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Functions and Mathematical Modelling with Spreadsheets

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Functions and Mathematical Modelling with Spreadsheets

Abstract

In this paper we intended to show how it is possible to encourage the use of computers in the learning of Mathematics, using development environments such as spreadsheets, with which teachers and students can develop interactive computer applications with graphical components and animation.

It also seeks to enhance the links between mathematics, technology and other sciences in order to enhance the power of mathematics for the simulation of physical phenomena.

The relationship between mathematics and observable physical phenomena are fundamental aspects for the implementation of an approach that favours the laboratory aspect of mathematics, together with the validation of hypotheses and mathematical modelling techniques (computer models) used to simulate the observed phenomena.

Keywords

Spreadsheets, functions, mathematical representations, programming

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1. Introduction

The use of digital technologies (DT) in teaching mathematics at different levels of education has become a common practice. Mathematical concepts that are often presented to the students in an abstract perspective are more easily understood when using appropriate digital technologies. These give students the chance to focus on reasoning, decision making and problem solving [1].

One of the most important decisions to be made is which specific digital tool to use. Then it is important to help students to explore the tools available to their full potential, rather than limiting their use to checking the results obtained or repeating the procedures already applied manually.

But a successful use of technology must be based on the understanding of the mathematical concepts involved. This comprehensiveness can also promote an innovative dynamic collaboration between teachers and students, the first providing the guidelines to achieve the desired result and reinforcing each step with the mathematical ideas involved, while the latter have the opportunity to test their math skills by programming the different steps and being rewarded with a final product of their own.

One can find on the Internet a large number of interactive features that essentially refer the student to the role of a spectator. Engaging students in building computational applications with very specific objectives in order to get them to use certain concepts, contributes, not only to the clarification of the concepts involved, but also to the development of students' critical thinking and organizational skills.

Using a spreadsheet like Excel or a similar one, we present in this paper a sequence of applications covering different levels of education and concerning the elementary study of functions.

2. The importance of the study of the functions

One of the goals of Mathematics is to explore concepts that allow us to study a great variety of concrete situations to which a system of ideas applies. Once identified and selected, these ideas are the basis for the construction of an abstract notion that is applicable, in particular, to the concrete situations that served as a model, as they share some essential characteristics. This is the case of the concept of function.

The variation of a quantity depending on the variation of another quantity is an important aspect in the study of the concept of function, but that study is incomplete if it is not possible to qualify and quantify this process of variation. In this article, we show how students can develop computer applications for the learning of functions at different levels of education, using some basic commands of a spreadsheet like Excel [2]

Since the concept of function is transversal to primary and secondary education, the choice is based on the families of functions studied in the different years of schooling: functions of direct proportionality (8th year), quadratic functions (10th year) and trigonometric functions (11th year).

It is important to highlight the systematic use of graphical representations of functions as a way of understanding and visualizing their fundamental aspects [3]

3. Applications developed by students and teachers for the study of functions. From basic to higher level.

3.1 Some examples of linear models

The function of direct proportionality is the first contact that students have with graphic representation of functions.

A geometric example is the linear variation of the perimeter value of a square with its side length value. Students should be encouraged to ask: will the variation of the area of a square with its side also be linear?

Starting with a blank spreadsheet, students can build an application to investigate this subject. The construction of a spreadsheet application should be accompanied by work with pencil and paper. Students initially plan what they want to get on a sheet of paper and only then implement it in the spreadsheet.

A relationship between variables can be represented by a table of values, a graph and an algebraic equation (Figure 1).

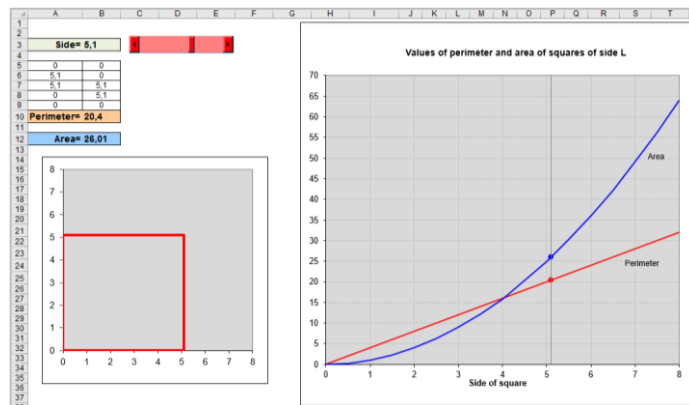


Figure 1: Variation of the values of the perimeter and the area of a square of varying side.

When students have completed the planning phase, established the necessary mathematical formulas and are fit for implementation on the spreadsheet, it is necessary to become aware of the procedures inherent to the spreadsheet itself. The classroom experience has demonstrated that students quickly learn the basic spreadsheet procedures. It is the teacher's job to monitor the students' work so that they take advantage of the use of this computational environment to effectively learn mathematics.

In this case, it is necessary to define the coordinates of the vertices of a square with a variable measure of the side and the coordinate tables relative to the graphs of the perimeters and areas as functions of the measure of the side. The first step is the drawing of a square with variable length side. To do so, students can follow the following instructions:

- Name the cell B3 "side";
- Place the length of side of the square in cell B3;

Build a table with the coordinates of the vertices of the square (in function of the length of the side).

To insert a scrollbar that allows to change, in a continuous way, the length of the side, the procedures and code are [4]:

View, Toolbar, Control toolbox, Scrollbar

```
Private Sub ScrollBar1_scroll()
ScrollBar1.Min = 0
ScrollBar1.Max = 80
Cells(3, 2) = ScrollBar1.Value / 10
End Sub
```

To obtain the graphs of the functions concerning the perimeters and areas as functions of the measure of the square side:

Place in cell B10 the expression for the perimeter of the square ($=4*B3$)

Place in cell B 12 the expression for the area of the square ($=B3^2$)

Select the table data for the perimeter and for the area and construct the correspondent dispersion graphs.

After building the application, students can observe that, for a square with side measuring four units, the values of the area and the perimeter coincide, as well as observe from which length of the side of the square the area becomes larger than the perimeter.

They may also verify that, contrarily to the perimeter function, the area function does not vary proportionally with the length of the side.

After the preceding task, students can be challenged to construct an application to explore another situation:

Consider the perimeters of the rectangles with constant width equal **to four** measure units. The length of the rectangles, x , varies.

Can we say that the perimeter function of such rectangles is directly proportional to their length?

Students should construct an interactive and animated application that allows the simultaneous visualization of the rectangle and respective table of coordinates of the vertices, the perimeter function graph and its analytical expression.

Figure 2 shows the application interface to study this subject.

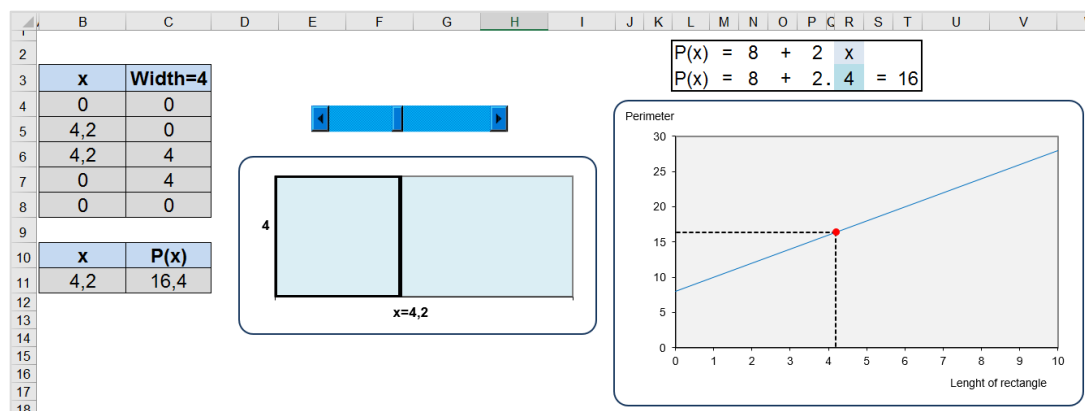


Figure 2: Variation the perimeter of a fixed-width rectangle.

3.2 Systems of linear equations

The study of systems of first degree equations in two variables is part of the program of mathematics at basic education level, being also foreseen its geometrical interpretation as well as the resolution of problems.

Studying the existence of solution of a system of linear equations is equivalent to analyzing the relative position of two coplanar straight lines and determine, if possible, their intersection.

Starting with a graphical approach can be a good option to motivate the students for the algebraic process. We propose an applied context as follows.

Suppose that two drivers A and B start simultaneously, and in opposite directions, a route between two locations that are 300 km apart. They have different speeds and intend to meet during the trip. Assuming that the speed of each one remains constant, choose the meeting point in order to avoid waiting time for any of the drivers. Students can explore this question using the interactive application whose interface can be seen on (Figure 3)

This application requires the construction of graphs reflecting, at any time, the distance of each driver to the respective starting point, as well as the introduction of scroll bars allowing the change of speed of the drivers, allowing this way the analysis of the different situations created. The spreadsheet tools to be used are those used to construct the application in the preceding paragraph so, once defined what is needed to be done from a mathematical point of view, students can build it easily.

The definition of the analytical expressions of the functions involved can raise some questions and the teacher's monitoring is crucial. As we are dealing with constant speeds, the distances are directly proportional to the time spent on route, and the speed is the constant of proportionality. But, as it is intended to represent in the same system of axes two pathways in opposite directions, designating by v_a and v_b the speed of cars departing from the point of coordinates $(0, 0)$ and $(300, 0)$ respectively, the formula for the distances as functions of time is $d(t) = v_a t$ and $d(t) = 300 - v_b t$.

Para determinar as coordenadas do ponto de encontro dos dois carros basta resolver o sistema de equações

$$\begin{cases} d = v_a t \\ d = 300 - v_b t \end{cases}$$

find its solution, $\left(\frac{300}{v_a + v_b}, \frac{v_a \cdot 300}{v_a + v_b} \right)$ and introduce these expressions in cells T8 and V8, respectively, of the spreadsheet.

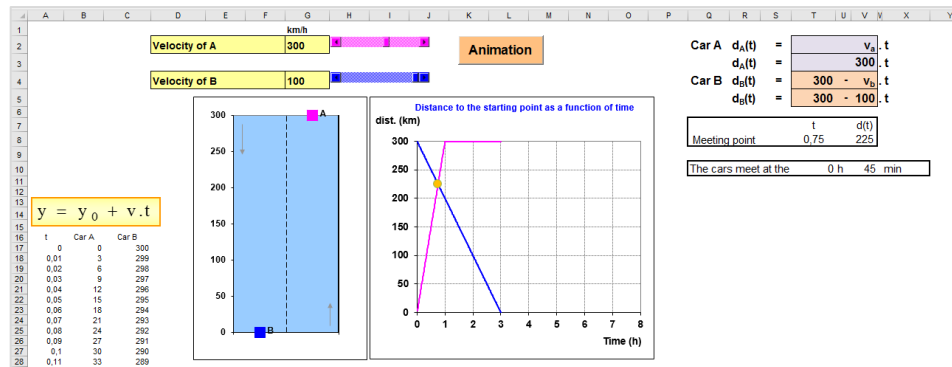


Figure 3: Graphical representation of the variation of distance of two cars according to the time.

The environment provided by the application allows students to observe the importance of different representations for the same problem. In this case it is possible to implement a methodology that provides, on the one hand, the use of the application previously elaborated and presented to the students and, on the other hand, the possibility to complete the application with small details, namely the determination of the solution of the system of equations as well as the determination of the moment in which the cars meet.

3.3 Study of quadratic function

The study of quadratic functions defined by $y = ax^2 + bx + c$ where $a \in \mathbb{R} \setminus \{0\}$ e $b, c \in \mathbb{R}$, their properties and graphical representation, are part of the mathematics national curriculum, and it is expected to study their properties and their graphical representations.

It is also recommended the graphic exploration of families of quadratic functions by varying the parameters of their analytical expressions, as well as the study of applications to the modeling of physical phenomena. An important aspect for the use of the spreadsheet in this topic is the possibility of displaying multiple graphical representations, allowing students to experiment and discover the effect of changing the parameters in the graphs.

The use of Excel in this subject promotes a systematic association with computer applications that complements a clear analytical approach, leading students to understand the parameters involved in the analytical expressions and to assign them a specific geometric meaning. Table 1 presents, in the first column, the analytical expression of the quadratic function in different ways, and in the second column, their respective graphical representations (parabolas) built up into Excel. In the first case, the analytical expression is written in terms of the parameter a and of the coordinates of the vertex of the parabola (x_v, y_v) .

In the second case, the analytical expression shows the three parameters a , b and c .

No 3º caso a expressão analítica evidencia o parâmetro a e os zeros x_1 e x_2 da função.

In the first case, the analytical expression is written in terms of the parameter a and of the coordinates of the vertex of the parabola, (x_v, y_v) .

In the second case, the analytical expression shows the three parameters a , b and c .

In the third case, the analytical expression shows the parameter a and zeros x_1 e x_2 of the function.

Table 1: Graphical representation of a quadratic function in different ways

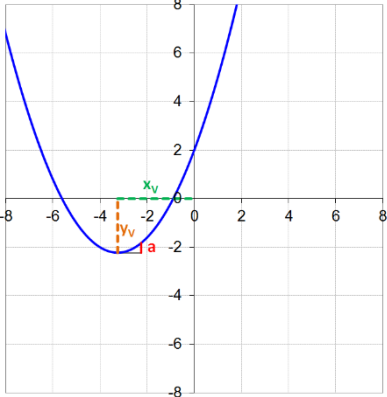
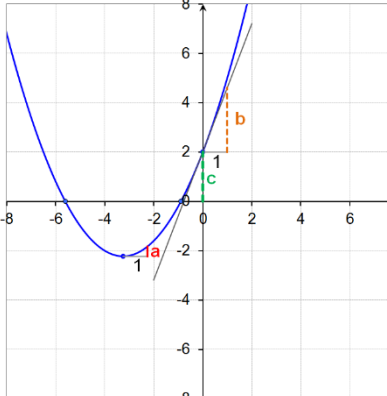
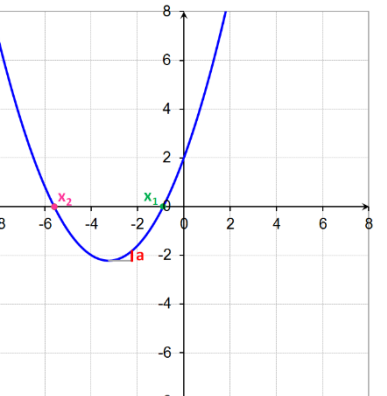
Analytical expression	Graphical representation
$f(x) = a(x - x_v)^2 + y_v$	
$f(x) = ax^2 + bx + c$	
$f(x) = a(x - x_1)(x - x_2)$	

Figure 4: Graphical representation of a quadratic function. Geometric meaning of the different parameters.

Assigning interactivity to the first graph of the previous figure, through three scrollbars (Figure 5) and starting with $x_v = y_v = 0$, we see the effect of the variation of the parameter a in the parabola format (more or less "elongated") and in the direction of the parabola concavity (up, down). Setting a and y_v and varying x_v , it appears that the parabola shape holds but the graphic undergoes a horizontal translation. Similarly, setting a and x_v and varying y_v , it appears that the graphs retain their shape but undergo a vertical translation.

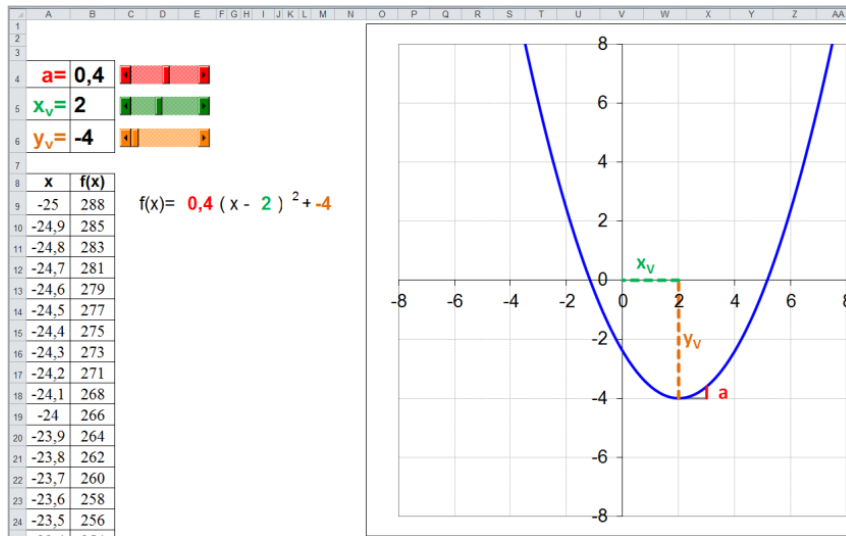


Figure 5: Application developed in Excel to display the graph corresponding to the function $y = a(x - x_v)^2 + y_v$

Likewise, introducing interactivity to the second graph of the figure 3, by three sliders (Figure 6) we can observe the effect of varying the parameters a, b and c in the graphs of functions.

Consider a unit straight line segment, parallel to the abscissa axis and with one end coincident with the vertex of the parabola. The value of parameter a can be viewed geometrically as the length of the line segment perpendicular to the first, at the other end.

The parameter c corresponds to the ordinate at the origin, i.e. the parabola intersects the axis of the ordinates at the point (0, c).

Finally, the parameter b corresponds to the slope of the line tangent to the graph of the function at the point (0, c).

From the point of view of the construction of this application, the objective of showing the geometric meaning of the parameters a, b and c leads to a deeper study of the quadratic function.

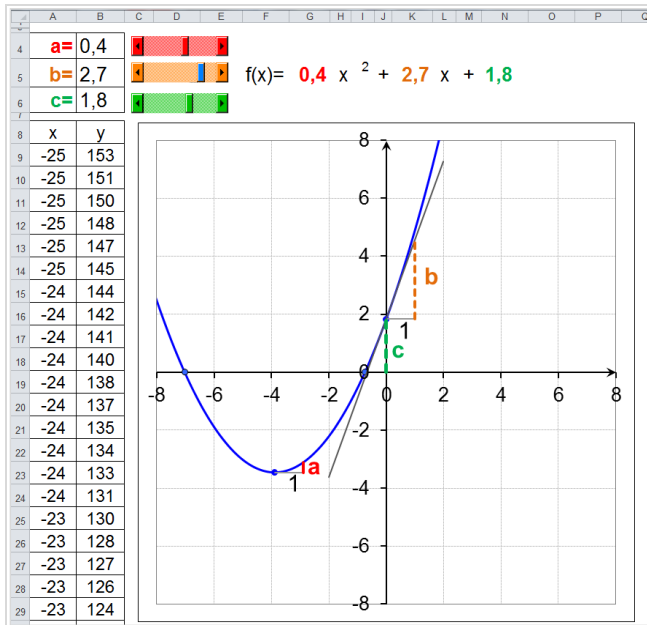


Figure 6: Application developed in Excel to display the graph corresponding to the function $y = ax^2 + bx + c$

Finally, in Figure 7 we present the application interface developed to represent a quadratic function using its zeros.

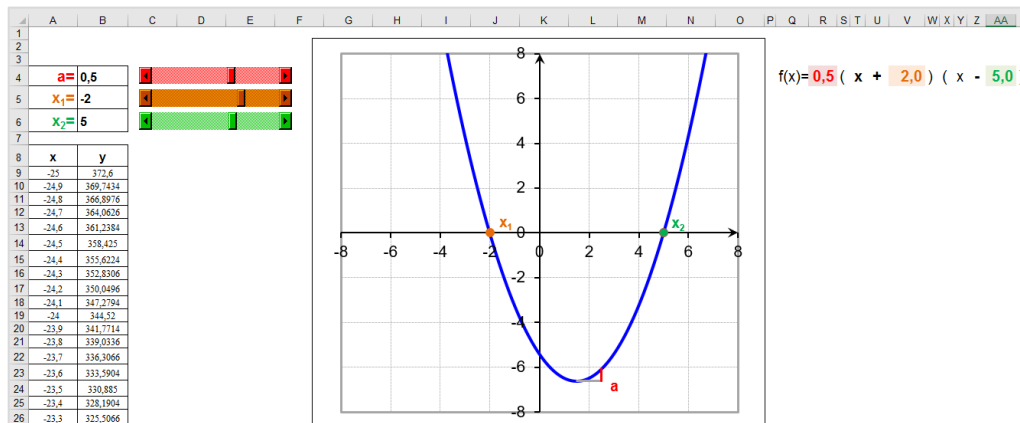


Figure 7: Application developed in Excel to display the graph corresponding to the function $y = a(x - x_1)(x - x_2)$

With the support of the applications presented above, it is possible to develop a study like the following:

1. Consider the parabola P, which is the graph of $f(x) = 4x^2$ and the point A(1, 4) on P.

- Using the quadratic-vertex.xls application, describe how one can get the parabola P', which is the graph of the function $g(x) = -4(x-3)^2+2$, from the parabola P through the transformations on the plane
- What is the point A' on parabola P' that corresponds to the point A on parabola P?

- Using the quadratic-vertex application, write the analytical expression of the function whose graph is obtained from the graph of f by a translation defined by the vector components $(-2, 4)$.
2. Consider the parabola which is the graph of the quadratic function $f(x) = x^2 + 2x$.
- Use the quadratic-canonical.xls application to determine the coordinates of the vertex Q and verify your answer, by determining, analytically, the coordinates of Q .
 - Consider the quadratic function $g_1(x) = 2x^2 + b_1x + c_1$, $b_1, c_1 \in \mathbb{R}$. Use the application quadratica-canonical.xls to calculate b_1 and c_1 to make sure that the vertex of g_1 and the vertex of Q are the same and verify your answer, analytically.
 - Consider the quadratic function $g_2(x) = 3x^2 + b_2x + c_2$. What should the values of b_2 and c_2 be in order to guarantee that the vertex of this function is the same as Q and verify your answer, analytically?
 - Consider the family of functions $g_m(x) = mx^2 + b_mx + c_m$. What should the values of b_m and c_m be so that the parabola vertex coordinates are equal to the coordinates of Q ?
3. Consider the quadratic function $h(x) = 3x^2 + 4x + c$ with c real number. Use the quadratic-canonical application to determine, in an approximate way, the values of c for which the function h has no zeros and determine analytically all values of c in these conditions.

3.4 Connections between Algebra and Geometry

In order to explore relationships between algebra and geometry, it is possible to develop a computational application that allows the study of the variation of a function associated with a specific geometric problem. The spreadsheet allows the simultaneous visualization of different representations and the addition of relevant information to the plots, such as, line segments and relevant points. This possibility plays a great role in the work of students as it stimulates and enhances mathematical thinking.

The following problem involves determining the roots of equations with degree greater than two which makes it suitable for a graphic treatment supported by a computer or a graphic calculator:

Consider the function $f(x) = -x^2 + 4x$ and point $A(1,0)$ (see Figure 8). Determine the function g that associates to each value of x the distance between the point A and the point $P(x, f(x))$. Knowing that there exist only two points on the graph of f distant in one unit from A , determine its abscissa exactly or approximately to one decimal place (adapted from Bivar et al, 2012) [5].

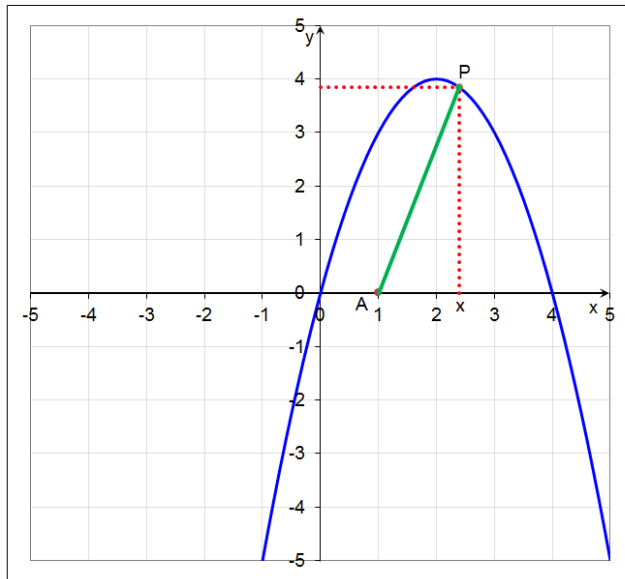


Figure 8: Plot of the function $f(x) = -x^2 + 4x$ points A and P and the line segment [AP].

The analytic expression of the function $g(x)$ is

$$g(x) = \sqrt{(x-1)^2 + (-x^2 + 4x)^2} = \sqrt{x^2 - 2x + 1 + (x^4 - 8x^3 + 16x^2)} = \sqrt{x^4 - 8x^3 + 17x^2 - 2x + 1}$$

In a spreadsheet, one can build an application where the function f , the point A, the line segment [AP] and the function g can be represented. Afterwards, using a scrollbar, the value of the abscissa of the point P can be controlled.

The function g will be restricted to the range $[0, 4]$ which matches the range where the function f is positive. In the context of the problem, as we change the abscissa of P, the length of the line segment [AP] also changes.

The challenge that lies ahead for the students is to develop a dynamic application where these relationships can be visualized and where several variations and hypothesis regarding the problem can be established.

Figure 9 shows the application interface. It is important to notice that the spreadsheet contains several representations: On the left, an associated table with the coordinates of the points necessary for the graphical representation of the function f (ranging from -25 to +25, with a step of 0,1). In the center lies the graphical representation of g and a scrollbar that allows controlling the value of the abscissa of point P. Changes in the abscissa of P produce immediate changes on the plots.

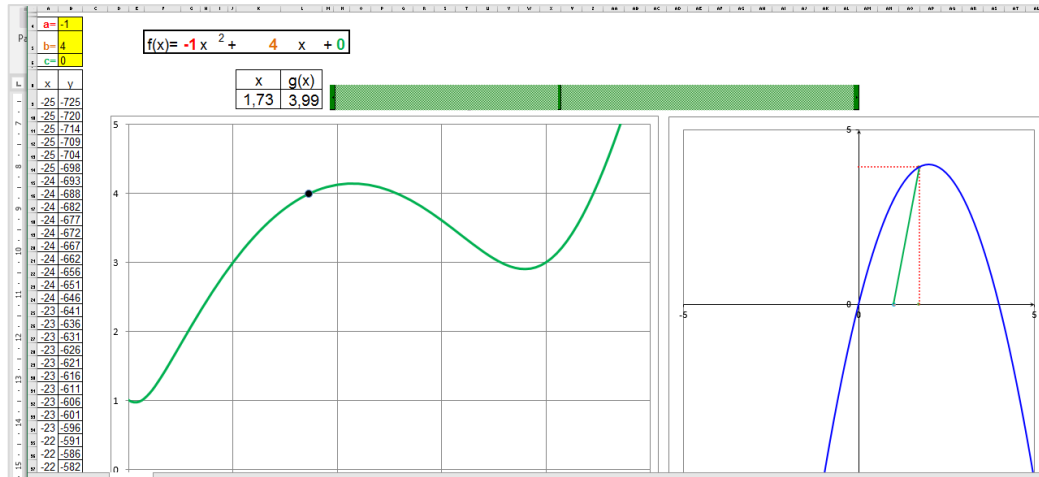


Figure 9 : Interface of application.

While developing this application in the classroom, side-by-side with the students, the mathematical content of the problem is greatly enhanced.

The possibility of building two graphs simultaneously, as well as the dynamic part of the application, broadens the knowledge that can be gained, as it allows the visualization of the way in which the length of the segment [AP] changes.

This approach promotes an integration of both the mathematical and the programming work, which allows a better planning of the work to be done in the classroom.

Using this application, some mathematical ideas can be further explored, such as:

- How many points in f are one unit distance away from point A and how many are four units of distance away from point A?
- Is there any point in f that is less than one unit of distance away from A?
- Study the monotony and existence of maxima and minima of g and analyze and interpret its meaning in the context of the situation;

An extension of this activity consists in generalizing the problem by considering a quadratic function $f(x) = ax^2 + bx + c$ $a, b, c \in \mathbb{R}$. The function g , which associates, to each value of the abscissa, the distance between point A and point P of abscissa x will be given by $g(x) = a^2x^4 + 2abx^3 + (2ac + b^2 + 1)x^2 + (2bc - 2)x + c^2 + 1$. In Figure 10, the results for different parameters of the quadratic function are shown.

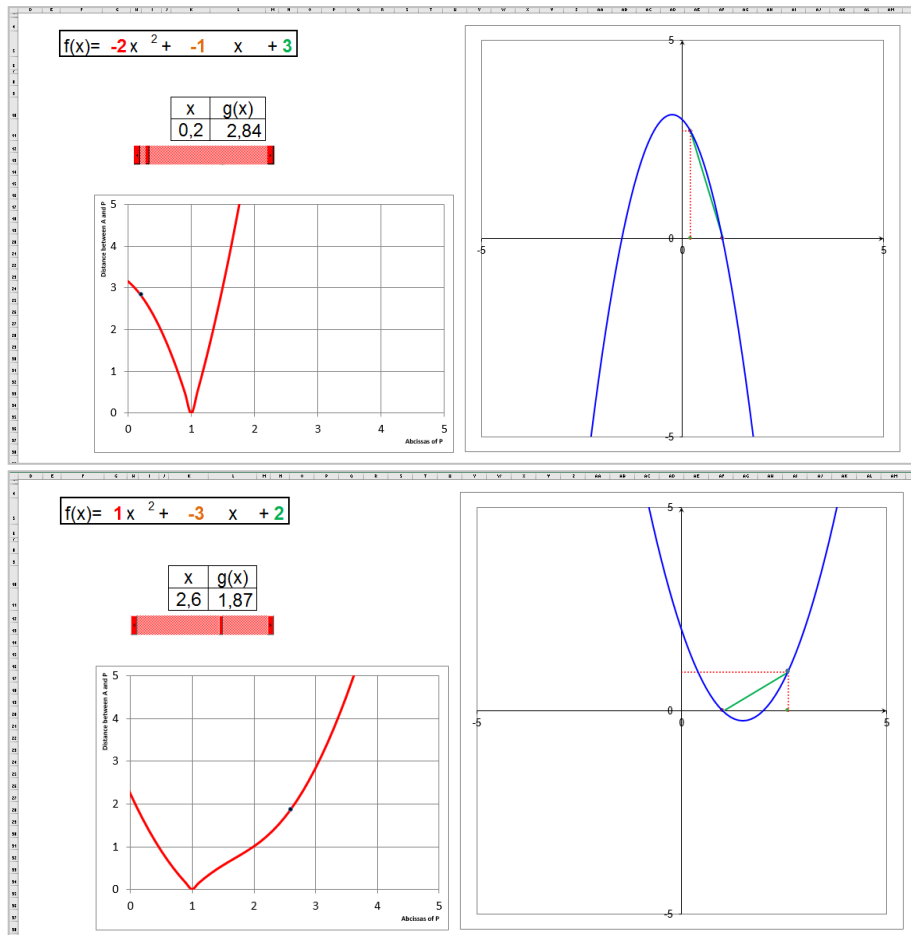


Figure 10: Extension of the problem for the generic case of a quadratic function. Two situations.

3.5 Trigonometric functions

Trigonometric functions are usually studied in secondary school and they have a wide range of applications, namely in the modelling of periodical phenomena.

Next, we describe an activity that was developed with prospective teachers. It consists of the development of a dynamic application on a spreadsheet, aiming the visualization of a schematic design of an Eolic windmill and of the movement of its rotating blades.

The sequence of steps to be followed demands an organized way of thinking and the application and use of mathematical concepts such as the choice of an appropriate Cartesian set of axes, suited to the position of the windmill, trigonometry formulae to simulate the rotation of the blades and the sin function to study the variation of the height of the end edge of each blade with time.

It is worthwhile to mention that, on the follow-up of this activity, a group of students developed an application, designed to the study of the problem of the movement of a giant spinning wheel, such as the ones found in some amusement parks. More precisely, the students modelled the relationship between the height (measured from the ground) of a passenger and time. This activity was also developed with students from the 11th grade in secondary school. In this particular case, students initially studied the problem from a mathematical perspective and only then they developed the application on the excel spreadsheet. Regarding this subject, it is possible to see the study of this particular

problem and the corresponding dynamic application developed in the context of future teachers training, available

<http://www.matematica-interactiva.com/matematica/index.php?ref=104>

Schematic representation of an Eolic windmill

To represent, schematically, an electrical wind generator and simulate its blades rotation movement, firstly, it's necessary to "place" the turbine in a Cartesian coordinate system and build a table with the coordinates of the vertices of the triangle that represents a blade (drawn in the vertical position in red) (Figure 11), coordinates that refer to the other two triangles which represent the green and yellow blades are obtained through a rotation around the origin of, respectively, 120° and 240°. Coordinates of the base triangle vertices are entered directly.

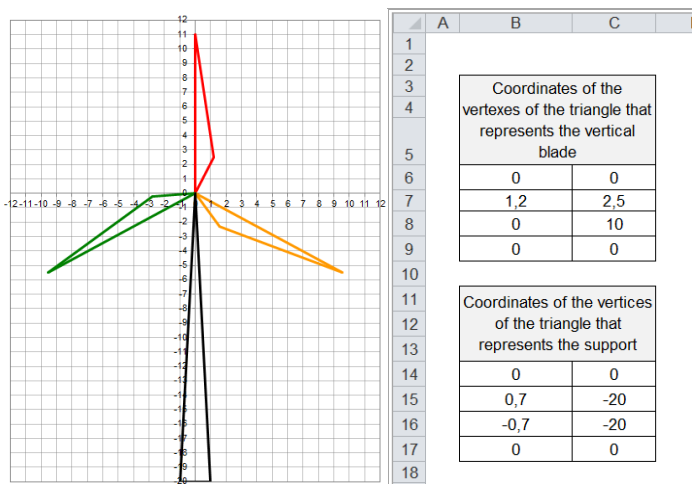


Figure 11: Schematic drawing of an electric windmill in a Cartesian set of axis and table with the coordinates of the vertices of the triangle representing the vertical blade and representing the base.

Visually, rotating an "object" produces a change of position but maintains distances to the center of rotation unchanged. The transformation parameter involved is the angle. As such, it is possible to establish a relationship between the coordinates of a point $P(x,y)$ and a point $P'(x',y')$ where P' is the point that is obtained after a rotation of P by an angle θ , with center of rotation on the origin of the Cartesian referential. In Figure 12, it is shown on the left side, that the rotation of amplitude θ of the triangle $[ABP]$ around the point A gives rise to the triangle $[AB'P']$. On the right, are shown the equations relating the coordinates of points P' and P .

After this first stage of mathematical work, two tables comprising the coordinates of the vertices of the triangles that represent the green and yellow blades are introduced on the spreadsheet. The angles will be of 120 and 240 degrees, respectively.

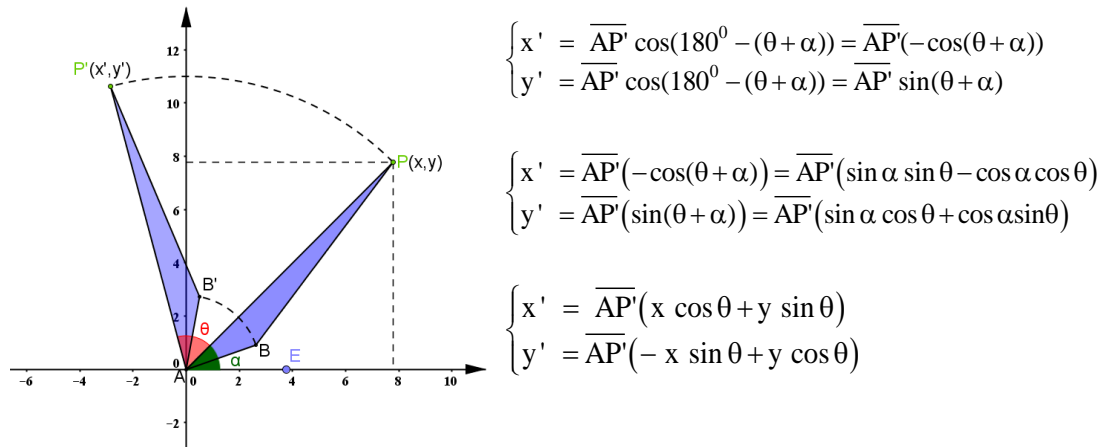


Figure 12: Rotating the point P around A with an angle of amplitude θ degrees.

Motion of the blades of the Electrical wind mill

To display the rotational movement of the mill blades, just keep in mind that if we vary the angle of rotation between 0° and 360° with an increase of 0.1 (for example), the coordinates of all vertices are updated and the effect of rotation is achieved by introducing in the application a slider connected to the cell containing the value of the amplitude of the rotation angle. By acting on the slider the value is constantly updated, and therefore the formulas (already introduced in the cells) as well. The final result appears in the graphical representation with the view of the movement of the blades, since the coordinates of the vertices are also being continuously updated.

This interdependence of the cells and between them and the graphics can be explored with clear advantages for the learning of mathematical concepts.

Plotting graphical representations

The goal at this point is to show how one can add a dynamic component to the application, but in the form of a short film. Just insert a command button that, when properly programmed, displays the rotational movement of the blades of a wind mill and at the same time the layout of the graphical representations of functions that describe the height of each vertex of the triangle (each of the blades) depending on the rotation angle.

In Table 2, it is presented the code corresponding to the command button and the description of the functions associated with the code.

Table 2 VBA Code

VBA code	DESCRIPTION OF FUNCTIONS RELATED TO THE CODE
PrivateSub CommandButton1_Click() n = 0 Pi = 3.1415927	Notes: n is a variable that indicates the line number in the worksheet.

<pre> For t = 0 To 360 n = n + 1 Cells(n, 4) = t Cells(n, 5) = 11 * Sin((t + 90) * Pi / 180) Cells(n, 6) = 11 * Sin((t + 210) * Pi / 180) Cells(n, 7) = 11 * Sin((t + 330) * Pi / 180) Next t End Sub </pre>	<p>The cycle For, Next allows one to perform the following:</p> <p>For each t between 0° and 360° (value of the rotation angle):</p> <ul style="list-style-type: none"> • increments a line; • “Writes” in cell (n, 4) the value of the angle: t; • “Writes” in the same line but columns 5, 6 and 7, the values corresponding to the three functions for every value of t corresponding to the three blades.
<p>Clicking on the command button, the columns are filled automatically. Once the graphical representation of the functions is connected to the columns that were filled, the desired effect of a movie is created.</p>	

One of the great advantages of using spreadsheets to develop these short films is that its users do not come themselves as programmers but as someone who "write formulas" and "build models" [6]. In fact, as one can see in this example, the computer skills are very few and they consist only of the knowledge of the operating cycle. However, mathematical modeling is of great importance because it is the establishment of mathematical expressions that define the desired functions that are the core of the problem. In Figure 13 it is presented the interface application developed with the command button incorporated.

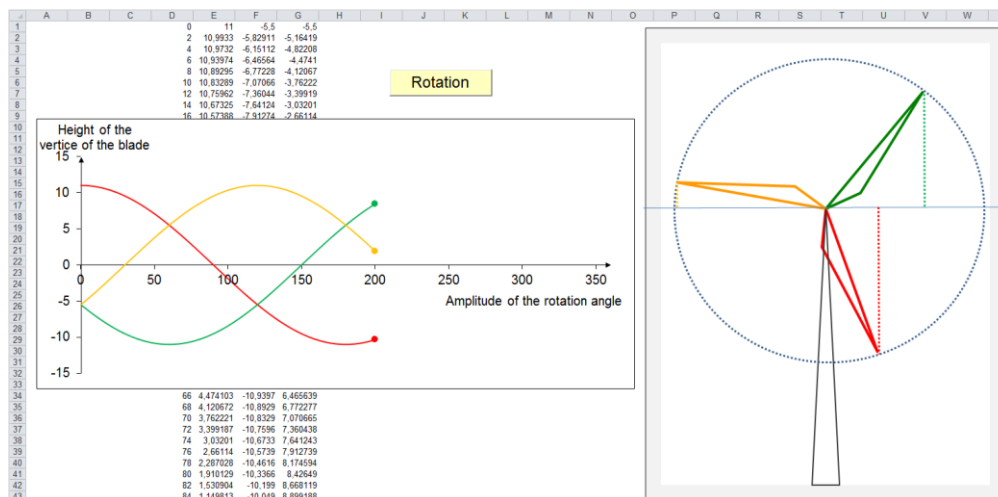


Figure 13: Application interface: Simulation of the motion of a windmill

Using the application, one can explore some questions that arise naturally:

What is the height of each of the blades at the start? And after a full turn?
 What is the meaning of the intersection points of the curves (two by two) in this context?

What is the period of the functions?

What are the values of the local maxima and local minima for each of the functions? What is the meaning of the increase, decrease and sign of the functions in this context?

As an extension of this activity it can be proposed changes to the code in order to obtain variations, such as change the length of the blades and change the number of blades.

3.6 Derivative function

The study of the derivative function made in secondary school provides the basis for the transition to higher education.

Many approaches to the study of the derivatives stimulate students' acquisition of automatic procedures instead of a global understanding.

The association of the derivative of a function f at a point x to the line slope of the tangent to the function curve at $(x, f(x))$, and the definition of the derivative function as the function that maps each x to $f'(x)$, promotes the development of the subject based on the use of derived tables. This trend can be reversed by taking advantage of representations that show these connections. The use of animated modules is a resource to be exploited, but be aware that benefits may be reduced if students are mere spectators. On the contrary, if students participate in the construction of these modules, the situation may change substantially [7].

In Figure 10 it is shown the interface of an application developed by students, for the implementation of the concept of derivative of a function at a point, equation of the tangent to the graph of a function at a point and their graphical representations. The great advantage of this application is in its construction, as it is the concretion of the procedures involved that mathematical knowledge is required. On the spreadsheet are represented the graphs of quadratic functions of the form $y=dx^2+e$ and linear functions $y = ax + b$. On the left side of the sheet one can find the scrollbars that allow varying the parameters of the analytical expressions. On the right side appear the tangency point coordinates whose abscissa can be changed with the correspondent scroll bar.

The purpose of the application is to choose a point of tangency and set the parameters of the quadratic function, find the equation of the tangent to the function in the chosen point. To do this, simply click on the scrollbars for the parameters of the equation of the line.

Then clicking on the command button "Equation of the tangent line" one can get the tangent line equation and compare it with the estimated values (Figure 14).

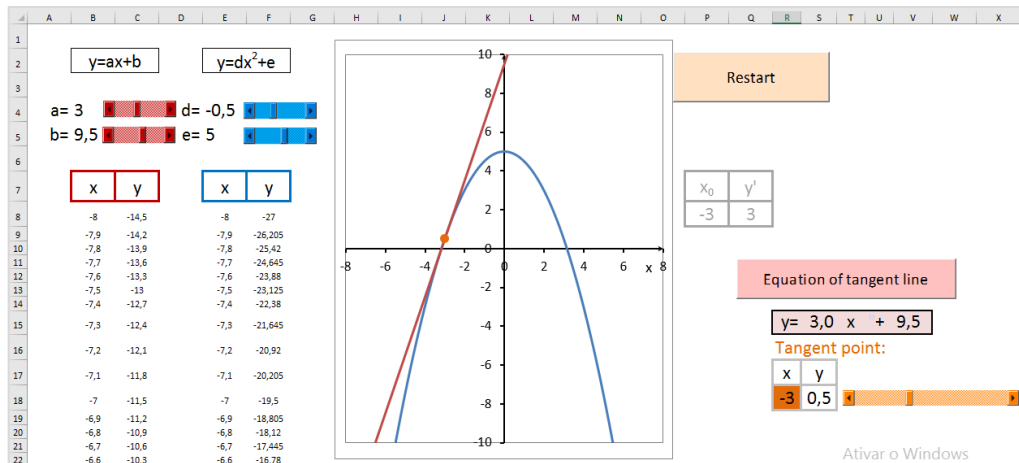


Figure 14: Application interface: Equation of tangent line.

The development of the application follows the guidelines below:

1. Construction of the graph of the function $y=ax+b$ with the possibility to change the parameters a and b ;
2. Construction of the graph of the function $y=dx^2+e$ with the possibility to change the parameters d and e ;
3. Construction of a table with the coordinates of the point of tangency with the possibility to change the value of the abscissa, using the scrollbar.
4. Addition of a command button to the interface, to compute the equation of the line tangent to the curve at the point of tangency.

Stages 1 and 2 consist basically of the construction of graphs and its construction process has already been discussed in point 1 of this article.

As a means to conclude stages 3 and 4, a table with the value of x_0 and the value of the derivative on that point is built. As Excel does not provide built-in functions to compute derivatives, the pre-calculated expression of the derivative of the quadratic function is introduced in the table. In cell P8 a value is entered for x_0 and in cell Q8 the expression $'=2*d*P8'$ is entered. Afterwards, the coordinates of the tangency point are entered together with a command button responsible for the computation of the slope and the value of y when $x=0$ corresponding to the equation of the tangent line to the function at the tangency point.

The code associated to the command button "Equation of tangent line" just consists of translating the mathematical computations for the slope and the "y" value at the origin to VBA.

```
Private Sub CommandButton1_Click()
Cells(15, 19) = Cells(8, 17) ' Approximation of Slope
Cells(15, 22) = Cells(18, 19) - Cells(8, 17) * Cells(18, 18) ' Zero intercept
End Sub
```

The application can be enhanced by adding the possibility to broad all the analysis already done to other functions as well as plotting the graph of the derivative. In Figure 15, the interface of the application is presented. One can find a new table that will be used to plot the derivative of the original function under study. The application has an extra button that allows the selection of the function to be

studied, from a pre-defined list that contains polynomial, trigonometric and exponential functions.

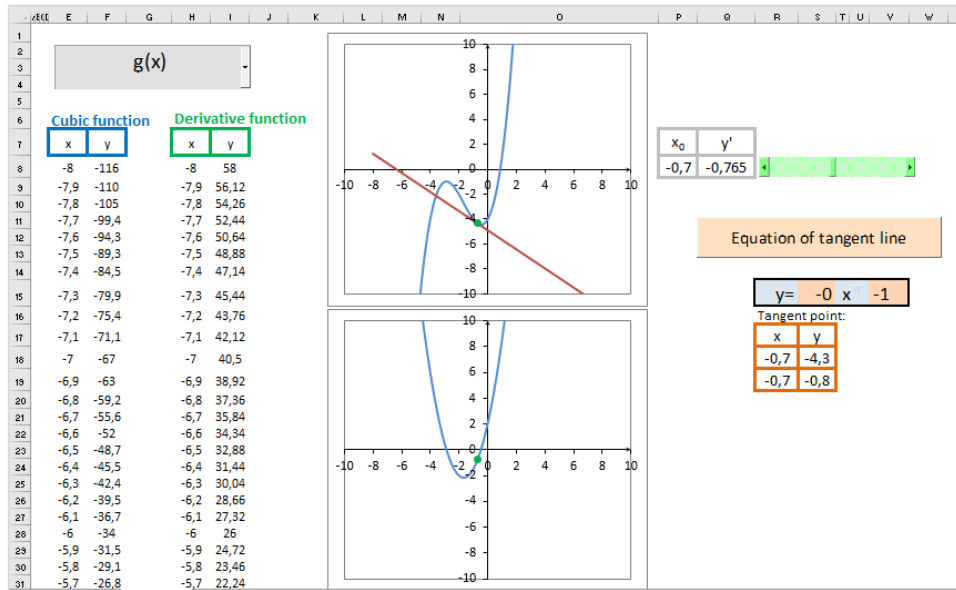


Figure 15: Application interface: Derivative of different functions.

Using this application, it is possible to relate the study of the sign of the derivative and the identification of the points which cause a sign change in the derivative, with the ranges of monotony and the extrema of a given function.

In the mathematics teaching program for secondary school, the study of rational functions is planned, together with the resolution of problems regarding the determination of tangent lines to the graphs of these functions. It is also expected that students can make use of derivative rules for both these and some more important functions. In this ambit, it is presented and proposed the development of an application on the spreadsheet that shows how the skills of symbolic manipulation and the capability of interpreting graphs can work together. The construction of the application is a good example of what may be an activity to be conducted with secondary school students.

The challenge that follows is based on the study a set of functions of the type

$$f(x) = \frac{k}{x-c} + (mx+b) \quad k, c, m, b \in \mathbb{R}$$

The first step is to build the graph with variable parameters (using scrollbars), then the graph of the derivative function

$$f'(x) = m - \frac{k}{(x-c)^2} \quad m, k, c \in \mathbb{R}$$

both graphs.

The application, whose interface is shown in Figure 16, presents the scrollbars used in parameter variation, the plots for the function and its derivative. The tangent lines for both graphs are also represented.

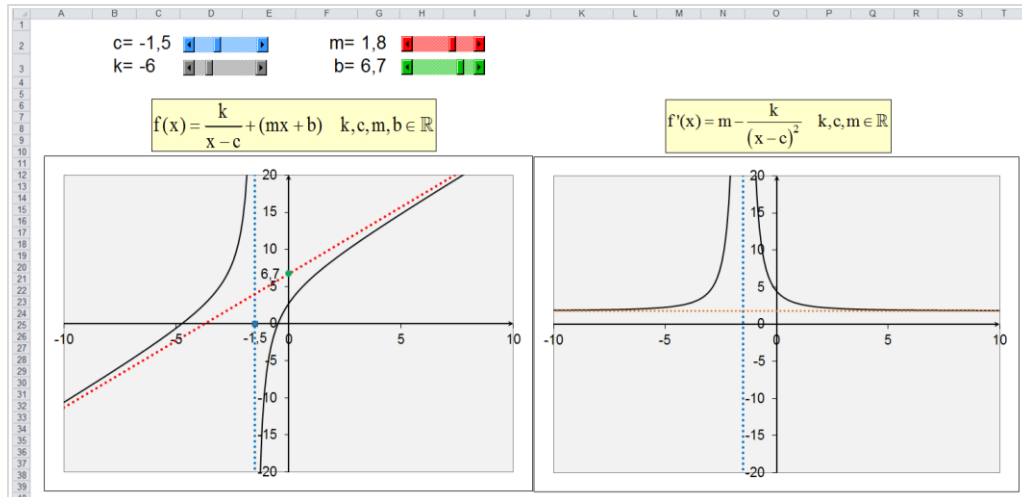


Figure 16: Application interface: Study of rational functions and their derivative functions.

The application allows the establishment of conjectures on the relationship between the parameters involved in the rational function and its derivative, such as:

- Relationship between the domains and codomains of $f(x)$ and its derivative;
- Relationship between the slope of the oblique asymptote of $f(x)$ and the derivative sign;
- Relationship between oblique asymptote of $f(x)$ and horizontal asymptote of $f'(x)$;
- Relationship between vertical asymptotes of the both graphs;
- The effect of the change of the ordinate value at the origin of the oblique asymptote of the graph of f on the graph of f' .

4. Numerical methods for differential equations solving. Study of the pendulum with a rigid rod

At this stage, the study of the oscillatory motion of a pendulum is deepened as a means to show the interest in using differential equations in the mathematical modelling of physical phenomena, as well as the interest in developing computational modules on a spreadsheet as a support tool for the modelling and analysis of the obtained results.

Currently, several studies refer to the importance of using a spreadsheet in numerical analysis. It is mentioned the intuitive character of this tool in implementing recursion (7).

The versatility of spreadsheets is well-noticed when it is intended to perform numerical calculations, as they promote efficiency and precision. The necessary formulae are easily entered and the needed procedures to implement the numerical method are conducted in an intuitive and organized way, allowing the student to develop a bigger understanding of the method. As the processing capabilities for graphics increase, it is possible to visualize a graphical representation of the values assumed by the variable in each successive iteration. To show how the most used numerical methods can be implemented on a spreadsheet, like Euler, Runge-Kutta and Heun methods, let us look at the following initial value problem where the unknown function is $u=u(t)$.

$$\begin{cases} \frac{du}{dt} = -ku & t \in [0, 5[\\ u(0) = u_0 \end{cases}$$

This equation is a first order linear differential equation, whose solution is a function. Since the exact solution is known, it is obviously not necessary to use numerical methods for solving the problem of initial values. However, the fact that we know the exact solution, allows us to gain confidence in the implementation of the methods and evaluate its accuracy and reliability.

Euler's method, Heun method and Runge-Kutta method

The Euler method is one of the simplest methods for the determination of the numerical solution of differential equations. It is a method that applies to first order linear differential equation or 1st order systems of differential equations. Let us now consider the initial value problem

$$u' = f(x, u) \text{ e } u(x_0) = u_0$$

In the numerical resolution of this problem, one should begin by discretizing the domain in intervals of equal length.

Integrating both sides of the differential equation for each interval yields the following equality

$$\int_{x_n}^{x_{n+1}} u'(t) dt = \int_{x_n}^{x_{n+1}} f(x, u(t)) dt$$

equivalent to

$$u(x_{n+1}) - u(x_n) = \underbrace{\int_{x_n}^{x_{n+1}} f(x, u(t)) dt}_{\text{Approximated by the area of the rectangle}}$$

and also

$$u_{n+1} = u_n + h \cdot u'_n$$

which is known by Euler formula.

The Heun method is an improvement of the Euler method. In each interval can be calculated a first approximation u_{n+1} by Euler's method ("prediction"), for then to be calculated u'_{n+1} :

$$u'_{n+1} = f(x_{n+1}, u_{n+1})$$

So the final value u_{n+1} can be calculated by the formula of Heun:

$$u(x_{n+1}) = u(x_n) + \underbrace{\frac{u'(x_n) + u'(x_{n+1}))}{2}}_{\text{Trapezoid area}} .h$$

The method of 4th order Runge-Kutta corresponds to a generalization of the previous method of Heun (trapezoid). The difference used is an average slope in the range and not just the slope at the extremes.

Here it is used a slope computed by a weighted averaging function of values derived from f, in the extreme and mid-range.

$$u_{n+1} = u_n + h \frac{(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4})}{6}$$

in which

$$k_{n1} = f(x_n, u_n); k_{n2} = f(x_n + \frac{1}{2}h, u_n + \frac{1}{2}hk_{n1}); k_{n3} = f(x_n + \frac{1}{2}h, u_n + \frac{1}{2}hk_{n2}); k_{n4} = f(x_n + h, u_n + hk_{n3})$$

Implementation on the spreadsheet

On the spreadsheet, one begins by introducing in a cell the value of h in order to discretize the domain with equal length intervals. Then set the values for the parameter k and the initial position u_0 . In this case it is considered $u_0 = 2$ and $k = 4$ (Figure 17).

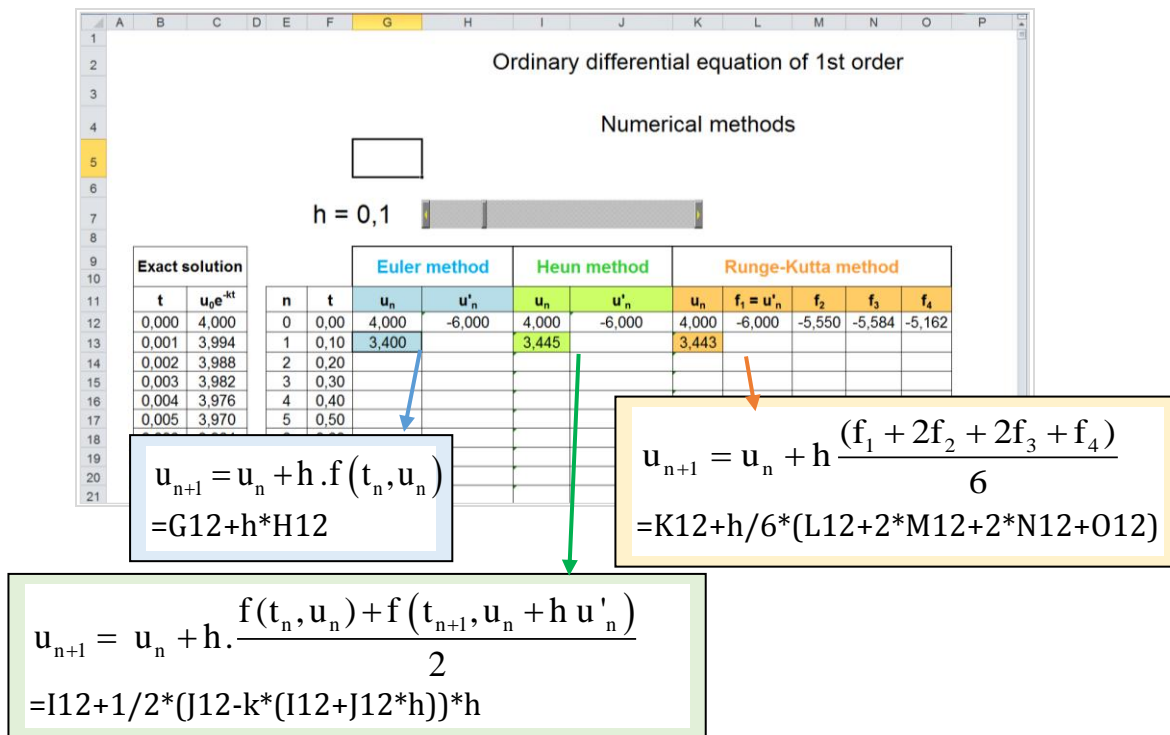


Figure 17: Implementation of the method of Euler, Heun and of the Runge-Kutta in a spreadsheet.

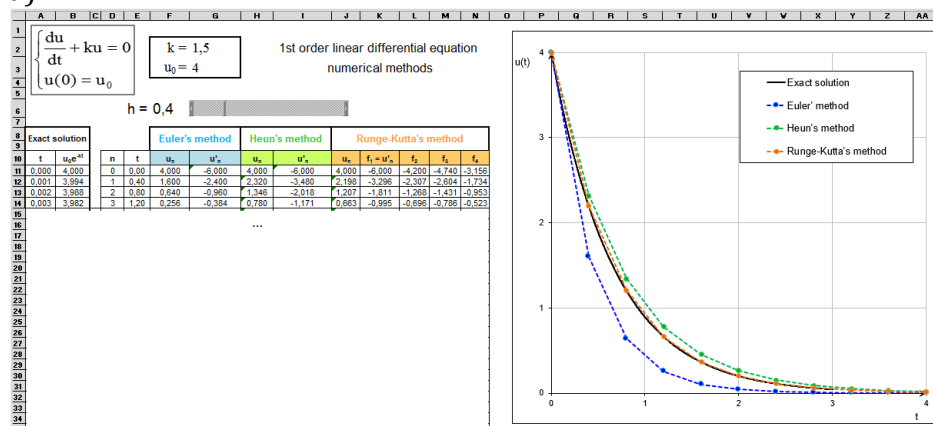
The initial value corresponds to u_0 and the value of the derivative at the initial time is the value of the derivative of the unknown function at $t = 0$.

In cells G12, I12 and K12 is introduced the value 4 and in the cells H12, J12, L12 is introduced $-kG12$, and $-kI12$, $-kK12$ respectively corresponding to the derivative of the unknown function at $t = 0$.

In cells G13, I13 e K13 is introduced the formulas from each of the three numerical methods.

After entering the formula, it is possible to use the iteration capabilities of the spreadsheet and "drag" the aforementioned formulae to obtain all the approximated values for the unknown function, in several points of the range. This iterative process is implemented almost instantaneously, providing the graphical representations of the obtained solutions and allowing the comparison of these with the exact ones [8]. It is still useful to introduce a scrollbar that allows the change of the value of h and observe approximations of the numerical solutions versus the exact solution.

a)



b)

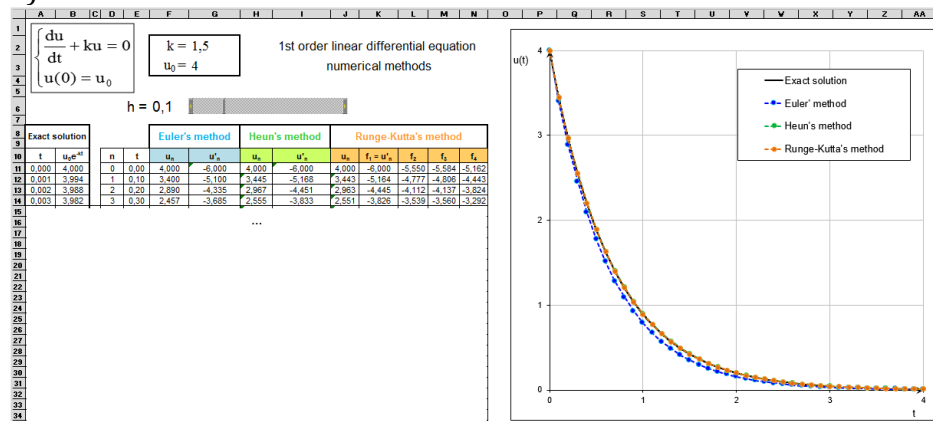


Figure 18: Using Excel for comparison of numerical methods for solving ordinary differential equations. Method of Euler, Heun and Runge-Kutta of 4th order.

Differential equation describing the motion of a classical pendulum. Programming a spreadsheet. Runge-Kutta method

The conduction of experiments is critical as a way to acquire more knowledge of the necessary formulae and also to acquire some critical sense regarding each numerical method's efficiency. The possibility of visualizing the plots with the solutions allows for the possibility to conclude that Runge-Kutta is a more effective method in terms of approximation to the original solution than the other methods Figure 18. For that reason, it shall be used to solve numerically (also using the spreadsheet) the differential equation that describes the motion of a classical pendulum.

The differential equation which describes the motion of a classical pendulum, of length L , where one end is fixed at a given point, around which the rod may rotate in a given plane, and on the other end is placed a body of mass m , is

$$mL \frac{d^2\theta}{dt^2} + c \frac{d\theta}{dt} + mg \sin\theta = 0$$

where

c represents the damping coefficient;
 m represents the mass of the suspended body;
 L represents the length of wire; and
 g represents acceleration of gravity.

The body is at point A and, when released, it oscillates to a position symmetrical to A, more specifically OF (without damping) (Figure 19). Since the displacement of the mass m occurs over a circular path of radius L , the distance travelled by the mass m on the trajectory is given by $s = \theta L$. To obtain the differential equation describing the oscillatory motion, it is enough to consider that, at each instant t , the sum of all forces applied to the mass m must be zero.

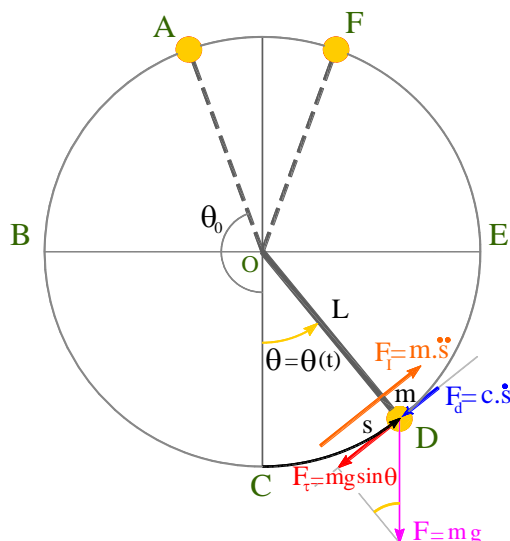


Figure 19: Equilibrium of forces in a classical pendulum.

This condition of balance of forces must occur both for the components of the forces in the radial direction and for the components of the tangential direction. For the study of the pendulum motion, it is sufficient to consider the balance of forces in the tangential direction (direction of motion). In this case it is the balance between the tangential components of the inertial force F_i , the damping force F_d and the gravitational force F_g , which is translated by the following equation

$$F_i + F_d + F_g = 0$$

where

$F_i = -m\ddot{s}$ is a fictitious force proportional to acceleration \ddot{s} and to the mass m (the negative sign indicates that the direction of inertia force is opposite to that of acceleration \ddot{s});

$F_d = -c\dot{s}$ is a force proportional to the speed \dot{s} and c is the coefficient of viscous damping;

$F_g = -mg\sin(\theta)$ is an exterior gravity force, g representing the acceleration of gravity.

The previous equation is equivalent to the following:

$$m\ddot{s} + c\dot{s} + mg\sin\theta = 0$$

that can be written in terms of the angle $\theta = \theta(t)$. In fact, given that $s = \theta L$ and, therefore, $\dot{s} = L \frac{d\theta}{dt}$ e que $\ddot{s} = L \frac{d^2\theta}{dt^2}$ comes:

$$mL \frac{d^2\theta}{dt^2} + c \frac{d\theta}{dt} + mg\sin\theta = 0$$

For problems involving second-order differential equations, the application of this method requires that the second-order equation is expressed by one of two first-order equations where the unknowns are the angular displacement θ and angular velocity $\dot{\theta}$.

In this case, by making $\dot{\theta} = \omega$ comes

$$\begin{cases} \dot{\theta} = \omega \\ mL\dot{\omega} + c\omega + mg\sin\theta = 0 \end{cases}$$

The problem of initial values is then written as follows

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = -\frac{c}{mL}\omega - \frac{g}{L}\sin\theta \\ \theta(0) = \theta_0 \quad e \quad \omega(0) = \omega_0 \end{cases}$$

Using Excel for graphical representations of the solution

Using a spreadsheet (Excel) it is possible to obtain the graph of the discrete numerical solution $\theta(t)$, which establishes the relationship between the time t , from the initial instant, and the angular position of a body of mass m .

One can also see the so-called phase plane which is simply the parametric representation of the angular position versus angular velocity at each instant of movement.

Since a particular solution of the problem is a function that depends on the parameters m (mass), c (damping), L (length), g (gravity), as well as of the starting position θ_0 and of the initial angular velocity ω_0 , we will show how to build graphics that can be modified interactively using scroll bars.

The organization of the worksheet is a fundamental process for the development of the computational module. It is the stage where the mathematical thinking on this subject is structured. For all these reasons, it is important to plan carefully the whole structure of the computing module, both in terms of the mathematics involved, and in terms of the desired visual aspect. It is important to define the starting cells that will be used for the parameters m , c , L and g and further to the initial conditions θ_0 and $\dot{\theta}_0$.

The spreadsheet is initially organized in order to enable the simultaneous display of the parameters m , c , g and L , and of the initial conditions. Next it will be introduced the value h in a cell (C13) in order to implement the numerical method (Figure 20).

The implementation of the Runge-Kutta method in the spreadsheet requires a redefinition of the problem of initial values in terms of the used notation.

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = -\frac{c}{mL}\omega - \frac{g}{L}\sin\theta \\ \theta(0) = \theta_0 \quad e \quad \omega(0) = \omega_0 \end{cases} \Leftrightarrow \begin{cases} \dot{\theta} = f(t, \omega, \theta) \\ \dot{\omega} = g(t, \omega, \theta) \\ \theta(0) = \theta_0 \quad e \quad \omega(0) = \omega_0 \end{cases}$$

The formulas of Runge-Kutta method can be rewritten to the present problem of initial values, taking into account that we are in the presence of a system of ordinary first order differential equations

$$f_1 = f(t_n, \omega_n, \theta_n) = \omega_n$$

$$g_1 = g(t_n, \omega_n, \theta_n) = -\frac{c}{mL} \omega_n - \frac{9,81}{L} \text{sen}(\theta_n)$$

$$f_2 = f\left(t_n + \frac{h}{2}, \omega_n + \frac{h}{2} g_1, \theta_n + \frac{h}{2} f_1\right) = \omega_n + \frac{h}{2} g_1$$

$$g_2 = g\left(t_n + \frac{h}{2}, \omega_n + \frac{h}{2} g_1, \theta_n + \frac{h}{2} f_1\right) = -\frac{c}{mL} \left(\omega_n + \frac{h}{2} g_1\right) - \frac{9,81}{L} \text{sen}\left(\theta_n + \frac{h}{2} f_1\right)$$

$$f_3 = f\left(t_n + \frac{h}{2}, \omega_n + \frac{h}{2} g_2, \theta_n + \frac{h}{2} f_2\right) = \omega_n + \frac{h}{2} g_2$$

$$g_3 = g\left(t_n + \frac{h}{2}, \omega_n + \frac{h}{2} g_2, \theta_n + \frac{h}{2} f_2\right) = -\frac{c}{mL} \left(\omega_n + \frac{h}{2} g_2\right) - \frac{9,81}{L} \text{sen}\left(\theta_n + \frac{h}{2} f_2\right)$$

$$f_4 = f(t_{n+1}, \omega_n + hg_3, \theta_n + hf_3) = \omega_n + hg_3$$

$$g_4 = g(t_{n+1}, \omega_n + hg_3, \theta_n + hf_3) = -\frac{c}{mL} (\omega_n + hg_3) - \frac{9,81}{L} \text{sen}(\theta_n + hf_3)$$

The spreadsheet can be arranged as shown in Figure 20. The formulas are introduced into cells and subsequently by "dragging", the remaining are filled up to a value T = 30.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1																
2																
3																
4																
5																
6																
7																
8																
9																
10																
11																
12																
13																
14																

Figure 20: Runge-Kutta method of 4th order to study the motion of a classical pendulum.

Graphical representations arise naturally by simply selecting the desired data and the graphing button. Figure 21 shows the computing environment provided by the developed application. It is possible to control the solution parameters and initial conditions. On the left side, there is a schematic representation of a pendulum and the phase plane. On the right side, there are the graphic representations of the position / time, speed / time and acceleration / time. With the button "Animation" one visualizes the pendulum motion and the appearance of the three graphs

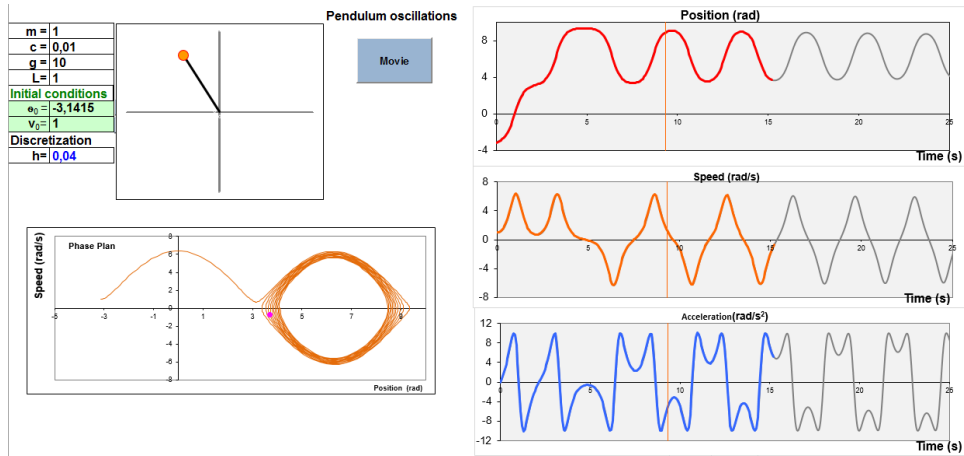


Figure 21: Interface of the pendulum application xls.

In the next figures, a motion diagram of an impactor, showing schematically the forces involved, is presented, together with, graphically (in the time domain and in the phase plane), the results obtained by applying the Runge-Kutta method four different situations:

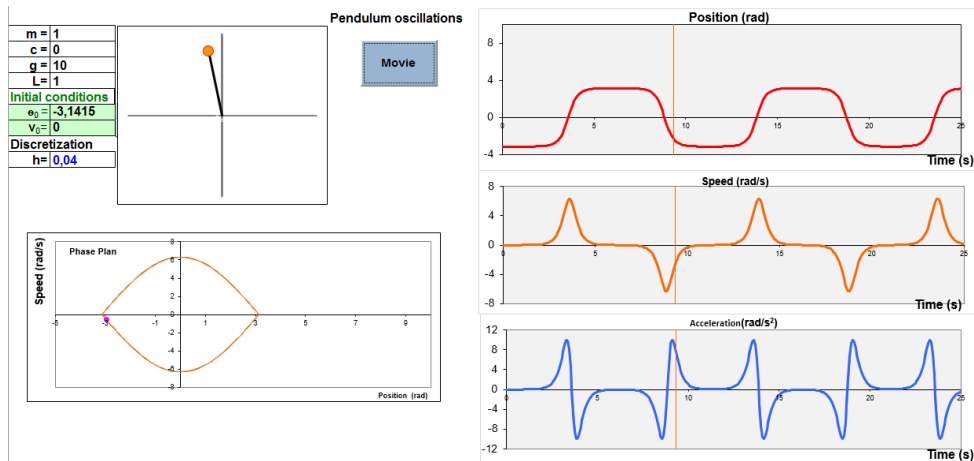


Figure 22: Damping and initial velocity both zero. The pendulum does not stop.

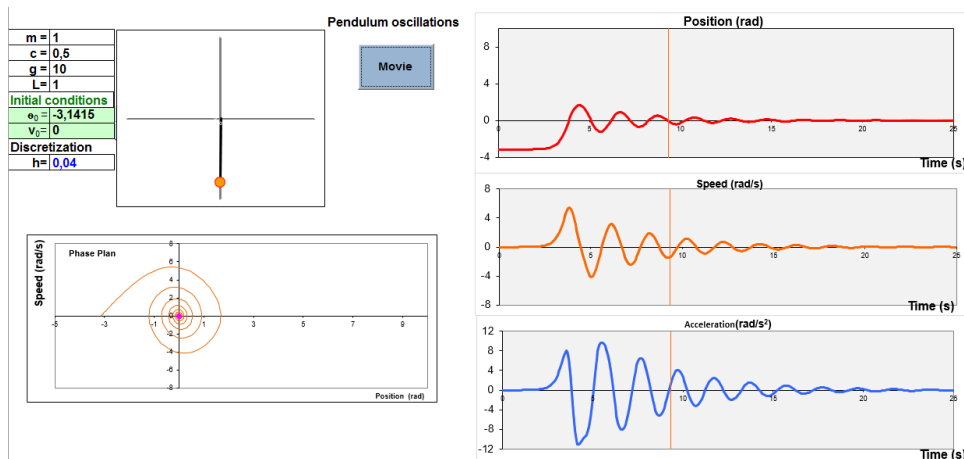


Figure 23: Damping nonzero and initial velocity null.

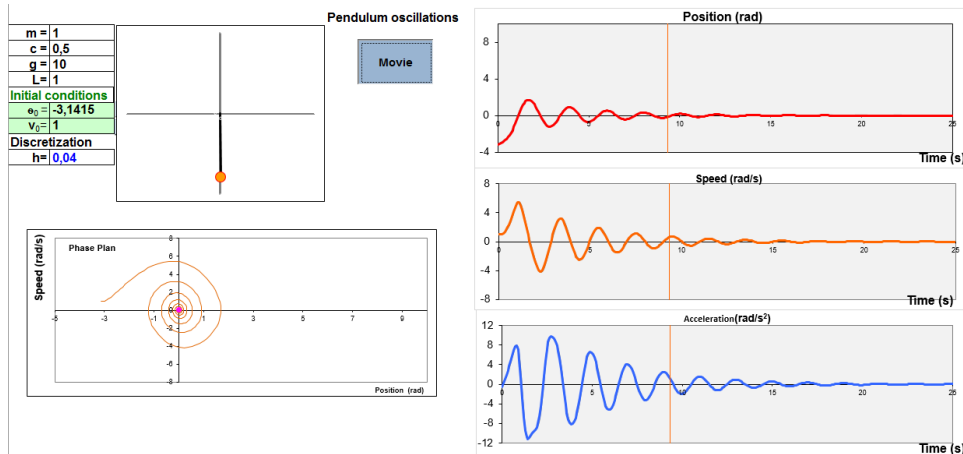


Figure 24: Damping and initial velocity both nonzero.

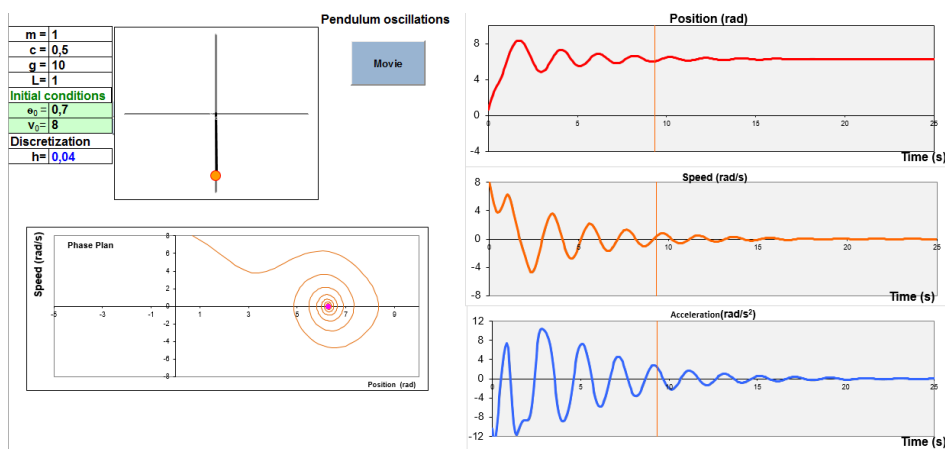


Figure 25: Damping and initial velocity both nonzero. The pendulum performs a revolution and continues in its oscillatory movement until stabilizing.

5. Conclusions

Traditionally, the dominant means of communicating mathematics are printed (static) texts and, occasionally, some (not too many) static pictures.

The software provides the use of a richer set of communication vehicles with dynamic characteristics, such as animations.

Until about ten years ago, the tools for creating documents with mathematical content were expensive and complicated to use. Only recent advances in software technology have given us tools accessible and more easily usable for the creation of such projects.

The interactive dynamic models presented in this paper can be a strong motivation for students who want to get a better understanding of the different possible dynamic behaviors of these models. The use of selectors allows a high degree of interactivity. They provide a faster, more dynamic and engaging way to demonstrate mathematical concepts than using transparencies or drawing on the board. When students construct objects in these software, they can drag points and lines with the mouse. As shapes and positions change, all mathematical relationships are preserved, allowing the examination of an entire set of similar cases in a matter of seconds. Teachers can use these programs across the mathematics curriculum, so they don't need different software for each class,

concept, or grade level they are teaching. The user-friendly interface of these software allows us to get a quickly acquaintance with its procedures so we can spend our time teaching mathematics, not software.

By using software tools which are available on most PCs, we have the important effect of showing that mathematical examples can be implemented without relying on specialized software tools. From a computational point of view, all operations we need are very simple. Therefore, the “conceptual mechanism” of these examples is accessible to those who want to learn mathematics deeply.

The user sees that interactive mathematical models can be created and used with tools accessible to everybody. This hopefully demonstrates that mathematical modelling is not first and foremost depending on very specialized tools, but upon insights, and can be done with standard tools.

Spreadsheets like in MS-Excel, allow teachers and students to easily build their own computational applications based on approaches which mainly focus on the search for the solution of mathematical problems/modelling. Nowadays, Excel is one of the most used spreadsheets. It is so popular that many of the existing open source spreadsheets are basically a copy of its main features. In these spreadsheets, we can find tools which allow the development of automated procedures (for example, animated effects) by using very simple programming commands.

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