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Investigation of divisibility in a spreadsheet environment

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Abstract
This classroom note is focused on the application of inquiry approaches to teaching divisibility in the set of whole numbers. The main attention is devoted to the composition of a sequence of questions implemented within the spreadsheet environment, the solution of which should encourage students to actively learn and discover the divisibility rule for number eleven. This is an extra-curricular topic in Slovakia aimed to be taught in extended Mathematics classes as many of their students encounter it in competitions. An important part of the development of the questions on workbook sheets is the implementation of feedback which provides evaluation of students' solutions and auxiliary instructions for the guidance of learning.

Keywords
inquiry-based teaching, divisibility, whole numbers, problem solving, spreadsheet, feedback.

Cover Page Footnote
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1. Introduction

Modernization trends in education often arise from the constructivist concept of learning. Constructivism is based on the recognition that students construct their own understanding of the world they live in through active interactions with the environment by reflecting on their experience. According to Fulier [7], in the implementation of the constructivist approach to teaching mathematics, teachers should emphasize active engagement of students in the process of gaining knowledge, i.e. students should discover mathematical relationships and rules through individual investigations. One of the methods to apply such an approach is the inquiry-based learning which is derived from the work of scientist, from scientific research. Similarly, a research or an investigation in mathematics learning is aimed at answering the identified research questions and using appropriate arguments to justify the findings.

Students' inquiry skills need to be developed gradually through using different inquiry activities until students are able to set up research question on their own and conduct an independent investigation to answer them. Banchi and Bell [3] distinguish four kinds of inquiry activities according to how much information is provided to students and how much guidance is provided by the teacher.

Confirmation inquiry - the role of students is to verify results which are known in advance. The goal of the teacher is to introduce students to the idea of an investigation; it could be to show them how to explore certain regularity. Students should be able to make various measurements, collect and organize data.

Structured inquiry - the teacher sets up the questions and gives the students instructions for each step of the investigation conducted. The students analyse the data obtained, organize data into neat tables, create graphs, formulate their findings and look for appropriate arguments to justify them.

Guided inquiry - together with the students, the teacher formulates the research questions and he/she could possibly provide some guidance for the inquiry activities. Students are to design the procedure of the investigation and the ways of reasoning for the answer to the research question.

Open inquiry - students have an opportunity to work as researchers. Based on the problem situation raised, they have to identify the research questions, carry on a sequence of inquiry activities, and seek answers and explanations.

At present, our research is devoted to the application of inquiry approaches to teaching mathematics at Slovak secondary schools. We have focused on suggesting series of activities based on structured inquiry to topics from the non-compulsory Slovak curriculum as there are not enough teaching materials available for them. However, these topics are usually taught in extended Mathematics classes. The basis for such activities could be worksheets containing a sequence of questions, solution of which should lead students to solve the problem raised and understand the discovered relationships. When designing the questions, it is desirable to bear in mind that students should work with different representation, experiment, generalize, and justify the discovered answers. If it is advantageous to use ICT in solving, then some questions from the worksheet could be linked to digital tools or perhaps the entire worksheet could be implemented in a digital environment. A spreadsheet may be a key component for the development of a stimulating learning environment for constructivist teaching [2]. In this article we describe a workbook prepared in a spreadsheet environment. As Sadri [8] notes, spreadsheet activities can provide practice in various areas of skills. They may enable students to explore properties of whole numbers and generalize numerical patterns using symbolic representation of data. Another advantage of the spreadsheet environment is the availability of investigation tools for exploring number patterns and development of algebraic thinking [5].

The aim of the workbook is to lead students to the discovery of the divisibility rule for number 11, to understand algebraic relationships and develop the skill to find appropriate arguments to justify the discovered properties of whole numbers. Allocating questions on different sheets of the workbook may provide several advantages. Abramovich et al. [1] stated that series of worksheets allows for the creation of learning environments of different levels of technological sophistication.

The main advantages of the spreadsheet environment in implementing the workbook can be summarized as follows:

- Motivating students by working in an unusual, engaging and interactive environment when solving mathematical problems.
- Ensuring that students progress through the series of worksheets as designed because the worksheets are displayed gradually, always after completing the previous one and hence ensuring they do not skip questions.
- Experimenting with whole numbers when exploring divisibility based on using simple formulas.
- Creating simple tables quickly by automatic fill, e.g. when creating continuous multiples of a number, in this particular case 11.
- Immediately evaluating students' solutions and the opportunity to provide students with brief auxiliary information if necessary.

2. The sequence of questions and their implementation in the workbook

The process of learning new concepts and relationships must be performed in connection to already known concepts and relationships based on practical activities providing students with new experience. When planning a teaching session, it is important to decompose the mathematical content into smaller tasks and follow-up questions which stimulate students to experiment and explore and guide them to discover new knowledge. While transitioning to generalization and abstraction students could be helped out by a suitably chosen sequence of questions solution of which requires the students to create and use different representations and models.

The core of the described workbook is a sequence of questions, the solution of these enables students to discover the properties of whole numbers that are divisible by 11.

The workbook was developed for the platform of Windows 7 and higher. It is implemented in the spreadsheet environment (MS Excel). We recommend to use the workbook in MS Excel 2007 and higher. The workbook contains macros; these need to be enabled. A guide how to enable them is available for example at the address https://www.youtube.com/watch?v=2Oq27zdbNJs.

Before using the workbook, the teacher should revise with the students writing whole numbers in an expanded form using units that are powers of 10 (i.e. the expanded form of a number with the standard notation *abcd* is: $a \cdot 10^3 + b \cdot 10^2 + c \cdot 10^1 + d$) and the meaning of the relationship "number *a* is divisible by number *b*". The divisibility by number 11 is taught only at the end of the time dedicated to divisibility and students should already be familiar with two types of divisibility rules: according to last digits (e.g. for numbers 2, 5, 10, 4, 8) and according to the sum of digits (e.g. for numbers 3, 9). To revise the selected rules for the divisibility and to understand standard forms in which variables represent unknown digits, the teacher could propose a few questions of the following types:

- Create a two digit whole number with one unknown digit "*d*" in any position from which you obtain a number divisible by 2 after substituting any value for "*d*".
- Create a two digit whole number with one unknown digit "*d*" in any position from which you cannot obtain a number divisible by 5 after substituting a value for "*d*".
- Create a three digit whole number with one unknown digit "*d*" in any position from which you obtain a number divisible by 4 after substituting any value for "*d*".
- Find which values can be substituted for "*d*" in the three digit whole number 7*d*2 with the unknown digit in the place of tens to obtain a number divisible by 3.

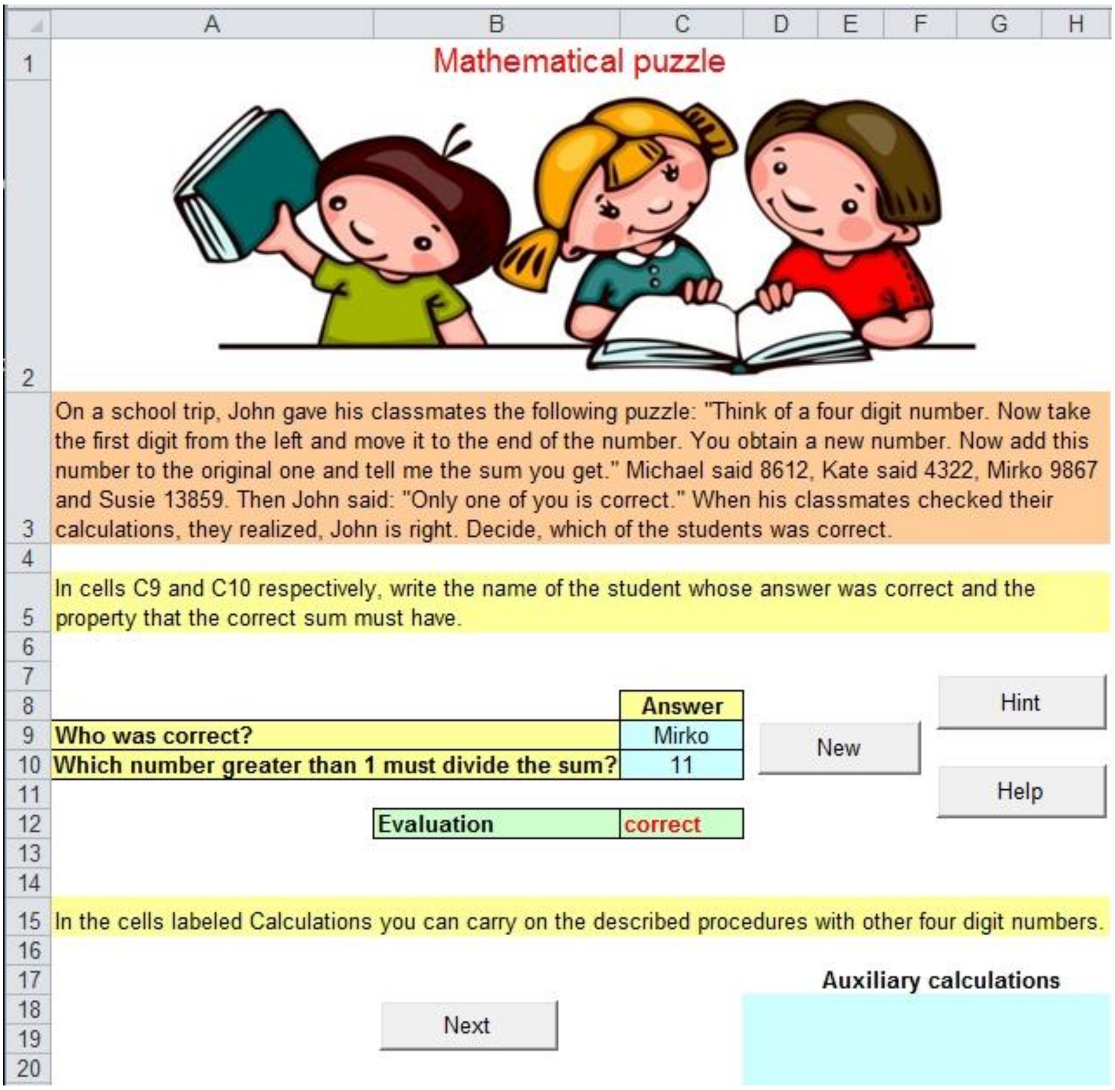

Figure 1: Q1 - motivational question

The workbook contains a sequence of eleven questions that are on separate sheets. Students can investigate divisibility of whole numbers and gradually move from exploring particular two, three and four digit whole numbers to exploring the standard notation of numbers containing variables in position of some digits. The solution of each question is automatically evaluated and when the answer is correct, the next sheet with the next question is displayed. Therefore, when you open the workbook the only things shown are the initial information and the motivational question Q1. Hiding the worksheets Q2 to Q11 is secured by the code written using VBA editor for Microsoft Excel objects ThisWorkbook. For example, the following code is entered to hide worksheet Q2:

```
Worksheets("Q2").Visible = xlVeryHidden
```

How to work with the workbook and how to write expressions in the workbook is explained on the information page. With the aim of guiding students to write expressions in the simplest form, the following rules are given.

- The expressions must not start with a minus sign.

- The expressions must not be written as multiples of simpler, correct expressions: *Do not write 2a+2b+2c or a+a+b+b+c+c if a+b+c is also correct.*
- The expressions must not contain brackets and spaces between characters.

At the beginning of the workbook we included a question that could awaken the interest of students and stimulate them to investigate the divisibility of the given whole numbers. Students can perform auxiliary calculations in the designated area of the cells. We expect that students will write simple formulas with the operation of division or even with the mathematical function MOD. Students have to enter their answers (see Figure 1) in the cells with a blue background. Other cells of the worksheet are locked. If a student cannot solve the motivational question, he/she can click on the Hint button and receives the instruction to perform the operations described in the question with particular four digit whole numbers. If a student uses the Help button, an explanation appears about how the question could have been solved using the expanded form of a four digit whole number in a standard form $abcd$ ($a \cdot 10^3 + b \cdot 10^2 + c \cdot 10^1 + d$). When solving the motivational question students still do not have to work with variables. If they experiment with particular four digit whole numbers or even if they just assume from the title of the workbook, they can easily see that a number obtained by the described procedure must be divisible by 11 and hence determine that only Mirko worked correctly. However, it would be beneficial to go back to this question at the end, once the rule is known to the students and give reasons for its solution.

To check the students' solutions, the following formula with logical function is saved in cell C12:

=IF(OR(ISBLANK(C9);ISBLANK(C10));"unfilled";IF(AND(C9="Mirko";

OR(C10=11;C10="by 11"));"correct";"incorrect")).

Entering correct data into cells C10 and C11 causes that the pressing of the Next button leads to displaying the next, so far hidden sheet called Q2 that contains the next question. The following macro is linked to the Next button:

```
Private Sub CommandButton4_Click()
    If Range("C12").Value = "correct" Then
    Worksheets("Q2").Visible = True
    Worksheets("Q2").Activate
    Else
    MsgBox "The table is not completed correctly!"
    End If
End Sub
```

The next part of the workbook focuses on exploring concrete numbers in terms of divisibility by 11. For illustration we chose the second question Q2 from the workbook. Students have to write five different whole numbers divisible by 11.

The entered whole numbers are evaluated in the neighbouring cells where individual numbers are checked for being divisible by 11, e.g. in the cell E6 this formula is used:

=IF(OR(D6<=60;MOD(D6;11)>0);0;1).

Then, using the function MODE.SNGL it is checked whether the written numbers are different from each other. In cells D12 and D13 number 1 is written in white letters. In cell F10 there is the formula: =MODE.SNGL(D6:D13). If a student writes five different integers which, according to the assessment have to be greater than 60, the result of the function MODE.SNGL is number 1. Otherwise, the result of this formula is a number that is repeated in the table in the area of cells D6:D10. In the cell E11 the sum of the values in the above mentioned neighbouring cells from E6:E10 is calculated. If it is equal to 5, all entered numbers are divisible by 11. The result of student's work evaluation is concentrated in cell D14, where the next formula is written:

=IF(OR(ISBLANK(D6);ISBLANK(D7);ISBLANK(D8);ISBLANK(D9);

ISBLANK(D10));"unfilled";IF(OR(F10>60;E11<5);"incorrect";"correct")).

The worksheet with the described question is shown in Figure 2. After correctly solving it and after pressing the Next button, the next question is displayed.

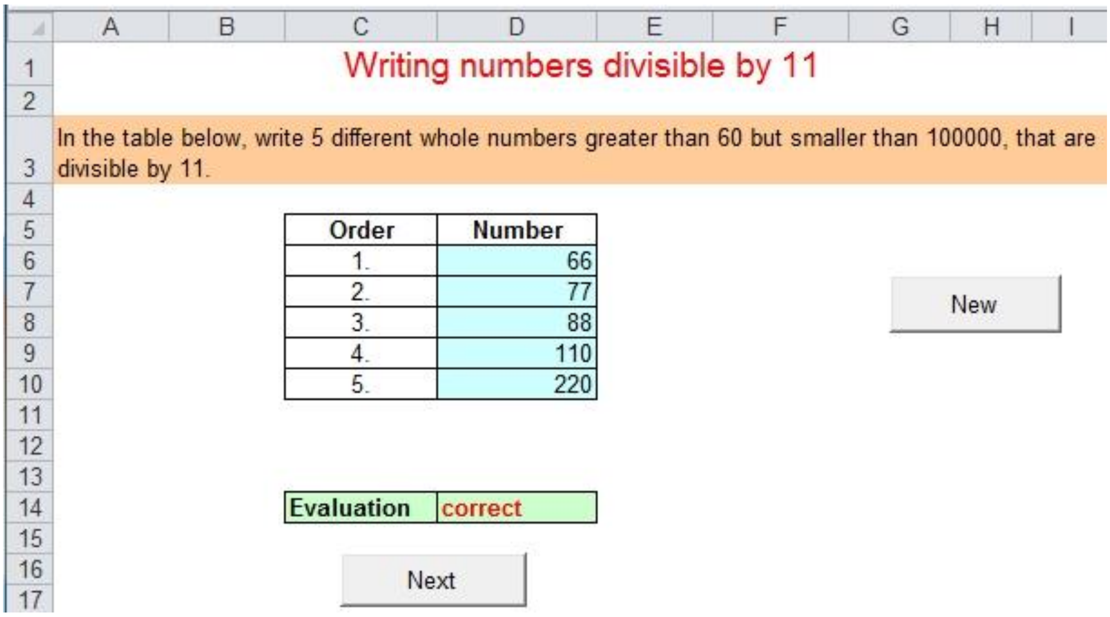

Figure 2: Q2 - writing numbers divisible by 11

Exploring concrete whole numbers concludes with the question Q4 where students have to find the smallest odd integer with all different digits that is divisible by 11. Working on this question student might start to notice what happens to the digits of whole numbers divisible by 11 when number 11 is added. The solution is number 143.

In the second part of the worksheet, students have to work with standard forms of whole numbers that contain variables in place of some digits. It starts with examining two digit numbers in the question Q5. Students have to find all the solutions for the unknown digit "*d*" in place of tens in the number with standard form *d*7 to obtain a number divisible by 11. After finding out there is only one such number (77), they pass on to the next question Q6 of the workbook.

In this question students have to substitute values for the missing digit in place of tens in three digit whole numbers and find all the ways how to create numbers divisible by 11. There is only one three digit number that satisfies the conditions and in this the missing digit is the sum of the two other digits. In Figure 3 the assignment and the solution of the question are shown.

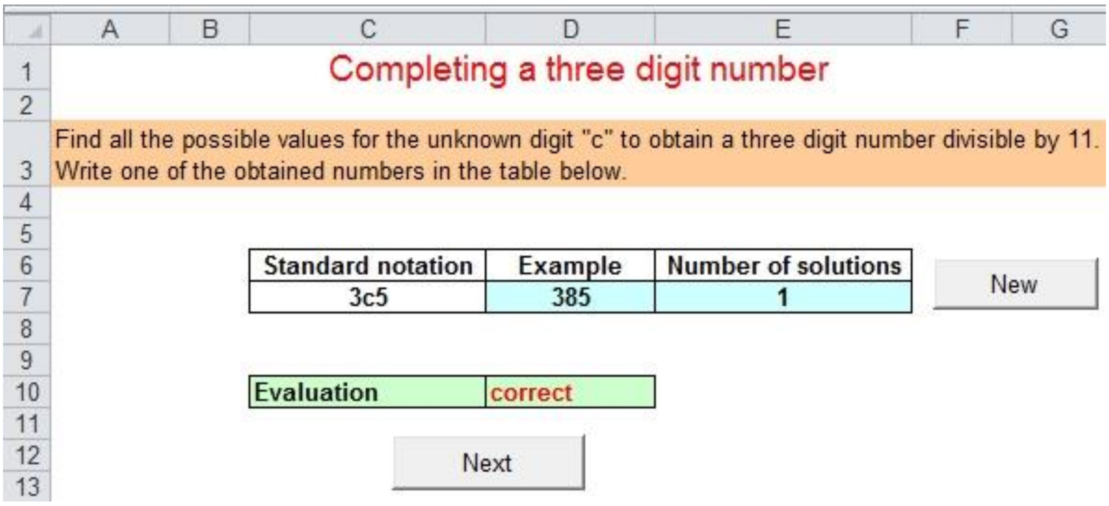

Figure 3: Q6 – investigation of three digit numbers

To check the students' solutions in the cell D10 the next formula is entered:

=IF(OR(ISBLANK(D7);ISBLANK(E7));"unfilled";IF(AND(D7=385;E7=1);"correct";

"incorrect"))

The following two questions Q7 and Q8 of the workbook are dedicated to the creation of three digit whole numbers divisible by 11, in which the only given digit is in the tens position. In the question Q7, the standard form of a three digit number is given as *a*8*b*. Students have to substitute values for the unknown digits and find a few three digit numbers divisible by 11. Then they have to formulate a relationship between the unknown digits in the units and in the hundreds position of the found numbers. They have to discover that the three digit number *a*8*b* is divisible by 11 only if $a = 8 - b$, where $b < 8$.

The question Q8 is analogous. In order for students not to generalize their findings by examining a number of special cases, they have to create three digit numbers divisible by 11 from the standard form *a*1*b*. The condition $b > 0$ is given in order for the students not to work with the number 110. After finding some suitable three digit numbers students can formulate the relationship between the numbers that can be substituted for the variables *a*, *b*: $a + b - 1 = 11$, where $b > 2$. The sheet with the solution of the question Q8 is shown in Figure 4. Evaluation of solutions and the function of the Next button are implemented in a similar way to that in the question Q6 illustrated in Figure 3.

After experimenting with numbers in standard form, students should try to justify their findings using the expanded form of number *a*1*b*: $a \cdot 10^2 + 1 \cdot 10^1 + b$. The Help button suggests writing number *a*1*b* using powers of 10 and expressing the power 100 as a sum of two numbers where one of them has to be the closest number to a hundred, divisible by 11. After the described rearranging, students should get the expression

$99a + a + 10 + b$. As number $99a$ is divisible by 11 for any value of a, then $a + 10 + b$ must be divisible by 11. Given the values of the variables a, b (a and b must be greater than 0) this expression must have the value of 22.

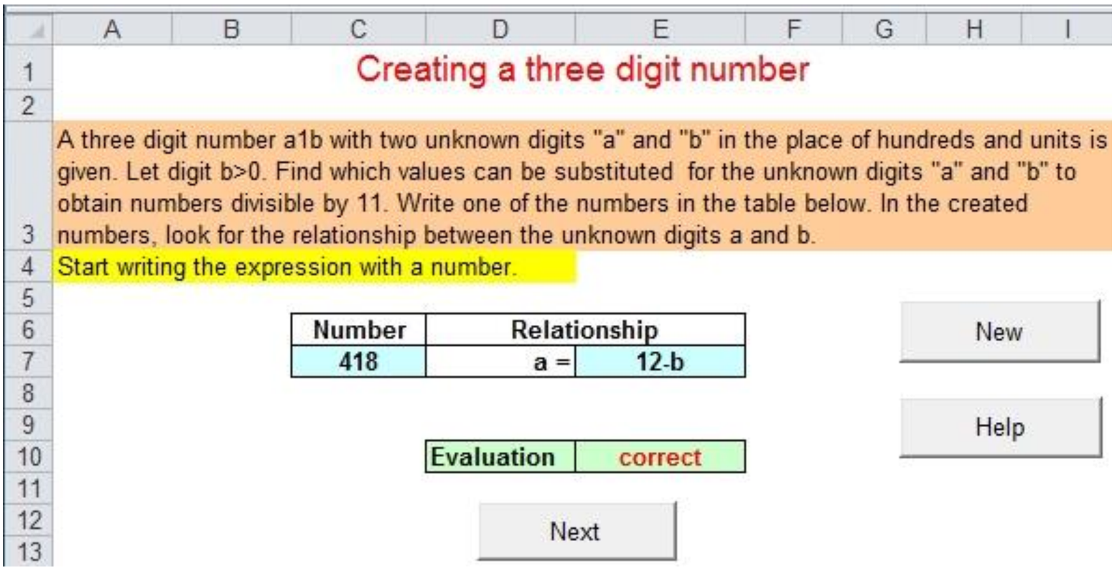

Figure 4: Q8 - finding the relationship between numbers

In the question Q9, the students have to find a condition of divisibility by 11 for a three digit number in the form *abc*.

Q9: Using variables a, b and c in the place of digits, we can write the standard form of three digit numbers as abc. When looking for the divisibility rule by number 11 for three digit numbers, it is useful to subtract one from the hundreds and add one to the tens in the extended form. Write an expression containing only variables a, b, c that must be divisible by 11.

The Help button in this question includes instructions for students to once again write the number *abc* in an expanded form with units that are powers of 10. When rearranging it is appropriate to add number 1 to the tens and subtract number 1 from the hundreds. If students perform the instructions correctly, they will get to the expression $99a + 11b + a - b + c$. If the resulting number has to be divisible by 11, the expression $a - b + c$ must be divisible by 11 too.
In the last two questions Q10 and Q11 the students have to work with four digit numbers. First they have to find the missing digits in numbers 327*d* and 7*d*83 to obtain a number divisible by 11. After solving this correctly, the students are presented with the last question Q11 of the workbook. The assignment and one correctly solved expression written in cell E7 are shown in Figure 5.

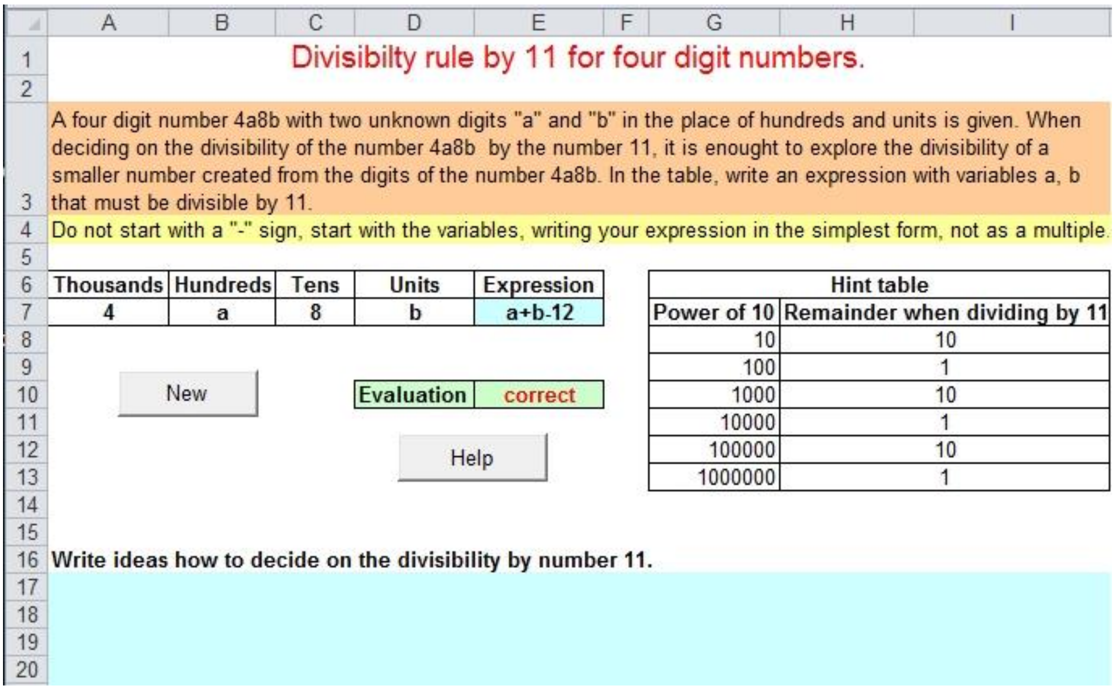

Figure 5: Q11 - formulating the discovered properties

Clicking on the Help button, students get instructions to use the Hint Table and observe the remainders when dividing the consecutive powers of 10 by 11 and how do these change. They should use the observed property in the expanded form of whole numbers in which they should alternately once add and then subtract a one. After entering a correct expression into cell E7 students should try to generalize their findings and formulate the rule for divisibility by 11. Their ideas may be written the designated area of cells.

In this question, it could be expected, the expression $a + b - 12$ will be entered in the cell E7. However, there are other, correct forms of the answer, like $a + b - 1$ or any multiple of it, e.g. $2a + 2b - 24$. Other than the rules given, we also tried to analyse the answers more precisely using logic functions and text functions, e.g. to test if the answer starts with the sequence of three characters *a+b*. In this question, we used the cells F6, F7, F8 for continuous calculations. The following formulas are entered into these auxiliary cells.

F6: =IF(OR(ISBLANK(E7);LEN(E7)<5);0;1)

F7: =LEFT(E7;3)

F8: =IF(F6=0;0;VALUE(MID(E7;4;LEN(E7)-3)))

In cell F7, we identified the first three characters from the left and in cell F8, the number that should be in the expression after the variables *a*, *b*. In cell E10, there's the final formula:

=IF(ISBLANK(E7);"unfilled";IF(F6=0;"incorrect";IF(OR(AND(F7="a+b";MOD(F8;11)=10);AND(F7="b+a";MOD(F8;11)=10));"correct";"incorrect")))

When using the workbook in class, a mathematics teacher should allow students to work independently. Students could work individually or in pairs. In class a teacher would primarily work as a consultant and observer. If he/she saw some students have long been devoted to dealing with one question, then he/she could further explain the question or give them a more detailed guidance than the ones incorporated in the workbook.

3. Discussion on the usability of the workbook in teaching mathematics

The described workbook was tested in three classes in the first year of secondary school (students aged 15–16 years). In two classes, students worked in pairs. Students had to solve the questions of the workbook in about 40 minutes. Solving problems from the first part of the workbook did not cause any difficulties for students. Only when exploring the numbers with variables in place of missing digits some students needed help from the teacher. The teacher often asked for an explanation why their answer is wrong. Most questions to the teacher lead to an explanation of the rearranging in the expanded form of the number given. Another problem was that students, after examining a few specific three digit numbers, determined that the sum of the digits in the place of units and hundreds is equal to the value of the digit in the tens place but could not extend this finding nor generalize it.

Another misconception identified was the conclusion that the difference of the sum of the digits in odd places and the sum of the digits in even places must be 0 or 11. In this case, it was suitable to guide the students to observe numbers larger than four digit numbers. The following dialogue occurred between a student and a teacher in one of the experimental classes.

T: According to your conclusion, what must the difference of the sum of the digits in odd places and the sum of the digits in even places be?

S: 0 or 11.

T: How did you come to this conclusion?

S: It is working for the numbers from the questions and I've tried it for some others.

T: Even bigger than four digits?

S (after looking through the sheets): No.

T: Is your conclusion correct for the five digit numbers 31 284 and 80 971?

S (after checking): Yes, as I've said it. For the first one, the difference is 0 and for the second one, it is 11.

T: OK. Now try to find the missing digit in the number 9*d*939 to obtain a number divisible by 11.

The student was thinking for a while and then used the numbers 0, 1, 2 for the missing digit and divided by 11 using the formula for division.

S: It is number 2.

T: And what is the difference of the sum of the digits in odd places and the sum of the digits in even places?

S: 22. Oh, I get it now, it could be 0, 11, 22, basically multiples of 11.

This was very common, that students could only formulate the rule only for four digit whole numbers even though in the last question Q11 a table is given with the remainders after dividing consecutive powers of 10 by 11, up to the power of 10^6. In the time available to formulate the rule for the divisibility by number 11 about a third of students did it correctly.

An important milestone for generalization was question Q9. In the first class that used the workbook, many students related their findings to calculations with specific numbers, in which they were observing the sum of the digits in odd places and compared it to the middle digit (the digit in the even place). Hence they wrote the expression $a - b + c$ in the form $a + c - b$ which was evaluated as incorrect in the first version of the workbook and needed to be changed. In the newest form of the workbook, this answer (and two others) is now accepted too. This pointed out that, using the workbook; students might need the teacher's help when writing expressions into cells. In some cases, students may properly solve the question, but when entering data to the table they might not follow the specified instructions and hence their answer will be evaluated as incorrect. That is why the teacher should emphasize the rules of writing the expressions to the students or if they ask why their solution is not correct, guide them to change it according to the rules given. Even if the students do not encounter a problem of this kind, in the final discussion, the teacher could encourage them to think about why such rules are given.

The questions from the workbook were aimed at observing and explaining changes in digits when continuously adding number 11 what many students had not done at all. When using the workbook, we also noticed a common misconception; some students believed that three digit numbers with the same digits (111, 222 ...) are divisible by 11. Therefore, we propose an alternative path that starts with the question: Determine whether number 770 is divisible by 11, then successively add 11 three times and describe how the digits have changed in all three numbers obtained. In the follow-up question students would work with four digit numbers. Determine whether number 4499 is divisible by 11, then add 11 and describe how the digits have changed. Then, to this new number add the smallest multiple of 11 to get a number greater than 4600 and divisible by 11. Describe the relationship between the digits in this final number. After solving the above proposed questions, a teacher with the students would formulate the rule for divisibility by number 11.

If a teacher does not want to devote a lot of time to teach this topic in school, he/she could, in our view, set this workbook as a home exercise. In a lesson he/she would then analyse selected students' solutions. Experience from the use of the workbook in school practice showed that some students have difficulties with variables in place of digits in the number notation and cannot independently use the expanded form of whole numbers in solving the questions from the second part of the workbook. Presumably, at home students should be able to solve the questions aimed at working with concrete whole numbers. If so, a teacher then would not need to introduce variables in a lesson and justify the divisibility rule for number 11, but could work with the experience of students with investigating the divisibility of particular whole numbers and submit a series of alternative follow-up questions for students.

Using the workbook in school practice has shown that it can support active learning. Implementation of feedback allows to quickly identify problems of students and it helps the teachers diagnose these and continually remove them. The possibility of using auxiliary information and instructions provides the students with individual learning paths and individual level of autonomy in the discovery of new findings depending on their knowledge and skills in mathematics.

4. Conclusion

Planning and implementing inquiry activities for learning of specific mathematical topics is a challenge for teachers. It requires creation of learning situations and conditions to allow students to experiment and actively discover new knowledge based on their experience and about how to improve understanding of mathematical concepts, patterns and relationships. The questions forming the basis of the described workbook together with the spreadsheet environment providing tools for calculations and an opportunity to incorporate interactivity and feedback, should be used with regards to student learning needs.

Acknowledgment

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