	1
11	0	11	1	11	1	11	0
12	1	12	1	12	1	12	1
13	1	13	0	13	0	13	0
14	1	14	1	14	1	14	1
15	1	15	1	15	1	15	1
16	1	16	1	16	1	16	0
17	1	17	1	17	0	17	1
18	1	18	1	18	1	18	0
19	1	19	1	19	1	19	1
20	1	20	1	20	1	20	1
21	2	21	1	21	1	21	1
22	1	22	1	22	1	22	1
23	1	23	1	23	1	23	0
24	2	24	2	24	1	24	1
25	1	25	1	25	1	25	1

Figure 8: Amounts that can be made with different values for A and B

Symmetry within the Postage Stamp problem

Turner (Turner, 2008) ends an interesting article about the Postage Stamp problem, which he casts in terms of coins rather than stamps, with the following:

> One thing I noticed when enumerating the numbers produced by various choices for coins A and B was that exactly half of the numbers between 1 and $N_{max} + 1$ are able to be created. For example, for A = 7, B = 5, $N_{max} = 23$ and we can make {5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, 24} which adds up to 12 numbers. That this seems to hold for any A and B is surprising.

The conditional formatting of the counts shows this to be the case, but it also shows another feature that is equally surprising. When we examine the pattern of highlighted and non-highlighted cells between 1 and $M(a, b)$, we see that it has symmetry (Figure 8). Every highlighted cell is matched by a non-highlighted cell in the following way:

> If cell k is highlighted, then cell $M(a, b) - k$ is not highlighted and vice versa.

To see why this should be the case, consider how a number k might be made as a linear combination of a and b.

If $k = ma + nb$, then the symmetrical value is $k' = M(a, b) - ma - nb$. A step-by-step process of repeatedly subtracting a and b from $M(a, b)$ shows that k' cannot be made as a linear combination of Stamps A and B. The process starts with:

> If $M(a, b)$ cannot be made as a linear combination of a and b then $M(a, b) - a$ cannot be made.
>
> If $M(a, b) - a$ cannot be made as a linear combination a and b, then $M(a, b) - 2a$ cannot be made.

This process continues until we have shown that $M(a, b) - ma$ cannot be made with a linear combination of a and b, after which we start the process of subtracting b. The result of this is that if $k = ma + nb$, then $M(a, b) - ma - nb = M(a, b) - k$ is not a linear combination of a and b.

A proof of the converse, namely that if $k \neq ma + nb$, then $M(a, b) - k$ is a linear combination of a and b, proceeds as follows. First, without loss of generality, let us assume that $a < b$. If k is not a linear combination of a and b, then neither is $k - a$ or $k - b$. We can continue subtracting a or b from k until we reach a situation where $k' = k - pa - qb < a$. At this stage, k' lies in that small interval between 1 and a which we know to be filled with numbers that are not linear combinations of a and b. Now consider $M(a, b) - k'$. If we add either a or b to this number, the result will be a number greater than $M(a, b)$ and we know that number to be expressible as a linear combination of a and b because $M(a, b)$ is the largest number that cannot be expressed as a linear combination of a and b. Thus $M(a, b) - k'$ is expressible as a linear combination of a and b. Also,

$$M(a, b) - k' = M(a, b) - k + pa + qb$$

Thus if $M(a, b) - k'$ is a linear combination of a and b then so is $M(a, b) - k$.

The above confirms the 'surprising' result noted by Turner, namely that half the numbers between 1 and $M(a, b)$ can be made as linear combinations of a and b, and shows that for each k that is one such number, there is a corresponding number, k', that cannot and that $k' = M(a, b) - k$.

5 Conclusion

There are two processes that are central to "doing mathematics" via a numerical approach. The first is generating data to ensure that hypotheses can be rigorously tested; the second is to identify any general patterns that the data display. From an educational perspective, the spreadsheet is an ideal medium for both these processes. It allows very large data sets to be generated without the need for time being spent hand-calculating the information. Through the use of conditional formatting, the spreadsheet enables the patterns that lie hidden within large sets of data to be made clear.

In this article, we have put the case for the use of spreadsheets in the classroom as an exemplary tool for giving students the opportunity to connect with mathematical concepts and problems. It does so by harnessing the power of visual learning, by making underlying patterns stand out from data and by reducing the burden of generating data. It is hoped that the next 10 years of mathematics education will see a much greater take-up of spreadsheets in the teaching of mathematics than has been the case over the past 10 years.

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