Spreadsheets in Education (eJSiE)

Volume 9 | Issue 1 Article 1

1-4-2016

Combinatorics of the Triangle Inequality: From Straws to Experimental Mathematics for Teachers

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Recommended Citation
Abramovich, Sergei (2016) Combinatorics of the Triangle Inequality: From Straws to Experimental Mathematics for Teachers, *Spreadsheets in Education (eJSiE)*: Vol. 9: Iss. 1, Article 1.
Available at: http://epublications.bond.edu.au/ejsie/vol9/iss1/1

Combinatorics of the Triangle Inequality: From Straws to Experimental Mathematics for Teachers

Abstract
This article demonstrates the importance of skills in asking questions and the value of spreadsheets in seeking answers in the spirit of experimental mathematics. This is accomplished through an activity with straws recommended for lower elementary mathematics classrooms that was expanded to a combinatorial inquiry into the triangle inequality. Whereas the article is a reflection on a mathematics content and methods course taught by the author to elementary teacher candidates, combinatorial explorations motivated by an unexpected question by one of the candidates can be recommended for mathematics education courses of higher ranks. As a result of these explorations, a family of integer sequences (not included into the OEIS®) has been introduced.

Keywords
teacher education, spreadsheets, experimental mathematics, combinatorial geometry

1. Introduction

One of the activities included in a curriculum guide by the Education Department of the state of New York [19], the context in which the author prepares teachers, recommends that students in grades 1 and 2 use straws to construct geometric figures. As presented, this hands-on activity is in support of the development of children's mathematical thinking and reasoning skills. Typically, straws available for this exploration are all the same and a variety of plane figures – triangles, quadrilaterals, pentagons, etc., can be constructed out of the straws. When this activity is discussed within an elementary mathematics content and methods course, mathematically unsophisticated teacher candidates perceive such use of straws as a "really funny" way of demonstrating how a particular figure looks like and, thereby, their joy about mathematics might end after a figure is constructed. On the contrary, the task with straws has the potential to extend this amateurish sense of a learning experience to enable, in the spirit of the experimental mathematics approach [2, 7, 8, 23], the discovery of knowledge, which is unlikely to be found in any part of the entire school mathematics curriculum.

What teacher candidates are supposed to learn in the modern university classroom is how to go beyond the seemingly mundane character of this (or similar) tasks they naively consider a "worthwhile exploration" simply due to its trendy hands-on setting and the use of everyday objects as mathematical thinking tools. In support of this learning objective, the paper intends to share some teaching ideas about possible uses of the task in different mathematics teacher education courses that span the whole pre-college curriculum. The paper may be of interest to the teachers of elementary school mathematics as it stems from a hands-on activity of using straws in the context of basic geometry. It may also be of interest to secondary mathematics teachers often considered the mathematical knowledge holders by their elementary colleagues who strive to answer rather innocent questions frequently asked by young children in the modern classroom. That is why, both groups of teachers need to know what kind of questions young children, when being encouraged, might ask and where answering (or thinking about how to answer) these questions can lead in the technological paradigm.

It should be noted that without using a spreadsheet, the teaching ideas of this paper would not likely come to fruition. The final result about the limiting behavior of certain number sequences associated with the original task, because of its quite unexpected mathematical simplicity in comparison with computationally complex (though elementary) learning context indicates that the opportunities for mathematical discovery by students and their teachers alike are rife. Furthermore, the final result demonstrates how the resulting elegance and even beauty of conclusions of experimental mathematics may serve as a confirmation of the validity of intermediate explorations. Put another way, a spreadsheet will be used to demonstrate how the computational experiment approach to precollege mathematics curriculum can motivate and enhance mathematical reasoning skills of prospective teachers of mathematics, something that they, in turn, need to develop in their own students. Indeed, schoolteachers as the major players of the mathematics education enterprise are responsible for developing in their students "the ability to reason mathematically, an appreciation of the beauty and power of mathematics, and a sense of enjoyment and curiosity about the subject" [10, p1].

2. Motivating "discovery" of the triangle inequality through questioning

As a reflection on the basic activity with straws, some simple questions can be asked first: How many straws are needed to construct a triangle, a square, a pentagon? Assuming that one typically uses three straws for triangle, four straws for square, five straws for pentagon, such queries follow the framework "single question – single answer", not leaving much room for the development of mathematical reasoning. In order to overcome the limitations of this (traditional) framework, elementary teacher candidates should be encouraged to demonstrate multiple ways of constructing a particular geometric figure. Such kind of encouragement to think deeper about mathematics follows the idea of "recognizing in a result something that can be turned into a question" [15, p98]. For example, after a triangle and a square have been constructed out of, respectively, three and four straws, this modest result may be turned into a question "posed in a routine way so as to obscure mathematical thinking" [15, p103]: Can one construct a square out of three straws and a triangle out of four straws? Whereas in response to the first part of the question one may return a smile (meaning, of course, not), its second part would most likely lead to head scratching followed by requesting a specific type of explanation [13].

A mathematical concept that emerges from the activities with straws as a means of explaining the failed construction of a triangle out of four straws is the so-called triangle inequality – the sum of any two sides of a triangle is greater than the third side. Surprisingly, the triangle inequality is a concept that everybody intuitively possesses without even realizing it (just watch how people cut across the grass rather than walk along the pavement to get to a building faster). At the same time, the concept is profound for it is kind of counterintuitive that moving from three straws to four straws screws up one's chances to construct a triangle. No wonder, four is the only integer greater than three that may not be decomposed into three like summands satisfying the triangle inequality.

3. An unexpected question seeking information

In the modern classroom, not only teachers ask questions but students are encouraged to ask questions as well. A need for change in the classroom pedagogy, away from passive learning with the only one voice heard (that is, the teacher's voice) was acknowledged in [24] where concern with "the underlying power relationship between teacher and child: the children seem to learn very quickly that their role at school is to answer, not to ask questions" (p279) was registered. Two decades later, Slater [22] referred to this kind of learning as a hidden contract between students and teachers: when students ask questions (educationally challenging or not), it is often considered as an attempt to break the contract on their part. Nonetheless, some students do ask questions and expect teachers, following the teaching standards of the 21st century [3, 9, 10, 16–18] to answer them "fluently and with little time" [6, p43]. Most recently, expectations for schoolchildren in the United States included skills to "listen or read the arguments of others … and ask useful questions to clarify or improve the arguments" [9, p7]. Teacher candidates need to have experience with asking and answering a variety of mathematical questions because it is one of the main characteristic features of mathematics when students may quite naturally ask questions, either seeking information or requesting explanation [13], for which even veteran teachers do not have clear and easy answers.

An example of such a question is as follows:

How many triangles – equilateral, isosceles, scalene – can be constructed if the longest side comprises three straws?

An elementary teacher candidate asked this question during the author's presentation of the activity with straws and its extension into the triangle inequality. This unexpectedly wonderful question (its original wording was slightly different) was possible to answer experimentally by actually constructing four triangles and describing them through the following integer triples satisfying the triangle inequality: (3, 3, 3), (3, 2, 3), (3, 1, 3), and (2, 2, 3). It was also possible to abstract from straws to square tiles (a common manipulative for the elementary classroom) as precursors of units (out of which whole numbers are built) to represent the four triangles in the form of tri-towers as shown in Figure 1.

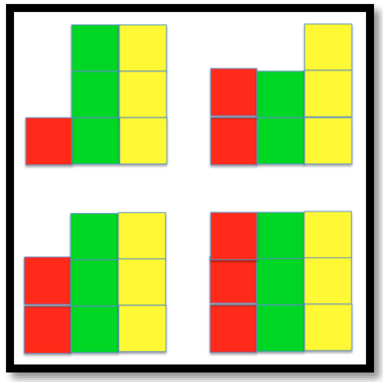


Figure 1. Abstracting from straws to square tiles.

It is obvious that replacing three by four (or five, six, etc.) would make experimenting with straws (and even with square tiles) too cumbersome. Apparently, the number of triangles would increase as the number of straws used for the longest side increases. But some teacher candidates continued being curious: If indeed increased, then by how many? An answer to this question was not immediately available and thus it was postponed till the next class.

4. Using a spreadsheet as a modeling tool

In order to address teacher candidates' curiosity, a spreadsheet was used to model the system of inequalities $k \leq m \leq n$, $k + m > n$ where n is the given positive integer, while positive integers m and k may vary. Such a spreadsheet is shown in Figure 2 where k and m are defined, respectively, in the ranges A3:A27 and B2:Z2; cell A2 displays the total number of triangles with the longest side built out of the given number of straws (n, displayed both in the body of the spreadsheet and in cell A1, which is slider-controlled to allow for the variation of n). As shown in Figure 2, when n = 25 in cell A1, the body of the spreadsheet displays this value of n each time the corresponding triple (k, m, n) satisfies the triangle inequality. For example, the cells A14, O2, and O14 display the triple (12, 14, 25). In addition, the spreadsheet can be programmed to show (the range AB3:AC27) the relation between n and the corresponding number of such triangles, $T(n)$.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	AA	AB	AC
1	25																												
2	169	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25		*n*	*T*(*n*)
3	1																									25		1	1
4	2																								25	25		2	2
5	3																							25	25	25		3	4
6	4																						25	25	25	25		4	6
7	5																					25	25	25	25	25		5	9
8	6																				25	25	25	25	25	25		6	12
9	7																			25	25	25	25	25	25	25		7	16
10	8																		25	25	25	25	25	25	25	25		8	20
11	9																	25	25	25	25	25	25	25	25	25		9	25
12	10																25	25	25	25	25	25	25	25	25	25		10	30
13	11															25	25	25	25	25	25	25	25	25	25	25		11	36
14	12														25	25	25	25	25	25	25	25	25	25	25	25		12	42
15	13													25	25	25	25	25	25	25	25	25	25	25	25	25		13	49
16	14														25	25	25	25	25	25	25	25	25	25	25	25		14	56
17	15															25	25	25	25	25	25	25	25	25	25	25		15	64
18	16																25	25	25	25	25	25	25	25	25	25		16	72
19	17																	25	25	25	25	25	25	25	25	25		17	81
20	18																		25	25	25	25	25	25	25	25		18	90
21	19																			25	25	25	25	25	25	25		19	100
22	20																				25	25	25	25	25	25		20	110
23	21																					25	25	25	25	25		21	121
24	22																						25	25	25	25		22	132
25	23																							25	25	25		23	144
26	24																								25	25		24	156
27	25																									25		25	169

Figure 2. There are 169 (A2) triangles with the largest side 25 (A1) linear units.

The details of spreadsheet programming (not intended for elementary teacher candidates) are included in the appendix. Pedagogically speaking, one can introduce a ready-made spreadsheet to this group not focusing on the development of skills in programming the software (something that may be appropriate for secondary teacher candidates or within a special course on the use of spreadsheets in mathematics education). What is much more important for prospective elementary teachers is to develop skills in using numerical evidence as support system in thinking mathematically; that is, to have experience in recognizing common properties that the numbers (otherwise unavailable) possess and establishing connections among them in a general form, independent on the concreteness of the support system. In other words, numerical evidence that a spreadsheet (or any other software) generates "can be used to support the reification … [and] draw attention to what is being stressed, … to make the abstraction shift in which the generality becomes object" [14, pp6 – 7].

5. Analysis of modeling data enables qualitative explanation

Observing the chart of Figure 3 (replica of the range AB3:AC27 in Figure 2) one can note that every second number in the bottom row is either a perfect square or a product of consecutive integers. For example, consider the triple (16, 20, 25) from this row (alternatively, cells AC9, AC10, and AC11 in the spreadsheet of Figure 2).

We have $16 = 4^2, 20 = 4 \times 5, 25 = 5^2$. How to explain that when the number 7, as a largest side, is replaced by the number 8, the number of triangles is increased by four? Modeling data provided by the spreadsheet not only facilitates explanation. The data creates a grade-appropriate opportunity for elementary teacher candidates to use decontextualized arithmetical activities such as partitioning of an integer in two summands [19] in application to geometry. One can see that the replacement of 7 by 8 adds eight new triangles with the side lengths satisfying the triangle inequality: $(1, 8, 8), (2, 8, 8), ..., (8, 8, 8)$. At the same time, some old triangles with the sum of the smaller sides *equal* to eight become extraneous (as, due to the triangle inequality, the sum has to be *greater* than eight). The number of such triangles is equal to the number of additive partitions of eight in two parts without regard to order and there are four ways of doing that: $8 = 1 + 7 = 2 + 6 + 3 + 5 = 4 + 4$. That is, the number of new triangles is equal to eight itself and the number of triangles which become extraneous is equal to the number of unordered additive partitions of eight in two parts. This *explains* why the transition from the largest side 7 to that of 8 increases the number of triangles by the difference 8 – 4 = 4. Put another way, the increase in the number of triangles can be calculated as follows: $4 \times 5 - 4^2 = 4(5 - 4) = 4$. Similar observations can be made in the case of other pairs of consecutive integers used as the largest side length. For example, the towers in the upper part of Figure 1 would disappear through the transition from 3-tile high tri-tower to that of 4-tile high because 1 + 3 = 4 and 2 + 2 = 4. All this motivates generalization, something that goes beyond the elementary level and can be discussed (along with the details of spreadsheet programming) in a mathematics education course for secondary teacher candidates. This discussion may include the whole story about the straws in order to demonstrate the importance of the method of conceptual ascend in the teaching of mathematics reflected in the modern day tenet, "What students can learn at any particular grade level depends on what they have learned before" [9, p5]. To this end, the remaining part of the article extends elementary content and it demonstrates different problem-solving avenues that a grade-appropriate generalization entails.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	6	9	12	16	20	25	30	36	42	49	56	64	72	81	90	100	110

Figure 3. The relation between the largest side and the number of triangles.

6. Moving from straws to formulas improves computational efficiency of a spreadsheet

Let $T(n)$ be the total number of integer sided triangles the largest side length of which is equal to n linear units. The task is to find a recursive relationship between $T(n + 1)$ and $T(n)$. The transition from n to $n + 1$ adds $n + 1$ new triangles with the side lengths

$$(n+1, n+1, 1), (n+1, n+1, 2), \ldots, (n+1, n+1, n+1).$$

At the same time, several triangles included in the count $T(n)$ disappear through this transition. For example, one such triangle has the side lengths $(n, n, 1)$. Indeed, when the first element is replaced by $n + 1$, the triple $(n + 1, n, 1)$ becomes extraneous because the triangle inequality is not satisfied. Likewise, all other triples with the sum of the second and the third elements equal to $n + 1$ become extraneous. There are INT($(n + 1)/2$) such triples. Therefore, the total gain in the number of triangles occurring through the transition from n to $n + 1$ is equal to $T(n+1) - T(n) = n+1 - INT(\frac{n+1}{2})$ from where a difference equation that defines a recursive relation between $T(n + 1)$ and $T(n)$ results

$$T(n+1) = T(n) + n + 1 - INT(\frac{n+1}{2}), T(1) = 1. \qquad (1)$$

Relation (1), a generalized computational tool, when verified within a spreadsheet yields the same numbers that are displayed in the bottom row of the chart of Figure 3. Better still, using relation (1) does not require two-dimensional spreadsheet modeling. However, it is due to a computationally inefficient approach of using the triangle inequality in modeling the sequence $T(n)$ that much more efficient model in the form of relation (1) has been developed. Furthermore, one can use relation (1) to develop a closed formula for the sequence $T(n)$. Because of the presence of the greatest integer function, the cases of n even and odd need to be considered separately as $INT(\frac{2k+1}{2}) = k$ and $INT(\frac{2k+2}{2}) = k+1$. It follows from (1) that

$$T(2k) = 1 + 2 + \ldots + 2k - 2(1 + 2 + \ldots + k - 1) - k = k(k+1)$$

and

$$T(2k+1) = 1 + 2 + \ldots + (2k+1) - 2(1 + 2 + \ldots + k) = (k+1)^2.$$

These formulas (the derivation of which is not difficult and, thereby, is skipped for the sake of brevity) confirm the results presented in the chart of Figure 2, and, besides offering an alternative to computing $T(n)$, provide a closed solution to the general form of the questions asked by an elementary teacher candidate.

7. Combinatorial geometry with colored straws

What if we have straws in different colors? Such straws, though not commonly available, may still be used in the classroom to make mathematics even more "akin to that of playing games … [seen by educators as] the spontaneous way in which children acquire much of their mastery over the environment" [12, pp80–81]. With this in mind, consider the case of two colors. How many different triangles can be constructed out of straws in two colors if none of the sides comprises more than three straws? As was shown above, when straws are the same size and color, there exist four triangles, namely,

(3, 3, 3), (3, 3, 2), (3, 3, 1), and (3, 2, 2).

How many triangles with the side lengths (3, 3, 3) can be made of the straws in two colors? When we have three straws in two colors, each straw (the unit of length) can be chosen in two ways and thus, by the rule of product [1], there are 2^3 different types of 3-straw sides. (For example, with white (W) and pink (P) straws the eight types are: WWW, WWP, WPW, WPP, PWW, PWP, PPW, PPP). In order to construct an equilateral triangle with 3-straw sides, one has to select three objects out of eight types allowing for the repetition of objects (e.g., when two or three sides are the same). Each such selection can be represented as a number with three ones (the number of objects selected) and seven zeroes (serving as separators among eight types – in the above list commas are the separators); the number of permutations of the digits in the number 1110000000 is equal to $\frac{(3+7)!}{3!\,7!}=120$. For example, the number 1110000000 (a permutation of ten digits) means that only one type (WWW) of a 3-straw side is used in the construction of a triangle and the number 1000000011 (another permutation of ten digits) means that one side of a triangle is the WWW type and its other two sides are both the PPP type. Additional information on the development of combinatorial formulas introduced in this section can be found in [25] and, for a more elaborate treatment used by the author in a mathematics content course for elementary teacher candidates, in [1].

In general, $E(n, p)$ – the number of equilateral triangles with the side length n constructed out of straws in p colors – is equal to

$$\frac{(3+p^n-1)!}{3!\left(p^n-1\right)!}=\frac{(p^n+2)!}{6\left(p^n-1\right)!}=\frac{p^n(p^n+1)(p^n+2)}{6}$$

as the number of permutations of digits in the number 11100…0 with three ones (the number of sides in a triangle) and $p^n - 1$ zeroes (serving as separators among p^n types of multicolored sides constructed out of *n* straws in *p* colors). That is,

$$E(n, p) = \frac{p^n(p^n+1)(p^n+2)}{6}. \qquad (2)$$

In particular, when $p = 2$ and $n = 3$ we have $\frac{2^3(2^3+1)(2^3+2)}{6} = \frac{8 \times 9 \times 10}{6} = 120$ triangles.

How many triangles with the side lengths (3, 3, 2) can be made of the straws in two colors? Firstly, there are 2^2 different types of 2-straw sides. Secondly, for equal sides, we have to select two objects out of 2^3 types, something that can be done in $\frac{(2+7)!}{2!\,7!} = 36$ ways – the number of permutations in the (9-digit) number 110000000 with two ones (the number of equal sides) and seven zeroes. For each such selection of a pair of 3-straw sides there are 2^2 selections of a 2-straw side. By the rule of product, there are $36 \times 4 = 144$ isosceles triangles with the side lengths (3, 3, 2) constructed out of straws in two colors.

In general, $I_1(n, p)$ – the number of isosceles triangles with the side lengths (*n*, *n*, *m*), $n > m$, constructed out of straws in *p* colors – is equal to

$$\frac{(2+p^n-1)!}{2!\left(p^n-1\right)!} \times p^m = \frac{(p^n+1)!}{2\left(p^n-1\right)!} p^m = \frac{p^{n+m}(p^n+1)}{2}.$$

That is,

$$I_1(n, p) = \frac{p^{n+m}(p^n+1)}{2}. \qquad (3)$$

In particular, when $p = 2$, $n = 3$, and $m = 2$ we have $\frac{2^5(2^3+1)}{2} = \frac{32 \times 9}{2} = 144$ triangles. Likewise, when $p = 2$, $n = 3$, and $m = 1$ we have $\frac{2^4(2^3+1)}{2} = \frac{16 \times 9}{2} = 72$ triangles with the side lengths (3, 3, 1) constructed out of straws in two colors.

How many triangles with the side lengths (3, 2, 2) can be made of the straws in two colors? That is, we have an isosceles triangle which base is the longest side. This time, we have to select two objects out of 2^2 types, something that can be done in $\frac{(2+3)!}{2!\,3!} = 10$ ways – the number of permutations of digits in the (5-digit) number 11000. For each such selection, there are 2^3 types of 3-straw sides. Therefore, by the

rule of product, there are 80 ($=10\times 8$) isosceles triangles with the side lengths (3, 2, 2) constructed out of straws in two colors. Once again, with white (W) and pink (P) colors, the types of equal sides are WW, WP, PW, and PP, so that the permutation 10001 denotes an isosceles triangle with the lateral sides of the WW and PP types (the base of which can be chosen from the eight types listed above).
In general, $I_2(n, p)$ – the number of isosceles triangles with the side lengths (n, m, m), $n > m$, constructed out of straws in p colors – is equal to $\frac{(2+p^m-1)!}{2!\left(p^m-1\right)!}\times p^n=\frac{(p^m+1)!}{2\left(p^m-1\right)!}p^n=\frac{p^{n+m}(p^m+1)}{2}$.
That is,

$$I_2(n, p)=\frac{p^{n+m}(p^m+1)}{2}. \quad (4)$$

In particular, when $p = 2$, $n = 3$, and $m = 2$ we have $\frac{2^5(2^2+1)}{2}=\frac{32\times 5}{2}=80$ triangles with the side lengths (3, 2, 2) constructed out of straws in two colors.
The case when the longest side comprises three straws does not allow one to construct a scalene triangle. With this in mind, consider the general case of constructing scalene triangles of side lengths (n, m, k) out of straws in p colors. Each of the sides can be constructed, respectively, in p^n, p^m, and p^k ways. By the rule of product, the number of such scalene triangles is equal to

$$S(n, p)=p^{n+m+k}. \quad (5)$$

8. Experimental mathematics as a method with ancient roots

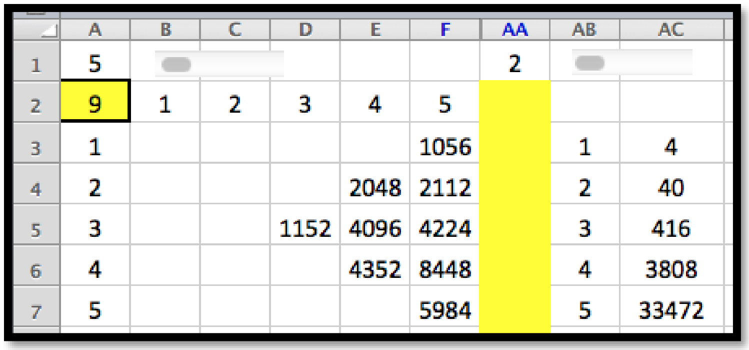
The spreadsheet of Figure 4, the programming of which is based on formulas (2) – (5), shows (cell AC5) the total of 416 (=120 + 144 + 80 + 72) triangles found above for $n = 3$ and $p = 2$. Unlike the sequence 1, 2, 4, 6, 9, … (Figure 3) included into the OEIS® (The On-line Encyclopedia of Integer Sequences, https://oeis.org/), already the sequence 4, 40, 416, 3808, 33472, … – the total number of triangles the longest side of which comprises respectively, 1, 2, 3, 4, 5, … straws in two colors – does not match anything in the table. Likewise, the sequence 10, 300, 9405, 271701, 7586055, … (that the spreadsheet of Figure 4 generates for $p = 3$) – the total number of triangles the longest side of which comprises, respectively 1, 2, 3, 4, 5, … straws in three colors – does not match anything in the table as well. Nonetheless, the last two sequences, along with the sequence $T(n)$ defined by formula (1), exhibit a similar behavior: if one forms (just as in the case of Fibonacci numbers) the ratios of two consecutive terms of the sequence, the ratios, as n increases, appear to converge to the third power of the number of colors. In general, the following technology-motivated proposition can be formulated.

Let f(n, p) denote the number of triangles – equilateral, isosceles, scalene – constructed out of equal straws in p colors when the longest side comprises n straws. Then

$$\lim_{n\to\infty}\frac{f(n+1,p)}{f(n,p)}=p^3. \tag{6}$$

To prove relation (6), one has to recognize that in order to find $f(n, p)$ for any given values of p and n, formulas (2) – (5) have to be used and the summation of geometric series with the terms p^i when i varies from 1 to n yields an algebraic expression with the leading term p^{3n}. One can see that whereas such demonstration of relation (6) is not complicated (once the computational algorithm has been constructed), it seems unlikely that, without computer-motivated query into the number of triangles constructed out of multicolored straws for which formulas (2) – (5) have been developed, this relation would come into existence.

This approach of enabling mathematical explorations through the use of technology has ancient roots as evidenced from a letter by Archimedes to Eratosthenes: "Certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said mechanical method did not furnish an actual demonstration. But it is of course easier, when the method has previously given us some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge" [5, p13]. Indeed, without a "mechanical method" that included a hands-on activity with multi colored straws followed by a spreadsheet-based computational experiment, an amazingly simple property of sequences (not even included into the OEIS®) that can be explained to a layperson may not come to light. In the modern terms, through such an explanation teacher candidates learn what it might mean for their students to develop "the ability to *contextualize* … [that is] to probe into the referents for the symbols involved" [9, p6, italics in the original]. In particular, through the process of contextualizing one can explain a computationally discerned property of the faster convergence of the ratios in (6) with the increase of p – the larger the number of colors, the larger is the number of triangles that can be constructed for the same value of n. That is, due to the inequality $p^{n+1}<(p+1)^n$, the number of triangles, as $p \geq 2$ increases, grows (exponentially) faster than with the increase of $n \geq 3$.



	A	B	C	D	E	F	AA	AB	AC
1	5						2		
2	9	1	2	3	4	5			
3	1					1056		1	4
4	2				2048	2112		2	40
5	3			1152	4096	4224		3	416
6	4				4352	8448		4	3808
7	5					5984		5	33472

Figure 4. The case of two colors yields the sequence 4, 40, 416, 3808, 33472, … .

9. Concluding remarks

This article has demonstrated how experimental mathematics for teachers supported by a spreadsheet can be integrated with hands-on activities recommended as means of developing young children's mathematical reasoning and thinking skills. Such skills include the keenness of asking mathematical questions that are aimed at changing the traditional educational milieu of the past. Although it is sometimes argued, "in the real world, most of the time, an answer is easier than defining the question" [11, p163], in order to be adequately prepared to deal with the real world, one must be given opportunities to satisfy natural curiosity by asking questions regardless what kind of answer they entail. In order to develop those skills in the pre-college classroom, such cognitive milieu must become a tradition in a teacher education classroom. When extended in a computer environment, the suggested activities can support content and method courses for prospective elementary as well as secondary mathematics teachers. By changing the traditional classroom culture of discouraging (or, at least, not encouraging) students to ask questions [22, 24], one can come across a challenging mathematical context for which the experimental mathematics approach is justified. Put another way, a task of constructing triangles out of straws, as the genesis of the discovery of new integer sequences, represents an example of hidden mathematics curriculum of teacher education [2].

The article demonstrated how numerical data and its proper analytic interpretation enable both types of questions – those seeking quantitative information and those requesting qualitative explanation – to be formulated and resolved. Indeed, comparing computer-generated integer triples the largest element of which is equal to n to those with $n + 1$ being the largest, makes it possible to explain qualitatively what happens with the corresponding triangles when n is increased by one. By the same token, a teaching idea of abstracting straws to square tiles as a precursor to numbers provides an alternative avenue for contextualizing the meaning of the triangle inequality.

By probing the triangle inequality with straws and using a spreadsheet to support the emerging inquiry, one can appreciate the notion that mathematics has evolved from concrete activities to abstract concepts through argument and computation [4]. In the digital era, a mathematical argument can emerge from the results of experimental mathematics. That is why having some kind of confirmation of the validity of the argument is of importance. In particular, such a confirmation may come from the elegance of conclusions of a mathematical experiment. When dealing with triangles, the number 3 may be considered a universal characteristic across all possible types of triangles, one of which is defined by the number of colors, p, diversifying their visual perception. Therefore, having p^3 as the attractor of the ratios of two consecutive terms of the sequence that counts the number of the corresponding triangles within a spreadsheet may serve as an informal confirmation of the correctness of the mathematical argument the underlie the computational experiment.

Experimental mathematics accommodates students of various interests and abilities by providing opportunities for mathematical explorations of different levels of complexity and encouraging computational experimentation in the spirit of scientific and engineering concept learning [21]. Towards this end, other situations dealing with triangle construction out of straws can be explored. For example, one can consider all kinds of triangles in which none of the sides comprises more than a given number of straws. Trying to use mathematics as an applied tool, one can also look for real life applications of the results beyond a whimsical application [20] to the construction of multi colored triangular frames. It is through connecting technology-enabled mathematical results to real life that experimental mathematics experienced by teacher candidates could affect the way that STEM (science, technology, engineering, and mathematics) disciplines are learned in school.

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10. Appendix: Spreadsheet programming details

1. The case of identical straws

Cell A1 is slider controlled; B2: = 1, C2: =IF(B2<$A1, 1+B2, " ") – replicated to cell Z2; A3: = 1, A4: =IF(A3<A$1,1+A3," ") – replicated to cell A27;
B3: =IF(OR(B$2=" ",$A3=" ")," ",IF(AND(B$2>=$A3, A1>=B$2, A1<B$2+$A3), A1," ")) – replicated to cell Z27; the range [AB3:AB27] is filled with natural numbers;
AC3: =IF(A1=0," ", IF(A1=AB3,A2,AC3)) – replicated to cell A27, because this formula includes a circular reference, one has to make it work and in the calculation dialog box to set at, say, 100 the number of times Excel iterates the formula.

2. The case of multi colored straws

Cells A1 and AA1 are slider controlled and given the names n and p, respectively; numbers in the ranges B2:Z2 and A3:A27 are given the names m and k, respectively; B3: =IF(OR(m=" ",k=" ")," ",IF(AND(m>=k,n>=m,n<m+k), IF(p=1, n, IF(AND (m=n,m=k), (p^n+2)*(p^n+1)*(p^n)/6, IF(AND(m=n,m>k), (p^(n+k))*(p^n+1)/2,
IF(AND(n>m,m=k), (p^(n+m))*(p^m+1)/2, IF(AND(n>m, m>k),p^(n+m+k))))))," ")) – replicated to cell Z27; the range [AB3:AB27] is filled with consecutive natural numbers;
AC3: =IF(n=0," ", IF(n=AB3,SUM(B3:Z27),AC3)) – replicated to cell A27, because this formula includes circular reference, one has to make it work and in the calculation dialog box to set the number of times Excel iterates the formula at, say, 100. Thus, as the value of n changes, the numbers in the range AC3:AC27 show the total number of triangles the longest side of which comprises n straws available in p colors.