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Figurate Numbers

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Abstract

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Keywords

Figurate numbers, recursion, differentiation

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1. Introduction

My wife and I have recently purchased some land in South Australia, whose road number is 575. It is a number that we like, as it is a palindrome and comprised of prime-number digits only. It is also has the property that

$$575 = 23 \times 25 = (24 - 1) \times (24 + 1) = 24^2 - 1$$

That is, 575 is a square number with just 1 missing. To our surprize, we were informed by the architect who is helping us develop the property, that 575 is also a 59-agonal number. This brief article explains what a 59-agonal is and suggests that a spreadsheet makes light work of checking that our architect was correct as his original intention had been to find another property numbered 575 where someone else of note had lived.

2. A definition

A figurate number is defined [1] to be:

"a number that can be represented by a regular geometrical arrangement of equally spaced points."

In this article we will look only at those numbers that are known as the n-agonal numbers. As an example, the 5-agonal or pentagonal numbers are found by counting dots in a sequence of nested pentagons shown in Figure 1. In Figure 1, we introduce the notation needed to described k-agonal number algebraically:

k – the number of sides of the arrangement of dots

n – the number of levels in an arrangement, starting at $n = 0$

$D(k, n)$ – the number of dots in an arrangement with k sides and n levels.

A similar sequence of diagrams for 6-agonal or hexagonal numbers, is shown as Figure 2 where the values for n and $D(k, n)$ are also included. Drawing special cases and choosing an appropriate notation is a problem-solving strategy that always helps at the early stages of an investigation.
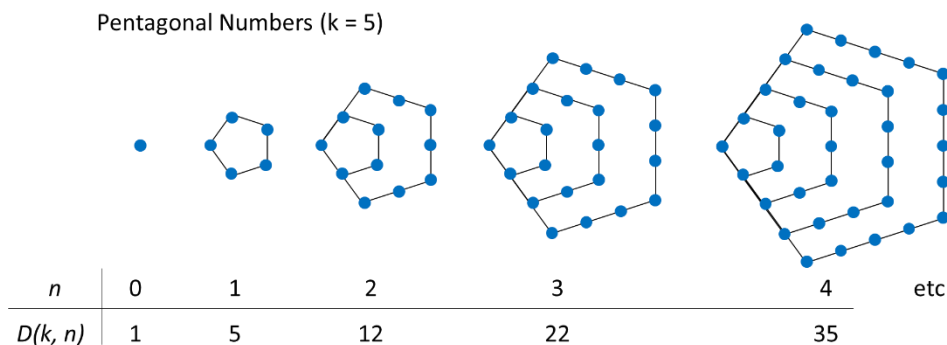
Pentagonal Numbers (k = 5)

n	0	1	2	3	4	etc
$D(k, n)$	1	5	12	22	35	

Figure 1: The figurate sequence for pentagonal numbers

Hexagonal Numbers (k = 6)

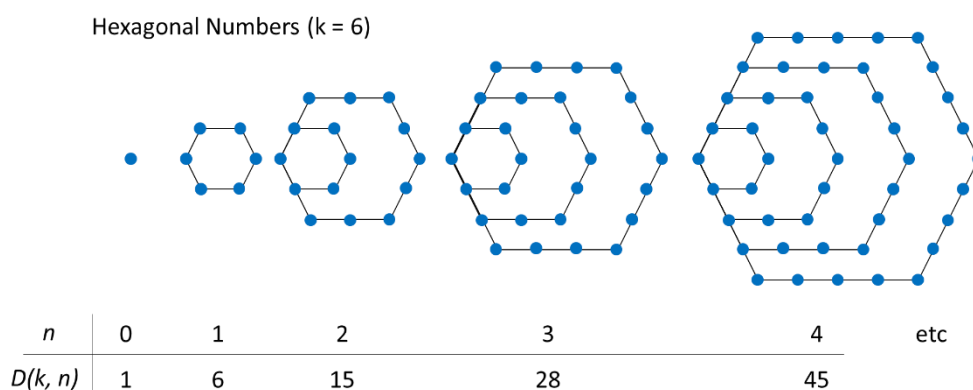| n	0	1	2	3	4	etc
$D(k, n)$	1	6	15	28	45	

Figure 2: The figurate sequence for hexagonal numbers

3. Visualising the growth pattern

Each new diagram in the series is made from the previous diagram by adding a either a pentagon or hexagon that is one larger than its predecessor but the new pentagon/hexagon also shares some of the dots with the previous diagram. In this section we look at three ways in which a visualisation of the growth pattern leads to a formula that can be used to calculate the values of $D(k, n)$.

We will focus on the hexagonal figurate numbers and describe the growth pattern for $D(6, n)$ as n increases, The same approach can be used to describe the changes to $D(5, n)$ as n increases.
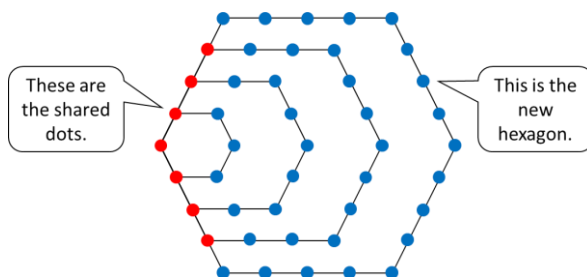

Figure 3: When a new hexagon is added, some of the dots are shared.

The highlighted dots in Figure 3 show that, when the fourth hexagon is added, the new number of dots is:

$$6 \times 4 \; new \; dots - 7 \; shared \qquad\qquad \textbf{Equation 1}$$

We can put together levels $0 - 4$ of this formula to find that:

$$\# \; dots \; at \; level \; 4 = 1 + 6(1 + 2 + 3 + 4) - (1 + \; 3 + 5 + 7)$$

Within this expression we see two summations, the first being $1 + 2 + 3 + \ldots$ which is the sum of natural numbers, which is a triangular number, and the second being the sum of odd numbers, which is a square number, and we can use these to convert Equation 1 into a formula that for the k-agonal figurate numbers. That is, the number of dots in a k-agonal configuration at level n is given by the formula:

$$D(k, n) = 1 + k \times \frac{1}{2} n(n + 1) - \; n^2 \qquad\qquad \textbf{Equation 2}$$

Another way to visualise the pattern is to see the arrangement of dots in terms of triangles and rays, as shown in Figure 4. At level 4, this view of the arrangement can be described as:

$$\# dots \; at \; Level \; 4 = 1 + 5 \; \times rays \; of \; 4 + 4 \; \times triangles \; of \; 6$$

This expression for the Level 4 configuration leads to the algebraic expression:

$$D(k, n) = 1 + (k - 1) \times n \; + (k - 2) \times \frac{1}{2} n(n - 1) \qquad\qquad \textbf{Equation 3}$$

Some algebraic manipulation is needed to show that Equation 3 is indeed the same as Equation 2.

A third way of describing the growth pattern comes from considering only the configuration of the new dots that are added at each level and this is illustrated in Figure 5. This view leads to a recursive formula, rather than an explicit formula of the type in Equations 2 and 3. For the case on $n \; = \; 4$:

$$\# dots \; at \; Level \; 4 = \# \; dots \; at \; Level \; 3 + (6 - 2) \times 4 + 1$$

In general terms, therefore, the recursive expression for $D(k, n)$ is:

$$D(k, n) = D(k, n - 1) + (k - 2)n + 1 \qquad\qquad \textbf{Equation 4}$$

It is the recursive form of Equation 4 that we will use to build a spreadsheet for figurate numbers.
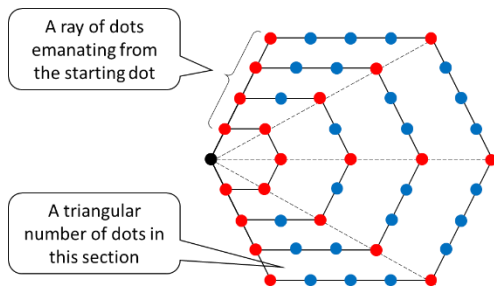

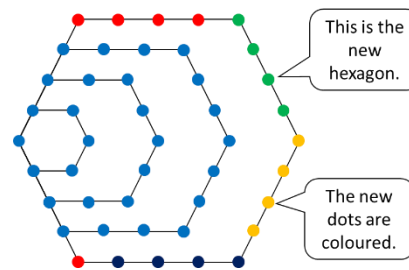Figure 4: Visualising the dots in rays and triangles

Figure 5: Focussing on the new dots that are added

	A	B	C	D	E	F
1						
2		n	D(k, n)	1st Diff	2nd Diff	
3		1	59			
4		2	174	115		
5		3	346	172	57	
6		4	575	229	57	
7		5	861	286	57	
8		6	1204	343	57	
9		7	1604	400	57	
10		8	2061	457	57	
11		9	2575	514	57	
12		10	3146	571	57	
13		11	3774	628	57	
14		12	4459	685	57	
15		13	5201	742	57	
16		14	6000	799	57	
17		15	6856	856	57	
18		16	7769	913	57	
19		17	8739	970	57	
20		18	9766	1027	57	
21		19	10850	1084	57	
22		20	11991	1141	57	
23						
24						

Figure 6: A spreadsheet to calculate k-agonal numbers

4. Making the spreadsheet

The following steps will make a spreadsheet that calculates the k-agonal figurate numbers and allow you to choose the value that k will take. The spreadsheet also calculates the first and second differences for $D(k, n)$.

1. Set up the headings as shown.

2. Make a list of the numbers 1 – 20 in Column B and name these as *level*.

3. Name cell C3 as k and make

   ```
   (C4) = C3 + (k - 2)*level + 1
   ```

4. Fill the formula in C4 down to C22.

By changing the value in C3 to be say 59, you will find that 575 is indeed a 59-agonal number or a penenta-enniagonal number to give it its proper name.

5. Differences and Differentiation

Whenever we construct a table of values in Excel, it is hard to pass up the opportunity to explore the first and maybe second differences.

1. Make `(D4)` = `C4` – `C3` and fill down.

2. Make `(E5)` = `D5` – `D4` and fill down.

Whatever value you chose for k, you will find that the values in Column E are all equal to $k - 2$. Why should that be?

Taking differences is similar to differentiating; particularly when the step-size of the variable is 1, as is the case for n in this context.

The formula for the first difference is $\dfrac{D(k,n+1)-D(k,n)}{1}$.

Whereas, the formula for differentiating would be $\dfrac{f(x+h)-f(x)}{h}$.

In this case, we can check that the spreadsheet confirms the algebra by differentiating the explicit form of $D(k, n)$ that we generated in Equation 2:

$$f(x) = 1 + \frac{kx^2}{2} + \frac{kx}{2} - x^2$$

$$\therefore \frac{df(x)}{dx} = kx + \frac{k}{2} - 2x$$

$$\therefore \frac{d^2 f}{d^2 x} = k - 2$$

Excellent! The spreadsheet showed the rule and the algebra confirmed it.

6. Postscript

The web page [3] provides lists of k-agonal numbers for $3 \leq k \leq 337$. Using the browser's search facility on this page for 575 shows that it occurs in three places. 575 is a 59-agonal number, 1575 is a 159-agonal number and 2575 is a 259-agonal number. The spreadsheet confirms this slightly surprizing feature of 575 and can be used to show that 3575 is a 359-agonal number and so on.

7. Acknowledgement

I am most grateful for the comments made by the referees of this article. Their suggestions for improvementsw have enabled the first draft to be altered in a way that will make the ideas expressed here educationally more useful.

8. References

1. Weisstein, Eric W. "Figurate Number." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/FigurateNumber.html
2. Weisstein, Eric W. "Hexagonal Number." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/HexagonalNumber.html
3. Barton, J. "Polygonal Numbers" viewed at http://www.virtuescience.com/polygonal-numbers.html

Suggestions for Activities

Figurate Numbers come in many forms. The examples given above build on a fixed shape, which is a regular polygon, and nest the polygons in such a way that all the polygons share a common vertex. For this type of figurate number, we have suggested the following steps to create a spreadsheet that lists k-agonal numbers and display the first and second difference:

1. Set up the headings as shown in Figure 6..
2. Make a list of the numbers 1 – 20 in Column B and name these as `level`.
3. Name cell C3 as `k`.
4. For the recursive form, make:
   `(C4) = C3 + (k − 2)level + 1`
   For the explicit formula form of Equation 2, make:
   `(C4) = 1 + k*level*(level + 1)/2 − level^2`
5. Fill the formula down Column C.
6. Make `(D4) = C4 − C3` and fill down.
7. Make `(E5) = D5 − D4` and fill down.

The simplest k-agonal numbers are the triangular numbers and the square numbers.

Question: How can you check that the spreadsheet you have made will generate both those sets of numbers?

Another type of figurate number is made with 3-D shapes. The one you may know best are the cubed numbers, but pyramids can also be used with triangular, square, pentagonal or hexagonal bases.

Question: What other types of figurate numbers can be created and how can you use a spreadsheet to find how they grow in size?