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The Arbitrage Pricing Model: A Pedagogic Derivation and a Spreadsheet-Based Illustration

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Abstract

This paper derives, from a pedagogic perspective, the Arbitrage Pricing Model, which is an important asset pricing model in modern finance. The derivation is based on the idea that, if a self-financed investment has no risk exposures, the payoff from the investment can only be zero. Microsoft Excel plays an important pedagogic role in this paper. The Excel illustration not only helps students recognize more fully the various nuances in the model derivation, but also serves as a good starting point for students to explore on their own the relevance of the noise issue in the model derivation.

Keywords

asset pricing, arbitrage pricing model, arbitrage pricing line, arbitrage portfolio

Cover Page Footnote

The author wishes to thank the anonymous reviewers for helpful comments and suggestions.

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1 Introduction

The Arbitrage Pricing Model of Ross (1976) is a major advancement in modern finance. The model is also simply known as the APT, which stands for the Arbitrage Pricing Theory. The objectives of this paper are two-fold. The first objective is to derive the APT from a pedagogic perspective, by using only familiar mathematical tools, with the help of intuitive reasoning. The second objective is to illustrate numerically the analytical steps in the model derivation by using Microsoft Excel^{TM}.[1] Being a numerical rendition of the corresponding analytical materials, the Excel illustration is intended to help students recognize more fully the nuances of the model derivation and dispel any remaining mystery. (Readers who are already familiar with the model and its significance in the finance literature may skip the remainder of this section.)

Originally intended as an alternative to the Capital Asset Pricing Model (CAPM) of Sharpe (1964) and Lintner (1965), by considering multi-factor asset pricing instead, the APT has had profound impacts on financial research. It has inspired many theoretical and empirical studies. The online posting by Professor Robert A. Korajczyk at Kellogg School of Management, Northwestern University, U.S.A., has listed about 400 articles that are related to the APT and multi-factor models in the finance literature, as of August 26, 2014.[2]

The APT has generated keen interest among investment practitioners as well. In an article in *Financial Analysts Journal*, Roll and Ross (1984) have offered some practical perspectives on the theory. That article was selected for the same journal's 1995 special issue, *50 Years in Review*, as one of the top-four articles in the decade of 1975-1984. In view of its academic and practical significance, the APT has been covered in standard finance textbooks at intermediate and advanced levels, with or without including model derivations. For example, model derivations can be found in Copeland, Weston, and Shastri (2005, Chapter 6), Elton, Gruber, Brown, and Goetzmann (2014, Chapter 16), and Levy and Post (2005, Chapter 11), but not in Bodie, Kane, and Marcus (2014, Chapter 10) and Hillier, Grinblatt, and Titman (2012, Chapter 6).

[1] For the remainder of this paper, whenever the name Excel or any of its computational tools is mentioned, its trademark is implicitly acknowledged.

[2] The electronic link is http://www.kellogg.northwestern.edu/faculty/korajczy/htm/aptlist.htm .

Central to the APT is a linear pricing relationship, known as the Arbitrage Pricing Line, which relates the expected return of each security to some sensitivity measures, in response to the corresponding economic factors. Derivations of the APT are based on an intuitive idea that, if a self-financed investment has no risk exposures, the payoff from such an investment can only be zero. Otherwise, there will be arbitrage profits to be exploited. It is the absence of arbitrage profits that enables a pricing relationship of securities to be established. Traditionally, to implement such an idea in an analytical setting, for the purpose of deriving an arbitrage-free pricing relationship, requires some mathematical knowledge that is unfamiliar to typical finance students. This alone is already a strong enough reason for some finance textbooks to omit any model derivation, as students with inadequate mathematical preparedness will likely be unable to appreciate the analytical nuances involved.

The analytical materials, as presented in Section 3, are confined to those directly pertaining to the attainment of the Arbitrage Pricing Line. As the intended readers include also students, the model derivation is presented in considerable detail for them to follow on their own. Omitted from the coverage are issues peripheral to this specific task. For example, empirical issues as to how relevant economic factors are identified are not covered; neither are statistical issues pertaining to joint probability distributions of such factors. Further, in view of the extensive textbook coverage of various basic properties and implications of the APT, including comparisons with those of the CAPM, there is no need to duplicate the same materials here. Interested readers are referred to the above-mentioned textbooks for details.

The task to establish an arbitrage-free pricing relationship starts with a linear equation — known as the return generating equation (or the return generating process) — for each of the many risky securities considered. These equations are for capturing the sensitivities of individual security returns to the underlying economic factors. The parameters representing such sensitivities are known as factor loadings. The part of security return that each return generating equation fails to capture is treated as random noise.

For the model derivation, there is a requirement that, in a portfolio investment setting, the number of securities in the capital market be large enough for any linear combinations of the random noise terms in the individual return generating equations to be attenuated effectively. Indeed, effective attenuation of random noise is a crucial condition for the existence of self-financed risk-free portfolios within the APT framework. The imposition of zero payoffs on such portfolios, in turn, facilitates the attainment of an arbitrage-free pricing relationship.

2 The Pedagogic Role of Excel

The Excel illustration is presented in Section 4. As the intended task is to illustrate the model derivation, rather than testing the model empirically, the use of artificial data is preferable. It is straightforward to use Excel to generate a large set of artificial data, for specifying the individual return generating equations. Such artificial data are then used to determine the remaining parameters in the derived Arbitrage Pricing Line. Here is a sketch of the Excel tools and operations involved:

The Excel function RANDBETWEEN is used to produce artificial data for all factor loadings in the return generating equation for each security. As this function returns a random integer in a given range of values, Excel scroll bars are used to specify such a range for data-entry convenience. A scaling factor is applied to the integers thus generated, so that the end results are all real numbers in a range of values suitable for use as factor loadings.

Although how random noise in each return generating equation is distributed is peripheral to the model derivation in this paper, to generate it for the purpose of a numerical illustration still requires that a probability distribution be specified. Under the simplest assumption that the noise has a uniform distribution with a zero mean, the use of the Excel function RANDBETWEEN is adequate. Alternatively, under the assumption of a normal distribution, the noise can be generated by nesting two Excel functions, NORMSINV and RAND. Here, NORMSINV returns the inverse of the standard normal cumulative distribution, and RAND — which returns a random number in the range of 0 and 1 — serves as the argument of the nested function. Under either assumption, a scaling factor is required for each randomly generated number, for specifying the severity of the noise.

In the process of deriving an arbitrage-free pricing relationship, two different types of portfolios of securities are constructed. While one type involves self-financed investments, the other type involves portfolios where investment funds are required instead. In the former case, each portfolio is intended to respond to none of the economic factors. In the latter case, each portfolio is intended to be responsive to a specific economic factor and nonresponsive to the remaining ones.

Excel *Solver* is suitable for use to construct both types of portfolios.[3] The use of *Solver* here differs from that in typical computational settings, for which unique solutions are sought. As it will soon be clear, there are infinitely many ways to construct each portfolio that satisfies a given set of conditions within the APT framework. In view of such a feature, the allocation of investment funds for each portfolio depends on the initial values used in the *Solver* search. Interestingly, it is the lack of uniqueness in portfolio allocations that allows an arbitrage-free pricing relationship to be derived with only familiar mathematical tools; this is indeed a crucial feature in the derivation of the APT from a pedagogic perspective.

For computational convenience, two matrix tools in Excel are also used for constructing each of the above-mentioned portfolios. Specifically, with pertinent numerical data stored in arrays, the sum of products of the corresponding elements there can be computed directly by using the Excel function MMULT for matrix multiplications. Nesting MMULT with the Excel function TRANSPOSE — for matrix transposition — allows the matrices involved to have compatible dimensions for multiplications.

There is an Excel file to accompany the numerical illustration in Section 4. This file can readily be used by instructors to generate artificial data for various numerical exercises on the derivation of the APT for students. A good exercise is to investigate numerically, within the APT framework, whether there are any payoffs from self-financed portfolios that respond to none of the economic factors. A crucial requirement for the model derivation is that the number of risky securities in the capital market be large enough for the random noise terms in the individual return generating equations to be attenuated effectively in portfolio settings. The issue as to how large is large enough for the purpose of reaching a self-financed risk-free portfolio depends primarily on the severity of random noise in each return generating equation. The Excel illustration in Section 4 will serve as a good starting point for students to explore the noise issue themselves. The work involved will help students recognize more fully its relevance in the model derivation.

[3] *Solver* is a popular numerical tool for constrained optimization. It is one of the Microsoft Office Add-ins. Users requiring help with its loading and basic operations can access Microsoft Office Excel Help by entering "solver" as the keyword for the search there.

3 A Derivation of the Model

Like the CAPM, the APT is also a single-period model that requires the usual assumptions of homogeneous expectations of investors and a frictionless capital market, where risk-free lending and borrowing are at the same interest rate. In such a market, short sales of any risky securities will generate beginning-of-period investment funds for other securities. Under these simplifying assumptions, the starting point in the derivation of the APT is to capture the random return of each risky security over the period by means of a linear equation, known as the return generating equation.

For a market with n risky securities, the random return R_i of any security i, for $i = 1, 2, \ldots, n$, is characterized as depending linearly on K common economic factors, each of which has been mean-removed. The part of the security return that this linear equation is unable to capture is treated as random noise. For analytical purposes, what each factor represents need not be specified, and the number of economic factors involved can be any positive integer. A required condition, however, is that n must be much greater than K. Why such a condition is required for the model derivation will soon be clear.

3.1 The Return Generating Equation

For ease of exposition, let us start with the special case where $K = 3$. By labeling the three mean-removed factors as F_1, F_2, and F_3, the return generating equation for each security i is

$$R_i = a_i + b_{i1}F_1 + b_{i2}F_2 + b_{i3}F_3 + e_i. \tag{1}$$

Here, a_i is a coefficient. The remaining coefficients, b_{i1}, b_{i2}, and b_{i3}, commonly called factor loadings, are intended to capture the sensitivities of the random return of security i to the individual factors, which are themselves random variables. The use of double subscripts for these three coefficients, though cumbersome, is necessary; the first subscript is used to indicate the security involved, and the second subscript, the factor involved. The random noise term e_i, which is characterized as having a zero expected value, is the part of the security return that this linear relationship is unable to capture.

To facilitate an intuitive interpretation of a_i in equation (1), let us describe briefly how each of F_1, F_2, and F_3 has been reached. Mean removal for a random variable is to subtract its expected value from it. Originally, the three economic factors are $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_3$, with the corresponding expected values being $E(\boldsymbol{F}_1)$, $E(\boldsymbol{F}_2)$, and $E(\boldsymbol{F}_3)$. Here, we have used $E(\cdot)$ to denote the expected value of any random variable $(\cdot)$. As

$$F_1 = \boldsymbol{F}_1 - E(\boldsymbol{F}_1), \tag{2}$$

$$F_2 = \boldsymbol{F}_2 - E(\boldsymbol{F}_2), \tag{3}$$

$$\text{and } F_3 = \boldsymbol{F}_3 - E(\boldsymbol{F}_3) \tag{4}$$

are mean-removed random variables, their expected values — $E(F_1)$, $E(F_2)$, and $E(F_3)$ — are all zeros.

By taking expected values of the two sides of equation (1), we can write

$$E(R_i) = a_i + b_{i1}E(F_1) + b_{i2}E(F_2) + b_{i3}E(F_3) + E(e_i), \tag{5}$$

which directly leads to

$$E(R_i) = a_i. \tag{6}$$

That is, the intercept term a_i in the return generating equation for each security i is the expected value of R_i. This is because each random variable on the right hand side of equation (5) has a zero expected value. In statistical notation, it is usually labeled as μ_i. Thus, the return generating equation for each security i can be restated as

$$R_i = \mu_i + b_{i1}F_1 + b_{i2}F_2 + b_{i3}F_3 + e_i. \tag{7}$$

3.2 A Self-financed Risk-free Investment

Under the assumption of homogeneous expectations, all investors accept equation (7) as being valid and agree on the values of the coefficients involved, for $i = 1, 2, \ldots, n$. The assumption of frictionless short sales makes it possible for someone to invest in the n securities with a zero cash outlay. Suppose that a dollar amount w_i is allocated to security i. If w_i is positive, the corresponding investment in security i is the dollar amount w_i. If w_i is negative instead, security i is held in a short position, and an immediate cash inflow, which is equal to the magnitude of w_i, is generated.

To achieve a self-financed investment, for which the net dollar amount invested in the n securities is zero, the condition of

$$\sum\nolimits_{i=1}^{n} w_i = 0, \tag{8}$$

must be satisfied. Here, the use of a summation sign is for notational simplicity, with $\sum_{i=1}^n w_i$ standing for the sum of the n individual terms, $w_1, w_2, \ldots, w_n$. This being the case of one equation with many decision variables, there are infinitely many ways to assign $w_1, w_2, \ldots, w_n$ to the n individual securities for the equation to be satisfied.

For each set of $w_1, w_2, \ldots, w_n$, the random end-of-period payoff (in dollars) from the investment can be expressed as $\sum_{i=1}^n w_i R_i$, which is the sum of the n individual terms, $w_1 R_1, w_2 R_2, \ldots,$ $w_n R_n$. Given equation (7), we can write the investment's random end-of-period payoff as

$$\begin{aligned}\sum_{i=1}^n w_i R_i &= \sum_{i=1}^n w_i \mu_i + \left(\sum_{i=1}^n w_i b_{i1}\right) F_1 + \left(\sum_{i=1}^n w_i b_{i2}\right) F_2 \\ &\quad + \left(\sum_{i=1}^n w_i b_{i3}\right) F_3 + \sum_{i=1}^n w_i e_i. \end{aligned} \tag{9}$$

Here, the use of each summation sign is analogous to that described previously.

In addition to the condition that equation (8) provides, let us impose three more conditions in assigning $w_1, w_2, \ldots, w_n$ to the n individual securities. Specifically, under the conditions of

$$\sum_{i=1}^n w_i b_{i1} = 0, \tag{10}$$

$$\sum_{i=1}^n w_i b_{i2} = 0, \tag{11}$$

$$\text{and} \quad \sum_{i=1}^n w_i b_{i3} = 0, \tag{12}$$

the investment's random end-of-period payoff will be unaffected by any of F_1, F_2, and F_3. Thus, under such conditions, how these random variables are distributed is not a concern in the model derivation.

Equations (8) and (10)-(12) represent four simultaneous linear equations with n variables, which are $w_1, w_2, \ldots, w_n$. As long as $n \geq 5$, there are infinitely many ways to assign $w_1, w_2, \ldots, w_n$ to the n individual securities for these four equations to be satisfied. In the language of finance, such a self-financed investment has no systematic risk that is associated with the three underlying economic factors.

Once the conditions in equations (10)-(12) are imposed, equation (9) becomes

$$\sum_{i=1}^n w_i R_i = \sum_{i=1}^n w_i \mu_i + \sum_{i=1}^n w_i e_i. \tag{13}$$

Each $w_i e_i$ term in the second summation on the right hand side of equation (13) is part of the random component of the end-of-period payoff from the self-financed investment. As each w_i is not a random variable and each e_i, which is random, has a zero expected value, the corresponding $w_i e_i$ term must have a zero expected value too. The sum $\sum_{i=1}^{n} w_i e_i$, which contains both positive and negative terms to attenuate each other, represents the random component of the net payoff from the self-financed investment. However, to cancel out all the positive and negative terms in $\sum_{i=1}^{n} w_i e_i$ does require n to be very large. If n is large enough to make $\sum_{i=1}^{n} w_i e_i$ trivially small, then equation (13) can be approximated very well by

$$\sum_{i=1}^{n} w_i R_i = \sum_{i=1}^{n} w_i \mu_i. \tag{14}$$

As $\sum_{i=1}^{n} w_i \mu_i$ is the dollar amount of the expected end-of-period payoff, an immediate implication of equation (14) is that the end-of-period payoff is always as expected. That is, there is no randomness in $\sum_{i=1}^{n} w_i R_i$. In a market where no arbitrage profits are available, a self-financed investment without any risk must have a zero end-of-period payoff. This economic feature is crucial in the model derivation.

Given that the analytical task here is to establish eventually a beginning-of-period pricing relationship of securities based on available information only, the above economic feature is better captured by

$$\sum_{i=1}^{n} w_i \mu_i = 0, \tag{15}$$

which states that no arbitrage profits are expected. The use of

$$\sum_{i=1}^{n} w_i R_i = 0 \tag{16}$$

instead is unsuitable for establishing the same pricing relationship. This is because $R_1, R_2, \ldots,$ R_n are random variables.

3.3 An Economic Consequence and a Linear Pricing Relationship

It is important to recognize that, while equations (8) and (10)-(12) represent four specific conditions in assigning the dollar amounts $w_1, w_2, \ldots, w_n$ to the n individual securities, equation (15) shows an economic consequence of such conditions. To establish a pricing relationship to connect μ_i, b_{i1}, b_{i2}, and b_{i3}, for $i = 1, 2, \ldots, n$, it is also important that we do not start with a set of security return generating equations where arbitrage opportunities are readily available. To see this, suppose that we have two securities, labeled as securities j and k, for which $b_{j1} = b_{k1}$,

$b_{j2} = b_{k2}$, and $b_{j3} = b_{k3}$. In such a case, we must have $\mu_j = \mu_k$. Having $\mu_j \neq \mu_k$ instead will indicate the presence of arbitrage opportunities.

Under the assumption of a frictionless capital market, if $\mu_j > \mu_k$ for example, investors can expect arbitrage profits by purchasing security j with proceeds from short selling security k. Of course, though expected, arbitrage profits are not assured, because of the presence of random noise in the return generating equations for the two securities. However, the greater the number of such pairs of securities with matching factor loadings but different expected returns, the less impact will be random noise on potential arbitrage profits.

This example raises the following question: In the derivation of the APT, do we start with equation (7) where the values of μ_i, for $i = 1, 2, \ldots, n$, are known in advance? The answer is no. We must treat each μ_i there as a parameter that has yet to be determined. Given equation (7), each μ_i is related to the corresponding factor loadings b_{i1}, b_{i2}, and b_{i3} linearly. Thus, it is reasonable for us to expect the relationship to be of the algebraic form

$$\mu_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \lambda_3 b_{i3}, \tag{17}$$

where λ_0, λ_1, λ_2, and λ_3 are parameters. These parameters, which are common for all n securities, have yet to be determined.

Before proceeding to verify equation (17) and to determine λ_0, λ_1, λ_2, and λ_3, let us review the analytical steps in the model derivation so far, from an algebraic perspective. Given b_{i1}, b_{i2}, and b_{i3}, for $i = 1, 2, \ldots, n$, there are infinitely many feasible results of $w_1, w_2, \ldots, w_n$ from solving equations (8) and (10)-(12). This is because there are more unknowns than the number of available equations. Likewise, for each set of feasible $w_1, w_2, \ldots, w_n$, there are also infinitely many feasible results of $\mu_1, \mu_2, \ldots, \mu_n$ from solving equation (15). However, for the purpose of establishing a meaningful pricing relationship, the solved $\mu_1, \mu_2, \ldots, \mu_n$ must be unique. That is, the solution must be independent of how $w_1, w_2, \ldots, w_n$ are assigned.

Given equations (8), (10)-(12), and (15), where each sum is zero, we can always write

$$\sum\nolimits_{i=1}^{n} w_i \mu_i - \lambda_0 \sum\nolimits_{i=1}^{n} w_i - \lambda_1 \sum\nolimits_{i=1}^{n} w_i b_{i1} - \lambda_2 \sum\nolimits_{i=1}^{n} w_i b_{i2} - \lambda_3 \sum\nolimits_{i=1}^{n} w_i b_{i3} = 0, \tag{18}$$

regardless of the values of the multiplicative constants λ_0, λ_1, λ_2, and λ_3 for the individual sums. Equation (18) is equivalent to

$$\sum\nolimits_{i=1}^{n} w_i \left(\mu_i - \lambda_0 - \lambda_1 b_{i1} - \lambda_2 b_{i2} - \lambda_3 b_{i3}\right) = 0. \tag{19}$$

What is crucial here is that there are infinitely many ways to assign $w_1, w_2, \ldots, w_n$ and that equation (19) always holds regardless of how $w_1, w_2, \ldots, w_n$ are assigned. Thus, the linear combination of the five terms, $\mu_i - \lambda_0 - \lambda_1 b_{i1} - \lambda_2 b_{i2} - \lambda_3 b_{i3}$, as enclosed by the pair of parentheses in equation (19), must itself be zero before it is multiplied by w_i, for $i = 1, 2, \ldots, n$. This feature allows us to see why equation (17) must hold.[4]

3.4 The Roles of the Risk-free Security and Some Specific Portfolios

For the purpose of establishing a meaningful pricing relationship of securities, the parameters λ_0, λ_1, λ_2, and λ_3 in equation (17) cannot be left unspecified. To find a set of meaningful values of these parameters requires the availability of some additional information. From an algebraic perspective, as there are four parameters in total to be determined, four known values of some relevant variables are needed.

It is easy to determine λ_0. All it takes is to apply equation (17) to a risk-free security. With the risk-free security labeled as security f, equation (17) becomes

$$\mu_f = \lambda_0 + \lambda_1 b_{f1} + \lambda_2 b_{f2} + \lambda_3 b_{f3}. \tag{20}$$

As security f has zero sensitivity to any of F_1, F_2, and F_3, implying that $b_{f1} = b_{f2} = b_{f3} = 0$, we must have

$$\lambda_0 = r_f. \tag{21}$$

Here, we have substituted r_f, a commonly used symbol of the risk-free interest rate, for μ_f. Thus, if r_f is known, so is λ_0.

To determine λ_1, λ_2, and λ_3 requires that the expected returns of three specific portfolios be known. Let us use the determination of λ_1 as an illustration. For this task, let us construct a portfolio, labeled as portfolio $\underline{1}$, by using the same risky securities in the market. The portfolio is not self-financed; it requires investment funds. The portfolio is intended to have unit sensitivity

[4] Equation (17) can also be deduced by considering the following six vectors in an n-dimensional space: $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$, $\boldsymbol{\iota} = (1, 1, \ldots, 1)$, $\boldsymbol{b}_1 = (b_{11}, b_{21}, \ldots, b_{n1})$, $\boldsymbol{b}_2 = (b_{12}, b_{22}, \ldots, b_{n2})$, $\boldsymbol{b}_3 = (b_{13}, b_{23}, \ldots, b_{n3})$, and $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$. According to equations (8), (10)-(12), and (15), vector $\boldsymbol{w}$ is orthogonal to the five remaining vectors. Thus, these five vectors must be on the same hyperplane, satisfying the condition of $\boldsymbol{\mu} = \lambda_0 \boldsymbol{\iota} + \lambda_1 \boldsymbol{b}_1 + \lambda_2 \boldsymbol{b}_2 + \lambda_3 \boldsymbol{b}_2$, which is equivalent to equation (17). However, from a pedagogic perspective, as concepts of hyperplanes are required, the alternative approach here may be too abstract for many finance students.

to F_1 and zero sensitivity to each of the remaining economic factors. The details for constructing such a portfolio are as follows:

Let $x_1, x_2, \ldots, x_n$ be the proportions of investment funds for securities $1, 2, \ldots, n$, respectively, satisfying the condition of

$$\sum_{i=1}^{n} x_i = 1. \tag{22}$$

Here, each x_i can be of either sign. If zero, the corresponding security is not selected for the portfolio. These proportions of investment funds are commonly known as portfolio weights. The random return of portfolio $\underline{1}$, labeled as $R_{\underline{1}}$, is the weighted average of $R_1, R_2, \ldots, R_n$, with the corresponding weights being $x_1, x_2, \ldots, x_n$; that is,

$$R_{\underline{1}} = \sum_{i=1}^{n} x_i R_i. \tag{23}$$

Combining equations (7) and (23) leads to

$$\begin{aligned} R_{\underline{1}} &= \sum_{i=1}^{n} x_i \left(\mu_i + b_{i1}F_1 + b_{i2}F_2 + b_{i3}F_3 + e_i\right) \\ &= \sum_{i=1}^{n} x_i\mu_i + \left(\sum_{i=1}^{n} x_i b_{i1}\right) F_1 + \left(\sum_{i=1}^{n} x_i b_{i2}\right) F_2 \\ &\quad + \left(\sum_{i=1}^{n} x_i b_{i3}\right) F_3 + \sum_{i=1}^{n} x_i e_i. \end{aligned} \tag{24}$$

Under the additional conditions of

$$\sum_{i=1}^{n} x_i b_{i1} = 1, \tag{25}$$

$$\sum_{i=1}^{n} x_i b_{i2} = 0, \tag{26}$$

$$\text{and } \sum_{i=1}^{n} x_i b_{i3} = 0, \tag{27}$$

equation (24) reduces to

$$R_{\underline{1}} = \sum_{i=1}^{n} x_i\mu_i + F_1 + \sum_{i=1}^{n} x_i e_i. \tag{28}$$

Thus, portfolio $\underline{1}$ has unit sensitivity to F_1 and zero sensitivity to each of the remaining two economic factors.

As the expected values of the random variables F_1 and $e_1, e_2, \ldots, e_n$ on the right hand side of equation (28) are all zeros, the expected value of $R_{\underline{1}}$, labeled as $\mu_{\underline{1}}$, is $\sum_{i=1}^{n} x_i\mu_i$. Notice that μ_1 and $\mu_{\underline{1}}$ are not the same; while the former is the expected return of security 1, the latter is the expected return of the portfolio that has unit sensitivity to F_1 and zero sensitivity to each of the remaining two economic factors. Just like the construction of self-financed risk-free portfolios, there are also infinitely many combinations of $x_1, x_2, \ldots, x_n$ for equations (22) and (25)-(27) to be satisfied. However, regardless of how investment funds are allocated for the construction of portfolio $\underline{1}$, the expected portfolio return is considered to be known for the purpose of model derivation.

The portfolio construction here differs from the construction of self-financed risk-free portfolios considered earlier in that the number of securities involved here need not be large enough for the noise terms in the individual return generating equations to be attenuated effectively. This is because, in contrast to the condition in equation (14) for self-financed risk-free portfolios, there is no need for $R_{\underline{1}}$ and $\mu_{\underline{1}}$ — which are $\sum_{i=1}^n x_i R_i$ and $\sum_{i=1}^n x_i \mu_i$, respectively — to be essentially the same here. Indeed, the portfolio construction here can be based on any subset of the n securities consisting of at least five securities.

Given equation (17), we can write

$$\begin{aligned} \mu_{\underline{1}} &= \sum_{i=1}^n x_i(r_f + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \lambda_3 b_{i3}) \\ &= r_f + \lambda_1 \sum_{i=1}^n x_i b_{i1} + \lambda_2 \sum_{i=1}^n x_i b_{i2} + \lambda_3 \sum_{i=1}^n x_i b_{i3}. \end{aligned} \tag{29}$$

Under the conditions in equations (22) and (25)-(27) for the allocation of investment funds, equation (29) reduces to

$$\mu_{\underline{1}} = r_f + \lambda_1 \tag{30}$$

or, equivalently,

$$\lambda_1 = \mu_{\underline{1}} - r_f. \tag{31}$$

In the latter expression, λ_1 can be interpreted as an excess return, which is the expected return of of portfolio $\underline{1}$ in excess of the risk-free interest rate. If r_f and $\mu_{\underline{1}}$ are known, so is λ_1.

We now extend the above illustration to determining λ_k, for $k = 1$, 2, and 3. For each k, we form a portfolio $\underline{k}$ that has unit sensitivity to F_k and zero sensitivity to any of the remaining factors. Portfolio $\underline{k}$ is intended to satisfy the conditions of

$$\sum_{i=1}^n x_i = 1, \tag{32}$$

$$\sum_{i=1}^n x_i b_{ik} = 1, \tag{33}$$

$$\text{and } \sum_{i=1}^n x_i b_{ih} = 0, \text{ for } h \neq k. \tag{34}$$

Here, $h \neq k$ covers all cases of $h = 1$, 2, and 3, except for $h = k$. In the case of $k = 3$, for example, $h \neq k$ covers both $h = 1$ and $h = 2$.

Let $\mu_{\underline{k}}$ be the expected return of portfolio $\underline{k}$. Just like the previously mentioned difference between the symbols μ_1 and $\mu_{\underline{1}}$, while μ_k without the underscore for the subscript k is the expected return of security k, $\mu_{\underline{k}}$ is the expected return of portfolio $\underline{k}$ instead. The above conditions for portfolio construction, when combined with equations (7) and (17), will lead to

$$\lambda_k = \mu_{\underline{k}} - r_f, \tag{35}$$

which allows each λ_k to be determined if $\mu_{\underline{k}}$ and r_f are known, for $k = 1, 2,$ and 3. Each λ_k can be interpreted as an excess return, which is the expected return of of portfolio $\underline{k}$ in excess of the risk-free interest rate.

With λ_0, λ_1, λ_2, and λ_3 determined, the model derivation is complete, for the special case where there are three economic factors in the return generating equation for each of the n securities. The result,

$$\mu_i = r_f + (\mu_{\underline{1}} - r_f)b_{i1} + (\mu_{\underline{2}} - r_f)b_{i2} + (\mu_{\underline{3}} - r_f)b_{i3}, \text{ for } i = 1, 2, \ldots, n, \tag{36}$$

is the Arbitrage Pricing Line. This is an equilibrium pricing relationship; it relates the expected return of each security i to the sensitivities of its random return to the three underlying economic factors, which are random variables themselves.

3.5 Some Analytical Materials in Matrix Notation

To facilitate some Excel-based computations and the explanations of the results in Section 4, we first write equation (17) equivalently as

$$\mu_i - \lambda_0 = \lambda_1 b_{i1} + \lambda_2 b_{i2} + \lambda_3 b_{i3}, \tag{37}$$

where λ_0, λ_1, λ_2, and λ_3 are unspecified parameters. This being a representative equation where i can be any of $1, 2, \ldots, n$, there are really n equations. In matrix notation, the n equations can be combined into

$$\begin{bmatrix} (\mu_1 - \lambda_0) \\ (\mu_2 - \lambda_0) \\ \vdots \\ (\mu_n - \lambda_0) \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & b_{n3} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}. \tag{38}$$

This matrix equation allows us to compute $\mu_1 - \lambda_0, \mu_2 - \lambda_0, \ldots, \mu_n - \lambda_0$ for a given set of factor loadings and given values of λ_1, λ_2, and λ_3, by using basic matrix tools in Excel.

While the factor loadings are constant in the model derivation, the computed values of $\mu_1, \mu_2, \ldots, \mu_n$ according to equation (38) depend on the values provided for λ_0, λ_1, λ_2, and λ_3. At the stage of the model derivation that confirms the validity of equation (17) and, equivalently, equation (38), parameters λ_0, λ_1, λ_2, and λ_3 are still arbitrary. However, the lack of uniqueness in the computed values of $\mu_1, \mu_2, \ldots, \mu_n$ notwithstanding, the sum $\sum_{i=1}^{n} w_i \mu_i$ is always zero.

For equation (38) to be a meaningful security pricing relationship, λ_0, λ_1, λ_2, and λ_3 cannot be left unspecified. By letting $\lambda_0 = r_f$ and $\lambda_k = \mu_{\underline{k}} - r_f$, for $k = 1$, 2, and 3, as established in the preceding subsection, we can write equation (38) as

$$\begin{bmatrix} (\mu_1 - r_f) \\ (\mu_2 - r_f) \\ \vdots \\ (\mu_n - r_f) \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & b_{n3} \end{bmatrix} \begin{bmatrix} (\mu_{\underline{1}} - r_f) \\ (\mu_{\underline{2}} - r_f) \\ (\mu_{\underline{3}} - r_f) \end{bmatrix}. \tag{39}$$

Equation (38), where λ_0, λ_1, λ_2, and λ_3 are arbitrary, encompasses equation (39) as a special case. Thus, as the computed sum $\sum_{i=1}^{n} w_i \mu_i$ is zero according to the former equation, it must also be zero according to the latter equation.

3.6 Extension to the General Case

We now extend the model derivation to a general case, where there are K underlying economic factors instead. To derive the corresponding Arbitrage Pricing Line, we start with the following return generating equation:

$$R_i = a_i + b_{i1} F_1 + b_{i2} F_2 + \cdots + b_{iK} F_K + e_i, \text{ for } i = 1, 2, \ldots, n, \tag{40}$$

where the K factors, $F_1, F_2, \ldots, F_K$, have been mean-removed. As before, each a_i can be interpreted as the expected return of security i and labeled as μ_i. Under the $K+1$ conditions of $\sum_{i=1}^{n} w_i = 0$, $\sum_{i=1}^{n} w_i b_{i1} = 0$, $\sum_{i=1}^{n} w_i b_{i2} = 0$, $\ldots$, $\sum_{i=1}^{n} w_i b_{iK} = 0$, the absence of arbitrage profits ensures also that $\sum_{i=1}^{n} w_i \mu_i = 0$. To achieve a self-financed risk-free investment, for which equation (14) holds, n must be much greater than K, so that the magnitude of $\sum_{i=1}^{n} w_i e_i$ can be considered to be trivially small. The idea is that, when more economic factors are involved, more conditions must be satisfied to achieve a self-financed risk-free investment. Thus, a greater K requires a greater n for the noise terms to be attenuated effectively.

As there are infinitely many ways to assign $w_1, w_2, \ldots, w_n$ to the individual securities for the above conditions to hold, $\mu_i, b_{i1}, b_{i2}, \ldots, b_{iK}$ for each security i are connected by

$$\mu_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_K b_{iK}. \tag{41}$$

Once $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_K$ are determined, by considering the risk-free security and K specific portfolios, the corresponding Arbitrage Pricing Line is

$$\mu_i = r_f + (\mu_{\underline{1}} - r_f) b_{i1} + (\mu_{\underline{2}} - r_f) b_{i2} + \cdots + (\mu_{\underline{K}} - r_f) b_{iK}, \text{ for } i = 1, 2, \ldots, n. \tag{42}$$

Here, each $\mu_{\underline{k}}$, for $k = 1, 2, \ldots, K$, is the expected return of a portfolio that has unit sensitivity to factor k and zero sensitivity to each of the remaining factors. In matrix notation, equations (41) and (42) can be written as

$$\begin{bmatrix} (\mu_1 - \lambda_0) \\ (\mu_2 - \lambda_0) \\ \vdots \\ (\mu_n - \lambda_0) \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1K} \\ b_{21} & b_{22} & \cdots & b_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nK} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_K \end{bmatrix} \tag{43}$$

and

$$\begin{bmatrix} (\mu_1 - r_f) \\ (\mu_2 - r_f) \\ \vdots \\ (\mu_n - r_f) \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1K} \\ b_{21} & b_{22} & \cdots & b_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nK} \end{bmatrix} \begin{bmatrix} (\mu_{\underline{1}} - r_f) \\ (\mu_{\underline{2}} - r_f) \\ \vdots \\ (\mu_{\underline{K}} - r_f) \end{bmatrix}, \tag{44}$$

respectively.

4 An Excel Illustration

The model derivation in Section 3, though requiring only familiar mathematical tools, is quite lengthy, as compared to analytical materials on many other topics in the standard finance curriculum. The Excel illustration in this section goes beyond showing the computations involved. It is also intended to help students recognize more fully the nuances of the analytical task. To make the Excel file accompanying this paper readily accessible to more readers, it has been saved as an Excel 1997-2003 workbook (which has an extension of .xls).

The long journey to derive the Arbitrage Pricing Line starts with specifying the return generating equation for each security. As the choice of the number of economic factors is not expected to affect students' understanding of the model derivation, the Excel illustration is confined to a three-factor case, for ease of exposition. A crucial requirement in the model derivation, however, is that the number of securities in the market must be large enough for the noise terms in the return generating equations to be attenuated effectively in a portfolio context. For the Excel illustration, n is tentatively set at 100.

Once the security return generation equations for all 100 securities are specified, the next step is the construction of a self-financed portfolio that responds to none of the three factors. Under the assumption that $n = 100$ is large enough for the noise terms to be attenuated effectively, the self-financed portfolio is deemed risk-free. The absence of arbitrage profits ensures a zero payoff for such a portfolio. It also establishes a linear relationship between μ_i and the factor loadings b_{i1}, b_{i2}, and b_{i3}, for $i = 1, 2, \ldots, 100$. This is equation (17), where the parameters λ_0, λ_1, λ_2, and λ_3 can have any values.

For the linear relationship to be meaningful in the context of security pricing, however, the four parameters there must be specified. This is where the risk-free security and three specific portfolios with unit and zero factor sensitivities are needed. They provide values of r_f, $\mu_{\underline{1}}$, $\mu_{\underline{2}}$, and $\mu_{\underline{3}}$ to make the eventual security pricing relationship meaningful.

As part of the Excel illustration, the validity of the assumption that $n = 100$ is large enough for reaching a self-financed risk-free portfolio is examined. The Excel file has been set up in such a way that the number of securities, the severity of random noise, and various other relevant numerical data, can easily be changed. Such flexibility will make it easier for students to explore the noise issue as exercises. Here are the details of the Excel illustration:

4.1 Specification of Each Return Generating Equation

Figure 1 displays the worksheet named "RetGenEq." It shows how the individual return generating equations are specified, for the case where $K = 3$ and $n = 100$. With column A providing the security label, columns B-D (starting from row 13) display the factor loadings b_{i1}, b_{i2}, and b_{i3}, for $i = 1, 2, \ldots, 100$, which are randomly generated under some uniform distributions. For conciseness in presenting a vast amount of illustrative data, rows 40-110, where the factor loadings and some other data for $i = 28, 29, \ldots, 98$ are stored, are not displayed in Figure 1. Likewise,

	A	B	C	D	E	F	G	H
1			Mean and spread			Spread		Worksheet
2		600			3000			name:
3		1200						RetGenEq
4				Mean and spread			For scaling	
5			500			2000		
6			1000					
7					Mean and spread			For scaling
8				400			6000	
9				800				
10							Column sums	
11							-148.2	0.231
12	Security i	bi1	bi2	bi3	ei(Uniform)	ei(Normal)	wi	xi
13	1	1.063	-0.155	0.21	0.02978	-0.018753	43.2	0.076
14	2	1.018	1.186	0.858	-0.01452	-0.003826	-46.8	-0.058
15	3	1.034	0.166	0.183	0.00965	-0.000151	58.8	0.067
16	4	1.763	1.004	1.096	-0.01075	0.0320166	-1.8	-0.02
17	5	0.972	0.541	0.845	-0.02981	0.0040394	22.2	0.042
18	6	-0.441	0.739	0.377	0.01762	-0.020315	-49.8	-0.093
19	7	1.169	0.781	0.277	-0.00197	-0.013353	39	-0.012
20	8	1.159	0.93	-0.129	-0.01918	0.0262519	-33	0.066
21	9	1.604	1.408	-0.25	-0.0233	-0.022516	0.6	-0.002
22	10	1.12	-0.272	-0.3	0.02277	-0.004919	46.2	0.033
23	11	1.07	0.277	0.92	0.00851	0.030072	47.4	-0.051
24	12	1.665	0.362	0.589	-0.00133	0.0099648	48.6	-0.006
25	13	1.292	0.134	0.772	-0.00198	0.0185176	40.2	0.088
26	14	0.325	0.644	1.148	-0.00391	-0.018543	-53.4	-0.047
27	15	1.128	0.917	1.198	0.02016	-0.013754	-56.4	0.017
28	16	-0.124	1.444	-0.264	-0.02583	-0.005054	-22.8	-0.017
29	17	1.216	-0.26	0.953	-0.01156	0.0087699	4.2	0.001
30	18	-0.586	0.239	-0.224	-0.02497	0.0087259	-9.6	-0.001
31	19	0.108	0.671	0.261	-0.0216	0.0241653	-31.8	-0.031
32	20	1.506	0.57	0.385	-0.01678	-0.005384	-38.4	-0.082
33	21	0.96	1.359	-0.079	-0.02615	-0.028311	52.8	0.09
34	22	0.056	1.194	-0.09	0.0252	-0.025957	-34.2	0.021
35	23	1.056	1.073	0.964	0.00145	0.0160453	-54	-0.032
36	24	1.776	0.755	-0.007	-0.02801	-0.016818	4.8	0.055
37	25	1.183	1.034	0.519	0.01702	0.0398574	-29.4	0.011
38	26	1.011	-0.08	1.15	-0.02404	-0.006899	46.8	-0.002
39	27	0.089	0.519	0.584	0.02971	-0.014527	59.4	0.031
111	99	0.721	0.859	0.681	-0.00053	-0.010966	16.8	0.002
112	100	0.384	1.136	0.309	-0.02463	-0.008136	9	-0.052

Figure 1 Numerical Specification of Return Generating Equations and Generation of Initial Values for Solver Searches.

the corresponding rows in all subsequent figures are also not displayed. However, these hidden data are still used in the computations involved.

The range of the factor loadings in each column is based on the values of the two cells in the same column, which are linked to the adjacent scroll bars.[5] Each b_{i1} in B13:B112 is generated over the range of $(m \pm s)/1000$, where m and s are the selected values from the two scroll bars labeled as "Mean and spread." With the two scroll bars set to cover integer values of $0 - 1000$ for m and $0 - 2000$ for s, the linked cell values of $m = 600$ in B2 and $s = 1200$ in B3 correspond to $(m - s)/1000 = -0.60$ and $(m + s)/1000 = 1.80$. Thus, the uniform distribution for generating each b_{i1} randomly is over the range of -0.60 to 1.80, and each cell formula in B13:B112 is =RANDBETWEEN(B$2-B$3,B$2+B$3)/1000.

The selection of m and s to generate b_{i2} and b_{i3}, for $i = 1, 2, \ldots, 100$, in C13:C112 and D13:D112, respectively, also involves scroll bars of analogous features, except for the linked cells. Thus, each b_{i2} in C13:C112 and each b_{i3} in D13:D112 can be generated in the same manner. The corresponding cell formulas are =RANDBETWEEN(C$5-C$6,C$5+C$6)/1000 and =RANDBETWEEN(D$8-D$9,D$8+D$9)/1000. In the illustration here, $m = 500$ and $s = 1000$, as displayed in C5 and C6, respectively, are for specifying each b_{i2}; $m = 400$ and $s = 800$, as displayed in D8 and D9, respectively, are for specifying each b_{i3}.

The factor loadings b_{i1}, b_{i2}, and b_{i3}, for $i = 1, 2, \ldots, 100$, which are part of the input parameters in the Excel illustration, could be entered manually to B13:D112 instead. However, to generate these 300 numbers randomly with the help of scroll bars and the Excel function RANDBETWEEN does reduce the burden in data entry for the current illustration. The approach here also provides great flexibility in generating new sets of input parameters for any subsequent student exercises to explore the nuances in the model derivation.

To complete the specification of the return generation equation for each security requires that a zero-mean noise term be generated as well. Columns E and F (starting from row 13) display the results from two different approaches to specify e_i, for $i = 1, 2, \ldots, 100$. Given the extensive use of the Excel function RANDBETWEEN elsewhere in Figure 1, the approach involving a zero-mean uniform distribution is self-explanatory. Each cell formula for E13:E112

[5] A scroll bar can be generated by selecting Insert, then Form Controls, under the Developer tab. The minimum and maximum values, the incremental change, and the cell link are based on user input. Changes to any of these settings are straightforward.

is =RANDBETWEEN(-E$2,E$2)/100000, where E$2 is linked to its adjacent scroll bar for covering integer values of $0 - 4000$. The greater the selected value from the scroll bar, the more severe is the noise.

The alternative approach to generate a zero-mean noise term involving the use of the Excel function NORMSINV requires an explanation. The function NORMSINV provides the inverse of the standard normal cumulative distribution. The standard normal distribution, by definition, is a normal distribution with a zero mean and a unit standard deviation. As an example involving such a distribution, let us consider cumulative probabilities of 2.5% and 97.5%. The corresponding departures from the zero mean, in terms of the numbers of unit standard deviations, are -1.9600 and 1.9600. Thus, cell formulas =NORMSINV(0.025) and =NORMSINV(0.975) return -1.9600 and 1.9600, respectively. That is, a 2.5% tail area on either side of a normal distribution corresponds to a departure of 1.9600 standard deviations from the mean.

The argument RAND() of the function NORMSINV provides a random number in the range of 0 to 1, which covers all cumulative probabilities. The formula =F$5*NORMSINV(RAND()) /100000 for each cell in F13:F112 generates a zero-mean noise term in the corresponding security return. The multiplicative factor F$5/100000, which represents the standard deviation of the normally distributed noise, is for specifying its severity. The greater the corresponding cell value, the more severe is the noise. For the illustration here, E2 and F5 are set at 3000 and 2000, respectively. In the former case, each randomly generated e_i is set to be within the range of ± 0.03 (or, equivalently, $\pm 3\%$); in the latter case, the standard deviation of the zero-mean normal distribution is set at 0.02 (or, equivalently, 2%).

Notice that the random numbers from the Excel functions RANDBETWEEN and RAND are automatically regenerated, whenever a computation takes place somewhere in the Excel file. Thus, to keep any set of randomly generated numbers constant for subsequent use, the corresponding cell contents will have to be pasted as their values. This can easily be accomplished by making a duplicate copy of the same worksheet, where all cell contents are subsequently replaced by the corresponding values.

4.2 Construction of Self-financed Risk-free Portfolios

The worksheet for Figure 1 also uses the Excel function RANDBETWEEN to generate some preliminary values of w_i and x_i, for $i = 1, 2, \ldots, 100$. The cell formula to generate each w_i in G13:G112 is =G$8*RANDBETWEEN(-100,100)/10000. The scroll bar that is linked to G8 is set to cover integer values of $0 - 10000$. As w_i represents the dollar amount that is assigned to security i, for the construction of a self-financed risk-free portfolio, the value in G8 serves as a scaling factor to adjust the invested amount. The cell formula to generate each x_i in column H (starting from H13) is =RANDBETWEEN(-100,100)/1000.

The values of w_i and x_i as displayed in Figure 1 are intended for use as initial values in *Solver* searches in subsequent steps. As indicated in G11 and H11 under the heading of "Column sums," such initial values do not satisfy the conditions of $\sum_{i=1}^{100} w_i = 0$ and $\sum_{i=1}^{100} x = 1$. However, these violations are not a concern; any values of the 100 decision variables in each case are suitable for initiating a *Solver* search for which there are infinitely many solutions.

All input parameters in subsequent figures are shaded in yellow. The worksheet, named "W," as displayed in Figure 2 is intended for two major tasks. Under the assumption that $n = 100$ is adequate for attenuating the noise terms effectively, the first task is to construct self-financed risk-free portfolios based on the set of factor loadings as displayed in B13:D112. This is the same set of randomly generated data already displayed Figure 1. The four columns of numbers in G13:J112 under the headings of "wi, wi*bi1, wi*bi2," and "wi*bi3" in row 12 are *Solver* results of w_i, $w_i b_{i1}$, $w_i b_{i2}$, and $w_i b_{i3}$, for $i = 1, 2, \ldots, 100$.

As there are infinitely many ways to construct portfolios satisfying equations (8) and (10)-(12), where $n = 100$, the initial values of w_i, for $i = 26, 27, \ldots, 100$, are retained in the *Solver* search. That is, only 25 values of w_i, for $i = 1, 2, \ldots, 25$, as displayed in G13:G37 need changes. For this task, a scroll bar linked to G6 is set to cover integer values of $0 - 200$. With G5 being the negative of the selected value in G6, a range of permissible values for these 25 values of w_i is established. In the illustration here, the selected range is from -100 to 100. In the *Solver* search, the target cell $G$10 is set equal to 0, by changing $G$13:$G$37, subject to the constraints of $G$13:$G$37<=$G$6, $G$13:$G$37>=$G$5, $H$10=0, $I$10=0, and $J$10=0.

The *Solver* results of $\sum_{i=1}^{100} w_i$, $\sum_{i=1}^{100} w_i b_{i1}$, $\sum_{i=1}^{100} w_i b_{i2}$, and $\sum_{i=1}^{100} w_i b_{i3}$ are as displayed in G10:J10. They are all zeros, subject to minor rounding errors. The use of scientific notation in

	A	B	C	D	E	F	G	H	I	J	K
1		rf	rf (%)						Worksheet name: W		
2		0.04	4								
3											
4		mu1 - rf	mu2 - rf	mu3 - rf			Range				
5							-100				
6		11	7	3	(%)		100				
7		0.11	0.07	0.03							
8											
9		mu1	mu2	mu3			Column sums				
10		0.15	0.11	0.07			3.9E-14	2.5E-14	5.5E-14	6.4E-14	2.5E-14
11											
12	Sec i	bi1	bi2	bi3	mu i - rf	mu i	wi	wi*bi1	wi*bi2	wi*bi3	wi*mu i
13	1	1.063	-0.155	0.21	0.11238	0.15238	50.935	54.144	-7.895	10.696	7.7615
14	2	1.018	1.186	0.858	0.22074	0.26074	-42.731	-43.501	-50.679	-36.664	-11.142
15	3	1.034	0.166	0.183	0.13085	0.17085	63.632	65.796	10.563	11.645	10.872
16	4	1.763	1.004	1.096	0.29709	0.33709	-17.032	-30.028	-17.1	-18.667	-5.7414
17	5	0.972	0.541	0.845	0.17014	0.21014	41.028	39.879	22.196	34.669	8.6216
18	6	-0.441	0.739	0.377	0.01453	0.05453	8.1848	-3.6095	6.0486	3.0857	0.4463
19	7	1.169	0.781	0.277	0.19157	0.23157	28.464	33.275	22.231	7.8846	6.5915
20	8	1.159	0.93	-0.129	0.18872	0.22872	-57.995	-67.216	-53.935	7.4813	-13.265
21	9	1.604	1.408	-0.25	0.2675	0.3075	-55.653	-89.267	-78.359	13.913	-17.113
22	10	1.12	-0.272	-0.3	0.09516	0.13516	-35.388	-39.635	9.6255	10.616	-4.783
23	11	1.07	0.277	0.92	0.16469	0.20469	69.903	74.797	19.363	64.311	14.309
24	12	1.665	0.362	0.589	0.22616	0.26616	35.809	59.622	12.963	21.092	9.531
25	13	1.292	0.134	0.772	0.17466	0.21466	52.422	67.73	7.0246	40.47	11.253
26	14	0.325	0.644	1.148	0.11527	0.15527	-1.8724	-0.6085	-1.2058	-2.1495	-0.2907
27	15	1.128	0.917	1.198	0.22421	0.26421	-54.394	-61.356	-49.879	-65.164	-14.371
28	16	-0.124	1.444	-0.264	0.07952	0.11952	95.754	-11.874	138.27	-25.279	11.445
29	17	1.216	-0.26	0.953	0.14415	0.18415	32.852	39.948	-8.5415	31.308	6.0497
30	18	-0.586	0.239	-0.224	-0.0545	-0.0145	47.061	-27.578	11.248	-10.542	-0.68
31	19	0.108	0.671	0.261	0.06668	0.10668	2.1368	0.2308	1.4338	0.5577	0.228
32	20	1.506	0.57	0.385	0.21711	0.25711	-55.006	-82.839	-31.353	-21.177	-14.143
33	21	0.96	1.359	-0.079	0.19836	0.23836	28.423	27.286	38.627	-2.2454	6.775
34	22	0.056	1.194	-0.09	0.08704	0.12704	-19.14	-1.0718	-22.853	1.7226	-2.4315
35	23	1.056	1.073	0.964	0.22019	0.26019	-46.053	-48.632	-49.415	-44.395	-11.983
36	24	1.776	0.755	-0.007	0.248	0.288	-37.861	-67.241	-28.585	0.265	-10.904
37	25	1.183	1.034	0.519	0.21808	0.25808	-38.682	-45.761	-39.997	-20.076	-9.9831
38	26	1.011	-0.08	1.15	0.14011	0.18011	46.8	47.315	-3.744	53.82	8.4291
39	27	0.089	0.519	0.584	0.06364	0.10364	59.4	5.2866	30.829	34.69	6.1562
111	99	0.721	0.859	0.681	0.15987	0.19987	16.8	12.113	14.431	11.441	3.3578
112	100	0.384	1.136	0.309	0.13103	0.17103	9	3.456	10.224	2.781	1.5393

Figure 2 Construction of Self-financed Risk-free Portfolios.

displaying these numbers is for showing the precision involved. For example, the number 3.90E-14 in G10 is 0.00...0390, where 3 is preceded by 13 zeros after the decimal point. (Hereafter, whenever a statement is made about a cell's computed value being zero, it is implicit that the value is subject to minor rounding errors.) This fact confirms the success of the *Solver* search. It is easy to verify numerically that, for the same set of factor loadings, repeated *Solver* searches with different initial values of w_i will lead to different search results. Regardless of the search results, the column sums in G10:J10 will always be zeros.

The second task of the worksheet for Figure 2 is to verify that, regardless of the *Solver* search results, there are always no arbitrage profits. To verify numerically that $\sum_{i=1}^{100} w_i\mu_i$ is always zero, some known values of r_f, $\mu_{\underline{1}}$, $\mu_{\underline{2}}$, and $\mu_{\underline{3}}$ — or, equivalently, λ_0, $\lambda_1+\lambda_0$, $\lambda_2+\lambda_0$, and $\lambda_3+\lambda_0$ in view of equations (21) and (35) — are required. The four scroll bars in columns B-D are for generating these values. Specifically, the scroll bar linked to C2, which is for $100r_f$, is set to cover integer values of $0-10$. In the illustration here, the selected r_f in B2 via the formula =C2/100 is 0.04 (or, equivalently, 4%). The three scroll bars for use to provide 100 times the values of $\mu_{\underline{1}}-r_f$, $\mu_{\underline{2}}-r_f$, and $\mu_{\underline{3}}-r_f$ are analogous; they are all set to cover integer values of $0-20$. The selected values of $\mu_{\underline{1}}-r_f$, $\mu_{\underline{2}}-r_f$, and $\mu_{\underline{3}}-r_f$, as displayed in B7:D7, are 0.11, 0.07, and 0.03 (or, equivalently, 11%, 7%, and 3%), respectively. The corresponding values of $\mu_{\underline{1}}$, $\mu_{\underline{2}}$, and $\mu_{\underline{3}}$, as displayed in B10:D10, are 0.15, 0.11, and 0.07 (or, equivalently, 15%, 11%, and 7%), respectively.

In view of equation (39), the values of μ_i-r_f, for $i=1,2,\ldots,100$, can be computed directly via basic matrix operations. In Figure 2, the computed results are displayed in E13:E112. The matrix operations, which require the "Shift, Ctrl, and Enter" keys to be pressed simultaneously for the selected block E13:E112, are via the formula =MMULT(B13:D112,TRANSPOSE(B7:D7)). This formula nests two matrix functions in Excel. The function TRANSPOSE is used to transform the 1×3 matrix in B7:D7 into a 3×1 matrix, so that it can be pre-multiplied by the 100×3 matrix in B13:D112. The values of μ_i, for $i=1,2,\ldots,100$, as displayed in F13:F112, are deduced by adding back the value of r_f to each cell in E13:E112.

With each μ_i determined, the corresponding value of $w_i\mu_i$ and hence the sum $\sum_{i=1}^{100} w_i\mu_i$ can easily be computed. The sum as displayed in K10 is zero. For the same set of factor loadings in B13:D112, there are two ways to check the robustness of the result in K10. One way is to perform the *Solver* searches repeatedly, with different initial values of $w_1, w_2, \ldots, w_{100}$ for each search. A simpler way is to vary the values of r_f, $\mu_{\underline{1}}$, $\mu_{\underline{2}}$, and $\mu_{\underline{3}}$, via the four scroll bars involved. Either way, any changes in the displayed values in column K will be readily noticeable.

From an algebraic standpoint, the set of values of r_f, $\mu_{\underline{1}} - r_f$, $\mu_{\underline{2}} - r_f$, and $\mu_{\underline{3}} - r_f$, as stored in B2 and B7:D7, can be viewed as one of the infinitely many ways to assign values of λ_0, λ_1, λ_2, and λ_3 for use in equation (38). Each set of these parameter values corresponds to a set of computed values of $\mu_1, \mu_2, \ldots, \mu_{100}$ in F13:F112. Regardless of what arbitrary values of these four parameters are used and how unrealistic some computed values of $\mu_1, \mu_2, \ldots, \mu_{100}$ may appear, the sum $\sum_{i=1}^{100} w_i \mu_i$ in K10 is always zero. The robustness of such a result is not surprising, as a non-zero sum in K10 would indicate arbitrage profits.

4.3 Construction of Portfolios with Unit and Zero Factor Sensitivities

The next step in the numerical illustration is to show how portfolios $\underline{1}$, $\underline{2}$, and $\underline{3}$ are constructed. Figure 3 is based on three separate worksheets for such a task, named "Port1, Port2," and "Port3." The *Solver* search in each of the three worksheets is based on the same set of factor loadings from B13:D112 of the worksheet for Figure 2. These factor loadings and the cells in B2, B7:D7, and B10:D10, which contain the same values of r_f, $\mu_{\underline{1}}$, $\mu_{\underline{2}}$, and $\mu_{\underline{3}}$ in Figure 2, are duplicated in each of the three worksheets at the corresponding cell locations. So are the computed values of $\mu_i - r_f$ and μ_i, for $i = 1, 2, \ldots, 100$, in E13:F112.

The three worksheets are set up in the same way, except for some differences in the *Solver* part. The first page of Figure 3, which is about the construction of portfolio $\underline{1}$, shows all relevant columns of the worksheet named "Port1." Columns G-K contain results that are specific to portfolio $\underline{1}$. To construct portfolio $\underline{1}$ with *Solver*, the target cell \$G\$10 is set equal to 1, by changing \$G\$13:\$G\$37, subject to the constraints of \$G\$13:\$G\$37<=\$G\$5, \$G\$13:\$G\$37>=\$G\$4, \$H\$10=1, \$I\$10=0, and \$J\$10=0. The *Solver* settings for portfolios $\underline{2}$ and $\underline{3}$ are the same, except for the last three constraints. Specifically, for portfolio $\underline{2}$, the last three constraints are \$H\$10=0, \$I\$10=1, and \$J\$10=0; for portfolio $\underline{3}$, the last three constraints are \$H\$10=0, \$I\$10=0, and \$J\$10=1 instead.

The second page of Figure 3 shows the results in columns G-K of the two worksheets that are specific to portfolios $\underline{2}$ and $\underline{3}$. The results in columns G-K of the worksheets for portfolios $\underline{2}$ and $\underline{3}$ are duplicated in columns L-P and columns Q-U, respectively, by means of some linking

	A	B	C	D	E	F	G	H	I	J	K
1		Rf					**G**	**H**	**I**	**J**	**K**
2		0.04									
3							Range		Worksheet name:		Port1
4							-0.5				
5							0.5				
6		mu1 - rf	mu2 - rf	mu3 - rf							
7		0.11	0.07	0.03							
8											
9		mu1	mu2	mu3			Column sums				mu1
10		0.15	0.11	0.07			1.00000	1.00000	8.7E-15	2.0E-14	0.15000
11											
12	Sec i	bi1	bi2	bi3	mu i - rf	mu i	xi	xi*bi1	xi*bi2	xi*bi3	xi*mu i
13	1	1.063	-0.155	0.21	0.11238	0.15238	0.22478	0.23894	-0.0348	0.0472	0.03425
14	2	1.018	1.186	0.858	0.22074	0.26074	-0.0613	-0.0624	-0.0727	-0.0526	-0.016
15	3	1.034	0.166	0.183	0.13085	0.17085	0.17393	0.17985	0.02887	0.03183	0.02972
16	4	1.763	1.004	1.096	0.29709	0.33709	-0.0039	-0.0069	-0.0039	-0.0043	-0.0013
17	5	0.972	0.541	0.845	0.17014	0.21014	0.12229	0.11887	0.06616	0.10334	0.0257
18	6	-0.441	0.739	0.377	0.01453	0.05453	-0.031	0.01368	-0.0229	-0.0117	-0.0017
19	7	1.169	0.781	0.277	0.19157	0.23157	0.01638	0.01915	0.01279	0.00454	0.00379
20	8	1.159	0.93	-0.129	0.18872	0.22872	0.06253	0.07247	0.05815	-0.0081	0.0143
21	9	1.604	1.408	-0.25	0.2675	0.3075	-0.0779	-0.125	-0.1097	0.01948	-0.024
22	10	1.12	-0.272	-0.3	0.09516	0.13516	0.17996	0.20156	-0.0489	-0.054	0.02432
23	11	1.07	0.277	0.92	0.16469	0.20469	0.06422	0.06871	0.01779	0.05908	0.01314
24	12	1.665	0.362	0.589	0.22616	0.26616	0.07858	0.13084	0.02845	0.04628	0.02091
25	13	1.292	0.134	0.772	0.17466	0.21466	0.21357	0.27594	0.02862	0.16488	0.04585
26	14	0.325	0.644	1.148	0.11527	0.15527	0.03958	0.01286	0.02549	0.04544	0.00615
27	15	1.128	0.917	1.198	0.22421	0.26421	-0.1979	-0.2232	-0.1814	-0.237	-0.0523
28	16	-0.124	1.444	-0.264	0.07952	0.11952	-0.0204	0.00253	-0.0294	0.00538	-0.0024
29	17	1.216	-0.26	0.953	0.14415	0.18415	0.18432	0.22413	-0.0479	0.17566	0.03394
30	18	-0.586	0.239	-0.224	-0.0545	-0.0145	0.1089	-0.0638	0.02603	-0.0244	-0.0016
31	19	0.108	0.671	0.261	0.06668	0.10668	0.02758	0.00298	0.01851	0.0072	0.00294
32	20	1.506	0.57	0.385	0.21711	0.25711	-0.0282	-0.0425	-0.0161	-0.0109	-0.0073
33	21	0.96	1.359	-0.079	0.19836	0.23836	0.03581	0.03437	0.04866	-0.0028	0.00853
34	22	0.056	1.194	-0.09	0.08704	0.12704	0.00182	0.0001	0.00217	-0.0002	0.00023
35	23	1.056	1.073	0.964	0.22019	0.26019	-0.018	-0.019	-0.0193	-0.0173	-0.0047
36	24	1.776	0.755	-0.007	0.248	0.288	-0.2258	-0.4011	-0.1705	0.00158	-0.065
37	25	1.183	1.034	0.519	0.21808	0.25808	0.01411	0.01669	0.01458	0.00732	0.00364
38	26	1.011	-0.08	1.15	0.14011	0.18011	-0.002	-0.002	0.00016	-0.0023	-0.0004
39	27	0.089	0.519	0.584	0.06364	0.10364	0.031	0.00276	0.01609	0.0181	0.00321
111	99	0.721	0.859	0.681	0.15987	0.19987	0.002	0.00144	0.00172	0.00136	0.0004
112	100	0.384	1.136	0.309	0.13103	0.17103	-0.052	-0.02	-0.0591	-0.0161	-0.0089

Figure 3 Construction of Portfolios with Unit and Zero Factor Sensitivities.

	A	L	M	N	O	P	Q	R	S	T	U
1		**G**	**H**	**I**	**J**	**K**	**G**	**H**	**I**	**J**	**K**
2											
3		Range		Worksheet name:		Port2	Range		Worksheet name:		Port3
4		-0.5					-0.5				
5		0.5					0.5				
6											
7											
8											
9		Column sums				mu2	Column sums				mu3
10		1.00000	4.8E-09	1.00000	-9.6E-09	0.11000	1.00000	-3.2E-09	-7.5E-09	1.00000	0.07000
11											
12	Sec i	xi	xi*bi1	xi*bi2	xi*bi3	xi*mu i	xi	xi*bi1	xi*bi2	xi*bi3	xi*mu i
13	1	0.00615	0.00654	-0.001	0.00129	0.00094	0.15209	0.16168	-0.0236	0.03194	0.02318
14	2	-0.0383	-0.039	-0.0455	-0.0329	-0.01	-0.0951	-0.0968	-0.1128	-0.0816	-0.0248
15	3	0.06728	0.06957	0.01117	0.01231	0.0115	0.17641	0.1824	0.02928	0.03228	0.03014
16	4	-0.0132	-0.0233	-0.0133	-0.0145	-0.0045	-0.1079	-0.1903	-0.1084	-0.1183	-0.0364
17	5	-0.0188	-0.0183	-0.0102	-0.0159	-0.0039	0.12146	0.11806	0.06571	0.10263	0.02552
18	6	0.13224	-0.0583	0.09773	0.04986	0.00721	0.04841	-0.0213	0.03577	0.01825	0.00264
19	7	-0.0541	-0.0632	-0.0422	-0.015	-0.0125	0.0455	0.05319	0.03553	0.0126	0.01054
20	8	0.04236	0.04909	0.03939	-0.0055	0.00969	-0.0821	-0.0952	-0.0764	0.01059	-0.0188
21	9	0.01411	0.02263	0.01987	-0.0035	0.00434	-0.2301	-0.369	-0.3239	0.05752	-0.0707
22	10	0.15608	0.17481	-0.0425	-0.0468	0.0211	0.06711	0.07517	-0.0183	-0.0201	0.00907
23	11	0.02176	0.02328	0.00603	0.02002	0.00445	0.15366	0.16442	0.04256	0.14137	0.03145
24	12	0.06461	0.10758	0.02339	0.03806	0.0172	0.15587	0.25952	0.05642	0.09181	0.04149
25	13	-0.024	-0.031	-0.0032	-0.0185	-0.0052	0.21676	0.28006	0.02905	0.16734	0.04653
26	14	0.23704	0.07704	0.15265	0.27212	0.03681	0.21158	0.06876	0.13625	0.24289	0.03285
27	15	-0.0695	-0.0784	-0.0637	-0.0832	-0.0184	-0.0486	-0.0548	-0.0445	-0.0582	-0.0128
28	16	0.37166	-0.0461	0.53668	-0.0981	0.04442	0.14271	-0.0177	0.20607	-0.0377	0.01706
29	17	-0.0196	-0.0239	0.00511	-0.0187	-0.0036	0.2625	0.3192	-0.0683	0.25016	0.04834
30	18	0.34334	-0.2012	0.08206	-0.0769	-0.005	0.15981	-0.0937	0.0382	-0.0358	-0.0023
31	19	0.23457	0.02533	0.15739	0.06122	0.02502	0.07507	0.00811	0.05037	0.01959	0.00801
32	20	-0.0419	-0.0632	-0.0239	-0.0161	-0.0108	0.12227	0.18414	0.0697	0.04708	0.03144
33	21	-0.0373	-0.0358	-0.0507	0.00295	-0.0089	-0.1617	-0.1553	-0.2198	0.01278	-0.0386
34	22	0.13758	0.0077	0.16428	-0.0124	0.01748	0.04446	0.00249	0.05309	-0.004	0.00565
35	23	0.14088	0.14877	0.15116	0.13581	0.03666	0.06705	0.0708	0.07194	0.06464	0.01745
36	24	-0.2299	-0.4083	-0.1736	0.00161	-0.0662	-0.4828	-0.8575	-0.3645	0.00338	-0.1391
37	25	0.01705	0.02017	0.01763	0.00885	0.0044	0.0916	0.10836	0.09471	0.04754	0.02364
38	26	-0.092	-0.093	0.00736	-0.1058	-0.0166	0.056	0.05662	-0.0045	0.0644	0.01009
39	27	-0.055	-0.0049	-0.0285	-0.0321	-0.0057	0.036	0.0032	0.01868	0.02102	0.00373
111	99	-0.083	-0.0598	-0.0713	-0.0565	-0.0166	0.06	0.04326	0.05154	0.04086	0.01199
112	100	-0.035	-0.0134	-0.0398	-0.0108	-0.006	0.085	0.03264	0.09656	0.02627	0.01454

Figure 3 Construction of Portfolios with Unit and Zero Factor Sensitivities (continued).

formulas. For example, on the second page of Figure 3, the formula for L13 is =Port2!G13, which duplicates the content of G13 in the worksheet named "Port2," and the formula for Q13 is =Port3!G13, which duplicates the content of G13 in the worksheet named "Port3."

For ease of exposition when describing the results and the computations involved in Figure 3, only the column headings in the three original worksheets are mentioned below. For clarity, the original column headings, G-K, are also displayed in G1:U1 of Figure 3. Each page of Figure 3 has two panels, which are separated by a vertical line. Each of the panels for portfolio-specific results shows the corresponding original worksheet name as well.

Three different sets of randomly generated x_i, for $i = 1, 2, \ldots, 100$, are used as the initial values for the three *Solver* searches. Changes to these values in each *Solver* search are confined to $i = 1, 2, \ldots, 25$ only. To avoid extreme outcomes in allocations of investment funds, the *Solver* result for each of these 25 values of x_i is confined to be in the range of -0.5 to 0.5, as indicated in G4:G5 for each portfolio.

As shown in G10:J10 for each portfolio, the conditions for constructing portfolios $\underline{1}$, $\underline{2}$, and $\underline{3}$ are all satisfied. Specifically, portfolio $\underline{k}$ has unit sensitivity to factor k and zero sensitivity to each of the remaining factors, for $k = 1$, 2, and 3. Based on each set of *Excel* search results of x_i in G13:G112, the values of $x_i \mu_i$ are stored in K13:K112. As confirmed in K10, the sum $\sum_{i=1}^{100} x_i \mu_i$ always matches the expected return of the corresponding portfolio. This result is robust, regardless of what initial values of x_i are used in the *Solver* searches.

Notice that, unlike the *Solver* searches for the construction of self-financed risk-free portfolios, where all 100 securities are involved, the *Solver* searches here can be based on any subset of the 100 securities. The only requirement is that each search must be based on five or more securities. The three worksheets for Figure 3 are set up in such a way that each *Solver* search can accommodate $26 \leq n \leq 100$. This can easily be achieved by using zeros for some or all initial values in G38:G112, where $x_{26}, x_{27}, \ldots, x_{100}$ are stored.

4.4 The Noise Issue

A crucial condition in the derivation of the APT is that the number of securities in the market must be large enough for the noise in the return generating equations for individual securities to be attenuated effectively in a portfolio context. However, the analytical materials in the model derivation cannot provide, and are not meant to provide, any guidance as to how large

n must be for this condition to hold. Intuitively, the more severe is the noise e_i in the return generating equation for each security i, a larger n is required for the sum $\sum_{i=1}^{n} w_i e_i$ in equation (13) to become trivially small.

As mentioned earlier, each randomly generated e_i for the Excel illustration here either is in the range of ± 0.03 for a uniform distribution or has a standard deviation of 0.02 for a zero-mean normal distribution. For such distributions, whether $n = 100$ is large enough to attenuate the noise effectively is assessed in the worksheet (named "Noise") for Figure 4. For this task, columns A-F of Figure 4 duplicate the corresponding data in Figure 2. Columns G-I of Figure 4 duplicate (in G13:I112) the same values of $w_1, w_2, \ldots, w_{100}$ in G13:G112 of Figure 2 and the same two sets of $e_1, e_2, \ldots, e_{100}$ in E13:F112 of Figure 1. In Figure 4, computed values of $w_1 e_1, w_2 e_2, \ldots, w_{100} e_{100}$ are displayed in J13:K112. The corresponding column sums, with each being $\sum_{i=1}^{n} w_i e_i$, are displayed in J4:K4.

When compared to the sums $\sum_{i=1}^{100} w_i$, $\sum_{i=1}^{100} w_i b_{i1}$, $\sum_{i=1}^{100} w_i b_{i2}$, $\sum_{i=1}^{100} w_i b_{i3}$, and $\sum_{i=1}^{100} w_i \mu_i$ in G10:K10 of Figure 2, which are all zeros (subject to rounding errors), each sum $\sum_{i=1}^{100} w_i e_i$ in J4:K4 of Figure 4 is far from being a zero. Repeated computations with different *Solver* results of $w_1, w_2, \ldots, w_{100}$ and different sets of random noise (from the same distributions) still result in each sum $\sum_{i=1}^{100} w_i e_i$ being non-zero. Thus, for the set of numerical data in the illustration, $n = 100$ does not seem to be adequate for attenuating the noise effectively.

As the model derivation hinges on the success of noise attenuation in a portfolio context, the issue as to how many securities are really needed for a self-financed portfolio to become risk-free deserves to be addressed. The Excel illustration here can serve as a good starting point for students to examine the noise issue more closely. Using the same Excel file accompanying this paper, students can explore on their own the impact of varying the severity of random noise in the return generating equations on the required number of securities in the market. Such an exercise will help them appreciate more fully the relevance of the noise issue in the derivation of the APT.

5 Concluding Remarks

What underlies the derivation of the APT is a simple idea. Specifically, a risk-free investment that requires no cash outlays must have zero payoffs; otherwise, arbitrage profits would be

	A	B	C	D	E	F	G	H	I	J	K
1		Rf							Worksheet name:		Noise
2		0.04									
3							Column sums				
4							3.9E-14	-0.0925	-0.2164	6.21775	-4.6639
5											
6		mu1 - rf	mu2 - rf	mu3 - rf							
7		0.11	0.07	0.03							
8											
9		mu1	mu2	mu3							
10		0.15	0.11	0.07							
11								Uniform	Normal	Uniform	Normal
12	Sec i	bi1	bi2	bi3	mu i - rf	mu i	wi	ei	ei	wi*ei	wi*ei
13	1	1.063	-0.155	0.21	0.11238	0.15238	50.935	0.02978	-0.0188	1.51686	-0.9552
14	2	1.018	1.186	0.858	0.22074	0.26074	-42.73	-0.0145	-0.0038	0.62046	0.16348
15	3	1.034	0.166	0.183	0.13085	0.17085	63.632	0.00965	-0.0002	0.61405	-0.0096
16	4	1.763	1.004	1.096	0.29709	0.33709	-17.03	-0.0108	0.03202	0.1831	-0.5453
17	5	0.972	0.541	0.845	0.17014	0.21014	41.028	-0.0298	0.00404	-1.223	0.16573
18	6	-0.441	0.739	0.377	0.01453	0.05453	8.1848	0.01762	-0.0203	0.14422	-0.1663
19	7	1.169	0.781	0.277	0.19157	0.23157	28.464	-0.002	-0.0134	-0.0561	-0.3801
20	8	1.159	0.93	-0.129	0.18872	0.22872	-57.99	-0.0192	0.02625	1.11234	-1.5225
21	9	1.604	1.408	-0.25	0.2675	0.3075	-55.65	-0.0233	-0.0225	1.29671	1.25307
22	10	1.12	-0.272	-0.3	0.09516	0.13516	-35.39	0.02277	-0.0049	-0.8058	0.17408
23	11	1.07	0.277	0.92	0.16469	0.20469	69.903	0.00851	0.03007	0.59488	2.10213
24	12	1.665	0.362	0.589	0.22616	0.26616	35.809	-0.0013	0.00996	-0.0476	0.35683
25	13	1.292	0.134	0.772	0.17466	0.21466	52.422	-0.002	0.01852	-0.1038	0.97073
26	14	0.325	0.644	1.148	0.11527	0.15527	-1.872	-0.0039	-0.0185	0.00732	0.03472
27	15	1.128	0.917	1.198	0.22421	0.26421	-54.39	0.02016	-0.0138	-1.0966	0.74813
28	16	-0.124	1.444	-0.264	0.07952	0.11952	95.754	-0.0258	-0.0051	-2.4733	-0.4839
29	17	1.216	-0.26	0.953	0.14415	0.18415	32.852	-0.0116	0.00877	-0.3798	0.28811
30	18	-0.586	0.239	-0.224	-0.0545	-0.0145	47.061	-0.025	0.00873	-1.1751	0.41065
31	19	0.108	0.671	0.261	0.06668	0.10668	2.1368	-0.0216	0.02417	-0.0462	0.05164
32	20	1.506	0.57	0.385	0.21711	0.25711	-55.01	-0.0168	-0.0054	0.923	0.29614
33	21	0.96	1.359	-0.079	0.19836	0.23836	28.423	-0.0262	-0.0283	-0.7433	-0.8047
34	22	0.056	1.194	-0.09	0.08704	0.12704	-19.14	0.0252	-0.026	-0.4823	0.49682
35	23	1.056	1.073	0.964	0.22019	0.26019	-46.05	0.00145	0.01605	-0.0668	-0.7389
36	24	1.776	0.755	-0.007	0.248	0.288	-37.86	-0.028	-0.0168	1.06049	0.63673
37	25	1.183	1.034	0.519	0.21808	0.25808	-38.68	0.01702	0.03986	-0.6584	-1.5418
38	26	1.011	-0.08	1.15	0.14011	0.18011	46.8	-0.024	-0.0069	-1.1251	-0.3229
39	27	0.089	0.519	0.584	0.06364	0.10364	59.4	0.02971	-0.0145	1.76477	-0.8629
111	99	0.721	0.859	0.681	0.15987	0.19987	16.8	-0.0005	-0.011	-0.0089	-0.1842
112	100	0.384	1.136	0.309	0.13103	0.17103	9	-0.0246	-0.0081	-0.2217	-0.0732

Figure 4 An Example of Inadequate Noise Attenuation.

available. However, seemingly innocuous uses of some basic finance terms to articulate this simple idea could still cause inadvertent confusions to students. In particular, although the term *rate of return* (or, simply, *return*) is not applicable to self-financed portfolios for which no investment capital — i.e., a zero in the denominator for each *return* computation — is involved, standard textbook explanations of the absence of arbitrage profits from such portfolios still routinely rely on its use.[6]

To deepen the perceived mystery of the APT to many finance students, textbook derivations of the model all require some mathematical knowledge that is likely unfamiliar to them. This paper, which has derived the Arbitrage Pricing Line from a pedagogic perspective, is intended to dispel any remaining mystery of the model. The model derivation has a crucial requirement. Specifically, the number of securities in the market must be large enough for self-financed risk-free portfolios to be constructed. Given such a requirement, as there are many more securities than the number of linear constraints for portfolio construction, there will be infinitely many ways to allocate investment funds among the available securities. Interestingly, it is the lack of uniqueness in the allocation of investment funds that facilitates a pedagogic derivation of the model.

Excel plays an important pedagogic role in this paper. It is also the same lack of uniqueness that allows Excel *Solver* to be used by students as a numerical tool to recognize the various nuances of the model derivation. The Excel file accompanying this paper has been set up in such a way that it is easy to generate different sets of data for the numerical tasks involved. Of particular importance is the noise issue. Although it is required in the model derivation that the noise be attenuated effectively in a portfolio context, whether the requirement is satisfied can be revealed only in numerical settings. Thus, the Excel illustration in this paper is a good starting point for students to gain valuable hands-on experience with the noise issue in the model derivation.

[6] For example, Levy and Post (2005, Chapter 11) state that "(s)ince the arbitrage portfolio involves no net investment and no risk, it must yield a zero expected return," (p. 372). Likewise, Copeland, Weston, and Shastri (2005, Chapter 6) state that "(i)f the return on the arbitrage portfolio were not zero, then it would be possible to achieve an infinite return with no capital requirements and no risk," (p. 178).

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