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Keywords

Spreadsheets; sequence; explicit formula; recursive formula; summation identities; multiple representations

Cover Page Footnote

I thank SiE editors and referees for their valuable suggestions and comments on the earlier drafts of the manuscript.

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Abstract

This article offers innovative and original mathematical vignettes in the treatment of number sequences along with recursive and explicit formulas in a technology-supported multi-representational pedagogical context. Multiple representations are used to explain (i) how to help students understand summation identities of the form "Left-Hand-Side (LHS) equals Right-Hand-Side (RHS)" and (ii) how to use spreadsheet activities to lead to the better understanding of how to develop proofs for these formulas. LHS and RHS can at times be thought of as the recursive and explicit forms, of the number sequence under consideration, respectively. Such a multi-representational pedagogy comprises (but is not limited to) physical manipulatives, diagrams or drawn representations, visual proofs, algebraic representations, formal proofs, and spreadsheets. The article concludes by emphasizing the fundamental role of spreadsheets for a thorough understanding of summation formulas or identities of the form LHS = RHS and the transition from numerical evidence to formal mathematical proof.

Keywords: Spreadsheets; sequence; explicit formula; recursive formula; summation identities; multiple representations.

1. Introduction

The focus of this in-the-classroom article is certain summation formulas and figurate numbers along with selected Fibonacci number relationships that can be taught in a multi-representational pedagogy comprised of a variety of representations such as physical manipulatives, diagrams or drawn representations, visual proofs, algebraic representations, formal proofs, and spreadsheets. The most important contribution of the article is the emphasis on the fundamental role of spreadsheets for a thorough understanding of such summation formulas, figurate number relationships or identities of the form LHS = RHS; and the role of spreadsheet activities in helping students understand and develop formal proofs for these formulas. Numerous articles exploring such formulas and identities in a spreadsheet environment are published. Among those, Baker and Sugden (2013, 2015) investigated patterns arising from Fibonacci number relationships by using standard spreadsheet functionalities. Baker (2016) presented a spreadsheet-based exploration of figurate numbers (pentagonal, hexagonal, and k-gonal numbers) and their growth patterns. The article by Abramovich, Fujii and Wilson (1995) incorporated use of various graphing technology including dynamic geometry software and spreadsheets to explore interesting properties of polygonal numbers. Abramovich (2011) discussed the role of spreadsheets in elementary number theory and teacher education.

2. Recursive and Explicit Formulas

Recursive and explicit formulas arise in a variety of context, such as summation formulas and figurate numbers. In particular, power-sum identities such as $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ have been known to ancient Greeks in closed form, who derived such expressions based on unity without the presence of a known measurement unit (NCTM, 1989). In fact, the geometrical and physical representation of figurate numbers (by points drawn on sand or pebbles) and the study of their properties were common in the early Pythagorean era (Heath, 1921; NCTM, 1989). Nicomachus of Gerasa (first century B.C.) is credited for the sum of odd integers formula $\sum_{i=1}^{n} 2i - 1 = n^2$, which he obtained via dot patterns forming symmetric L–shapes. Nicomachus (first century B.C.), Aryabhata (fifth century A.C.), and Al-Karaji (tenth century A.C.) are known for deriving the integral cubes summation formula $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.

The first complete investigation and the resulting properties of figurate numbers – also called polygonal numbers – were studied by Nicomachus of Gerasa (first century B.C.), which are given in the manuscript *Introductio Arithmetica*. Though his work included few of his original ideas, it is commonly acknowledged that Introductio arithmetica stood as an artistic collection of well described, clearly presented and explained definitions and statements with a lot of illustrations based on physical forms and visual proofs (NCTM, 1989; Nicomachus of Gerasa, 1926). Nicomachus of Gerasa's work that contains today's well–known "sum = product" identities arising from the geometry of the figures generated by dots and via line segments connecting these dots (Heath, 1921; NCTM, 1989, pp. 54-56; Nicomachus of Gerasa, 1926, pp. 230-262). Table 1 outlines some of these identities some of which can be written in the recursive and the explicit formats.

Description of the Identity	**LHS = RHS Form**	**Recursive Form**	**Explicit Form**
The sum of the first n consecutive positive integers is the n^{th} triangular number $\boldsymbol{T_n}$.	$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$	$\begin{cases} T_1 = 1 \\ T_n = T_{n-1} + n \end{cases}$	$T_n = \frac{n(n+1)}{2}$
The sum of the first n consecutive positive odd integers is the n^{th} square number $\boldsymbol{S_n}$.	$\sum_{i=1}^{n} 2i - 1 = n^2$	$\begin{cases} S_1 = 1 \\ S_n = S_{n-1} + (2n-1) \end{cases}$	$S_n = n^2$
The sum of any pair of consecutive triangular numbers is a square number.	$T_{n-1} + T_n = n^2$	N/A	N/A
Eight times any triangular number plus 1 is the square of an odd number.	$8T_n + 1 = (2n+1)^2$	N/A	N/A
The sum of the first n triangular numbers is the n^{th} tetrahedral number $\boldsymbol{\theta_n}$	$\theta_n = \sum_{i=1}^{n} T_i$	$\begin{cases} \theta_1 = 1 \\ \theta_n = \theta_{n-1} + T_n \end{cases}$	$\theta_n = \frac{n(n+1)(n+2)}{6}$
The sum of the first n consecutive positive even integers is the n^{th} pronic number $\boldsymbol{P_n}$.	$\sum_{i=1}^{n} 2i = n(n+1)$	$\begin{cases} P_1 = 2 \\ P_n = P_{n-1} + 2n \end{cases}$	$P_n = n(n+1)$
The n^{th} pronic number equals twice the n^{th} triangular number.	$P_n = 2T_n$	N/A	N/A
Four times any pronic number plus 1 is the square of an odd number.	$4P_n + 1 = (2n+1)^2$	N/A	N/A
The sum of the first n consecutive positive cube numbers is the square of the n^{th} triangular number.	$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$	$\begin{cases} T_1^2 = 1 \\ T_n^2 = T_{n-1}^2 + n^3 \end{cases}$	$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$
Six times any triangular number plus 1 is a hex number.	$6T_n + 1 = H_n$ $= 1 + 3n(n+1)$	N/A	N/A

Table 1: LHS = RHS Identities in Explicit and Recursive Formats

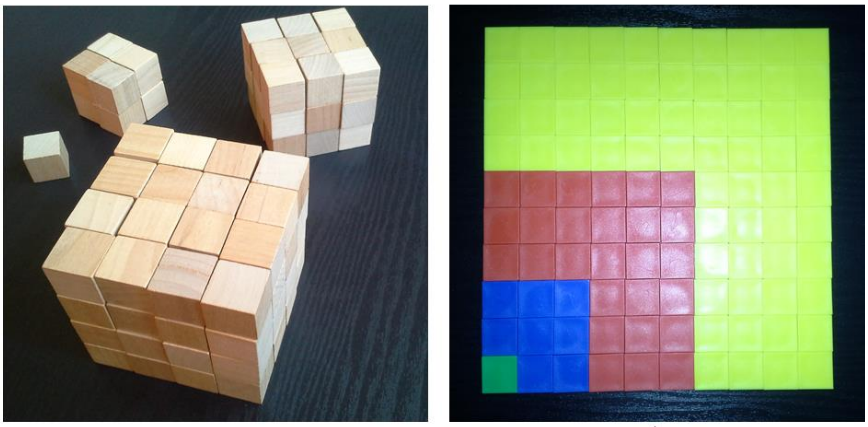
This article offers a teaching method that can be used to help students understand and distinguish between recursive and explicit formulas in a multi-representational pedagogy. Although some of the formulas listed in Table 1 cannot be categorized as either explicit or recursive within the algebraic representation, it could still be possible to model these equations as 'explicit' or 'recursive' in the language of Spreadsheets representations, as demonstrated below.

3. LHS and RHS in a multi-representational pedagogy

The notions of recursive formulas and explicit formulas are closely related to the LHS and the RHS of a given summation formula, respectively. The first identity that will be used to illustrate the multi-representational pedagogy is "The sum of the first n consecutive positive cube numbers is the square of the n^{th} triangular number," that is, $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.

3.1. Physical Manipulatives: Area as a Sum vs. Area as a Product

Actual 1-inch cubes can be used to demonstrate some of the perfect cubes of the number sequence $a_n = n^3$, as depicted in Fig.1a. Using color tiles, it is possible to visualize the sum of the first four cubes as the area of a growing rectangle. The area of the rectangle depicted in Fig.1b can be interpreted in two ways: (i) the area as a sum can be thought of as $\sum_{i=1}^{4} i^3$, which corresponds to the recursive form; (ii) the area as a product can be thought of as T_4^2, which corresponds to the explicit form of the summation identity.



a b

Figure 1: Adding perfect cubes

3.2. Algebraic Representation: Recursive vs. Explicit Forms

The LHS of the identity $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$ could be algebraically interpreted in the following recursive form: $\begin{cases} T_1^2 = 1^3 \\ T_n^2 = T_{n-1}^2 + n^3 \end{cases}$. This representation is in agreement with the growing rectangle pattern depicted in Fig.1b because $T_1^2 = 1^3, T_2^2 = 1^3 + 2^3 = T_1^2 + 2^3, T_3^2 = 1^3 + 2^3 + 3^3 = T_2^2 + 3^3, T_4^2 = T_3^2 + 4^3$. Similarly, the RHS of the identity $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$ can be written explicitly (closed form) as $1^3 = \left[\frac{1(1+1)}{2}\right]^2 =$

$1^2, 1^3 + 2^3 = \left[\frac{2(2+1)}{2}\right]^2 = 3^2, 1^3 + 2^3 + 3^3 = \left[\frac{3(3+1)}{2}\right]^2 = 6^2, 1^3 + 2^3 + 3^3 + 4^3 = \left[\frac{4(4+1)}{2}\right]^2 = 10^2$, in agreement with the area as a product of the growing rectangle in Fig.1b. An animated implementation of Fig.1b that illustrates the above discussion using the spreadsheet grid itself (with columns resized to form squares) and conditional formatting is included in Appendix A.

3.3. Spreadsheets Representation: Making Sense of Recursive & Explicit Forms

In the spreadsheet, natural numbers $n = 1,2,3,\ldots,20$ could be entered in Column A. The terms of the sequence, that is, the perfect cubes, are then entered in Column B via the syntax B1=A1^3, which is then dragged down to B20. Then, the operation in Column C is carried out to demonstrate the recursive aspect of the summation: Set C1=B1, followed by the syntax C2: =C1+B2 (my students call this "diagonal sum"), depicted in Fig.2a.

A	B	C
1	1	1
2	8	=C1 + B2
3	27	
4	64	
5	125	
6	216	
7	343	
8	512	
9	729	
10	1000	
11	1331	
12	1728	

a

A	B	C
1	1	1
2	8	9
3	27	36
4	64	100
5	125	225
6	216	441
7	343	784
8	512	1296
9	729	2025
10	1000	3025
11	1331	4356
12	1728	6084

b

A	B	C	D
1	1	1	=(A1*(A1+1)/2)^2
2	8	9	
3	27	36	
4	64	100	
5	125	225	
6	216	441	
7	343	784	
8	512	1296	
9	729	2025	
10	1000	3025	
11	1331	4356	
12	1728	6084	

c

A	B	C	D
1	1	1	1
2	8	9	9
3	27	36	36
4	64	100	100
5	125	225	225
6	216	441	441
7	343	784	784
8	512	1296	1296
9	729	2025	2025
10	1000	3025	3025
11	1331	4356	4356
12	1728	6084	6084

d

Figure 2: Visualizing sum of consecutive cubes

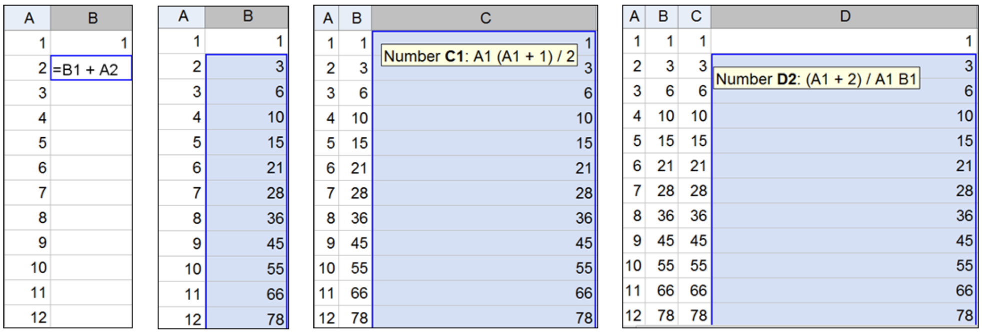
Alternatively, the same sum could be visualized by saying "add the cell to the left to the cell above, and then copy down." In fact, in actual usage, doing this with a mouse (or similar device) is a much better way to carry out the computations than typing in formulas. After dragging down to C20 as in Fig.2b, Column D is used to demonstrate the explicit form of the summation identity: Set D1=(A1*(A1+1)/2)^2 as in Fig.2c, and drag down to D20; observe that Columns C and D are equivalent (Fig.2d). This equivalence shows the LHS=RHS equivalence of the summation identity $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$ that is explored.

4. More examples

This section illustrates the LHS=RHS equivalence for some number sequences that I have used with my students in various classes in the past (Sequences and Series, History of Mathematics, Calculus, Methods of Teaching Secondary-School Mathematics, etc.). The figures of physical manipulatives and the spreadsheet screenshots are all taken from my own lecture notes and classroom videos. I remade some of the figures in the case where a screenshot taken from a certain classroom video was either incomplete or unclear.

4.1. Sum of the first *n* consecutive positive integers is the n^{th} triangular number

As usual, natural numbers $n = 1,2,3,\ldots,20$ are entered in Column A. Column B is then used to demonstrate the recursion $\begin{cases} T_1 = 1 \\ T_n = T_{n-1} + n \end{cases}$ in the following manner: Set B1: =A1, followed by the syntax B2: =B1+A2 as in Fig.3a, which is then dragged down to B20 as before (Fig.3b). The explicit (closed) form $T_n = \frac{n(n+1)}{2}$ could then be demonstrated in a similar manner as before by setting C1: =A1*(A1+1)/2 and applying it to the whole column (Fig.3c). It is important to emphasize that the values of Column C should depend on the values of Column A, hence, the correct syntax would be C1: =(A1*(A1+1))/2, as shown in Fig.3c. An alternative recursion for the triangular numbers could also given by $\begin{cases} T_1 = 1 \\ T_{n+1} = \frac{n+2}{n} T_n \end{cases}$. When entering this recursion in spreadsheets in Column D, the syntax to be used should depend on **both** the values of Column A and the values of Column B: Set D1=1, followed by the syntax D2: =((A1+2)/A1)*B1, which is then dragged down to D20 as before (Fig.3d)

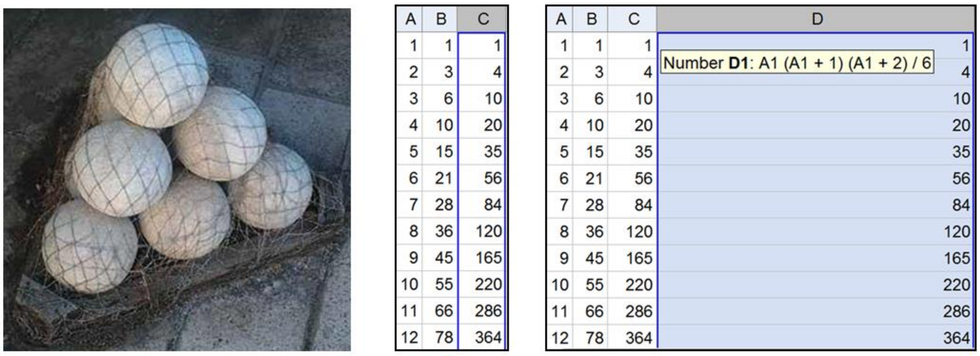


a b c d

Figure 3: Visualizing sum of consecutive integers

4.2. Sum of the first *n* triangular numbers is the n^{th} tetrahedral number

A tetrahedral number corresponds to a three-dimensional figure in the shape of a tetrahedron made of spheres. Each tetrahedral number can be expressed as the sum of consecutive triangular numbers. The n^{th} tetrahedral number θ_n is defined as the sum of the first *n* triangular numbers, whose algebraic representation is the equation $\theta_n = \sum_{i=1}^{n} T_n$. For example, the third tetrahedral number, $\theta_3 = T_1 + T_2 + T_3 = 1 + 3 + 6 = 10$ as shown in Fig.4a. One possible recursion for the tetrahedral numbers is given by $\begin{cases} \theta_1 = 1 \\ \theta_n = T_n + \theta_{n-1} \end{cases}$.



a b c

Figure 4: Visualizing tetrahedral numbers

The sequence of tetrahedral numbers in spreadsheets can be obtained recursively as follows. Upon entering the natural numbers in Column A, the triangular numbers in Column B could be entered either recursively or explicitly as demonstrated above in Section 3.1. In my case, my students used the recursive form for the triangular numbers and suggested a technique that I had not used before when it came to enter the tetrahedral numbers in Column C: (i) right-click on the top of Column B; (ii) select copy; (iii) paste it to Column C. These numbers recursively appeared on Column C as depicted in Fig.4b. To demonstrate the explicit form $\theta_n = \frac{n(n+1)(n+2)}{6}$, the syntax D1: =A1*(A1+1)*(A1+2)/6 could be used (Fig.4c).

4.3. Square pyramidal numbers

A square pyramidal number corresponds to a three-dimensional figure in the shape of a squarebased pyramid. The n^{th} square pyramidal number is defined as the sum of the first n square numbers. The arrangement shown in Fig.5 represents the fifth square pyramidal number, 55, which is the sum of the first five square numbers: 1, 4, 9, 16, 25.

a b

Figure 5: Boulets en Pierre, Musée Historique de Strasbourg, France

A	B	C
1	1	1
2	4	5
3	9	14
4	16	30
5	25	55
6	36	91
7	49	140
8	64	204
9	81	285
10	100	385
11	121	506
12	144	650
13	169	819

Number **C2**: B2 + C1

a

A	B	C	D	E	F
1	1	1	1	1	1
2	4	5	3	4	5
3	9	14	6	10	14
4	16	30	10	20	30
5	25	55	15	35	55
6	36	91	21	56	91
7	49	140	28	84	140
8	64	204	36	120	204
9	81	285	45	165	285
10	100	385	55	220	385
11	121	506	66	286	506
12	144	650	78	364	650
13	169	819	91	455	819

Number **F2**: E1 + E2

b

Figure 6: Visualizing square pyramidal numbers

Spreadsheets can be used to obtain the sequence of square pyramidal numbers {1, 5, 14, 30, 55, 91, 140, 204, 285, 385, …}. Moreover, it can be shown that each square pyramidal number P_n can be written as the sum of two consecutive tetrahedral numbers. For this purpose, Column A and Column B are used to enter natural numbers and square numbers, respectively. In Column C, set C1=B1 and C2: =B2+C1, which is dragged down to the bottom cell. Once again, as was the case before, this sum could be obtained by adding the cell to the left to the cell above, and then copying down; which is more efficiently executed with a mouse instead of typing these formulas. This way of obtaining the square pyramidal numbers is based on the sum of square numbers, as depicted in Fig.6a. The other way of obtaining the square pyramidal numbers is achieved via the addition of tetrahedral numbers (which can be obtained via triangular numbers as illustrated in Section 3.2 above). For this purpose, Column D and Column E are used to enter triangular numbers and tetrahedral numbers, respectively. In Column F, we use the syntaxes F1=1 and F2: =E1+E2, which is dragged down to the bottom cell as usual (Fig.6b). Remark that F1=1 and starting with F2, we added the consecutive tetrahedral numbers in pairs. We observe that Columns C and F are equivalent, as expected.

4.4. Identities involving triangular numbers

To demonstrate $8T_n + 1 = (2n+1)^2$ on spreadsheets, my students first defined the natural numbers and the triangular numbers in Columns A and B, respectively, as before. Squares of odd numbers are entered via the syntax C1: =(2*A1+1)^2, as in Fig.7a. Column D is then used to demonstrate $8T_n + 1$: Set D1: =8*B1+1 then drag down to D20; once again we observe the equivalence of Columns C and D (Fig.7b).

	A	B	C
1	1	1	9
2	2	3	25
3	3	6	49
4	4	10	81
5	5	15	121
6	6	21	169
7	7	28	225
8	8	36	289
9	9	45	361
10	10	55	441
11	11	66	529
12	12	78	625

Number **C1**: $(2A1 + 1)^2$

a

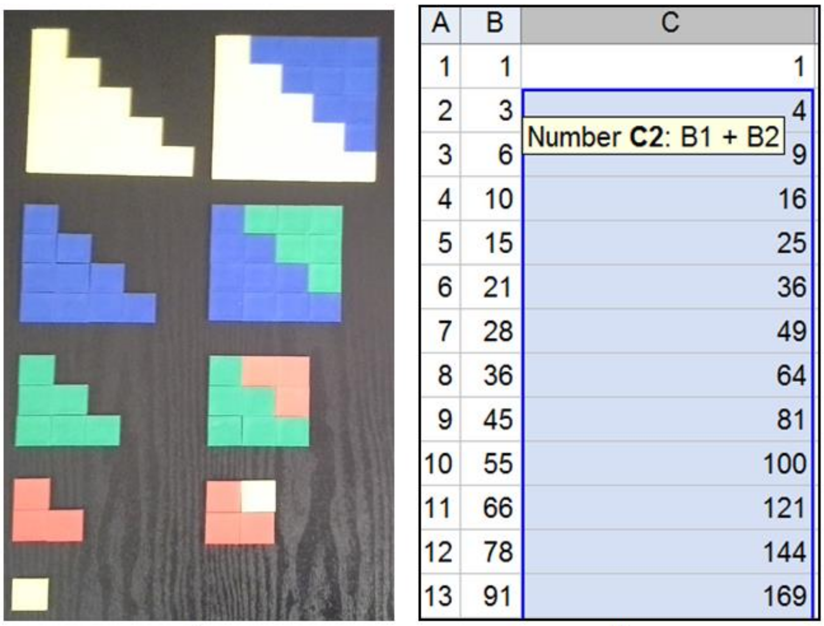
	A	B	C	D
1	1	1	9	9
2	2	3	25	25
3	3	6	49	49
4	4	10	81	81
5	5	15	121	121
6	6	21	169	169
7	7	28	225	225
8	8	36	289	289
9	9	45	361	361
10	10	55	441	441
11	11	66	529	529
12	12	78	625	625

Number **D1**: 8B1 + 1

b

Figure 7: Visualizing eight times a triangular number

The identity $T_{n-1} + T_n = n^2$ can be demonstrated both physically (using square color tiles) and via spreadsheets. Triangular numbers $T_1 = 1, T_2 = T_1 + 2, T_3 = T_2 + 3, T_4 = T_3 + 4, T_5 = T_4 + 5, \ldots$ can be arranged using different colors; adding them consecutively in pairs, we obtain a sequence of square numbers, as depicted in Fig.8a. To demonstrate the identity $T_{n-1} + T_n = n^2$ in spreadsheets, we proceed by entering the natural numbers and the triangular numbers in Column A and Column B, respectively. Column C is then used to visualize the sum of the consecutive pairs of triangular numbers via the syntaxes C1=1 and C2: =B1+B2, which is then dragged down to the bottom cell (Fig.8b). As also demonstrated for Fig.1b, an animated implementation of Fig.8a that illustrates the identity $T_{n-1} + T_n = n^2$ using the spreadsheet grid and conditional formatting is included in Appendix B.



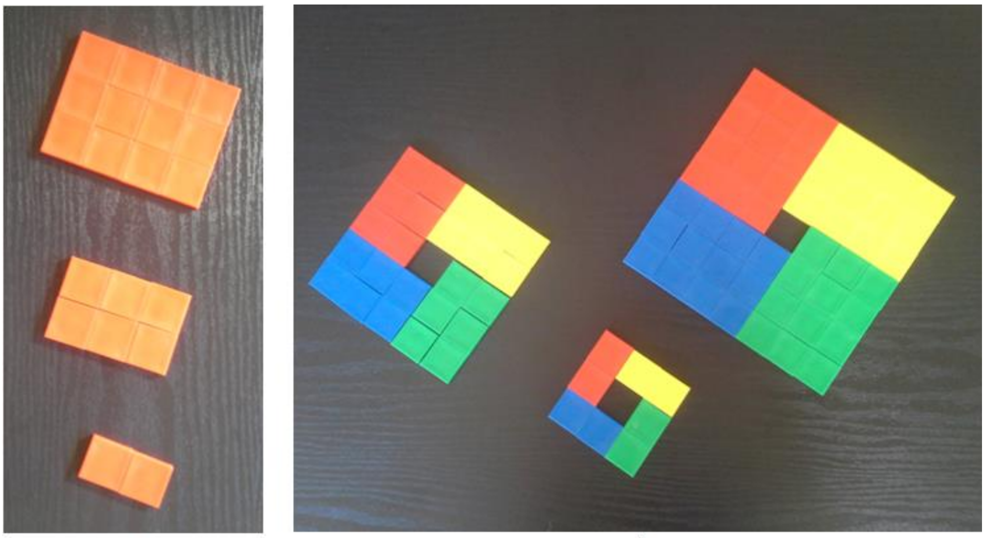
A	B	C
1	1	1
2	3	4
3	6	9
4	10	16
5	15	25
6	21	36
7	28	49
8	36	64
9	45	81
10	55	100
11	66	121
12	78	144
13	91	169

a b

Figure 8: Visualizing sums of consecutive triangular numbers

4.5. Identities involving pronic numbers

Pronic numbers P_n are the numbers of the form $n(n + 1)$ such as 2, 6, 12, 20, 30, etc. Nicomachus used the adjective "heteromecic" for these numbers (NCTM, 1989, p. 56). Pronic numbers can be represented as rectangles with integer dimensions differing by 1 (Fig.9a). For $n = 1,2,3$, the identity $4P_n + 1 = (2n + 1)^2$ can be represented with the arrangement of four congruent rectangles each made of a different color, as in Fig.9b. The areas of these rectangles could be interpreted in two ways: (i) the area as a sum can be thought of as $4P_n + 1$ respectively for $n = 1,2,3$; (ii) the area as a product can be thought of $(2n + 1)^2$ respectively for $n = 1,2,3$.



a b

Figure 9: Pronic numbers

To demonstrate the identity $4P_n + 1 = (2n+1)^2$ on spreadsheets, my students first entered the natural numbers in Column A as usual. Pronic numbers are then introduced in Column B via the syntax B1: =A1*(A1+1). Column C is then used to define the LHS of the identity via the syntax C1: =4*B1+1. Finally, squares of odd numbers are entered via the syntax D1: =(2*A1+1)^2, as before. The equivalence of Columns C and D demonstrates the identity $4P_n + 1 = (2n+1)^2$ (Fig.10).

A	B
1	=A1*(A1 + 1)
2	6
3	12
4	20
5	30
6	42
7	56
8	72
9	90
10	110
11	132
12	156
13	182

a

A	B	C
1	2	=4*B1+1
2	6	25
3	12	49
4	20	81
5	30	121
6	42	169
7	56	225
8	72	289
9	90	361
10	110	441
11	132	529
12	156	625
13	182	729

b

A	B	C	D
1	2	9	=(2*A1+1)^2
2	6	25	25
3	12	49	49
4	20	81	81
5	30	121	121
6	42	169	169
7	56	225	225
8	72	289	289
9	90	361	361
10	110	441	441
11	132	529	529
12	156	625	625
13	182	729	729

c

Figure 10: Visualizing pronic numbers

4.6. Hex numbers

Hex numbers $H_n = 1 + \sum_{i=0}^{n-1} 6i$ can be derived and also modeled in many ways: (i) One way of modeling hex numbers $H_1 = 1, H_2 = H_1 + 6, H_3 = H_2 + 12, H_4 = H_3 + 18, \ldots$ could be achieved by using dots placed on concentric hexagonal lattices starting from center outward (Fig.11); (ii) Another way of visualizing these numbers could be realized by relating them to triangular numbers via $H_1 = 1, H_2 = 1 + 6T_1, H_3 = 1 + 6T_2, H_4 = 1 + 6T_3, \ldots$ as shown in Fig.12.

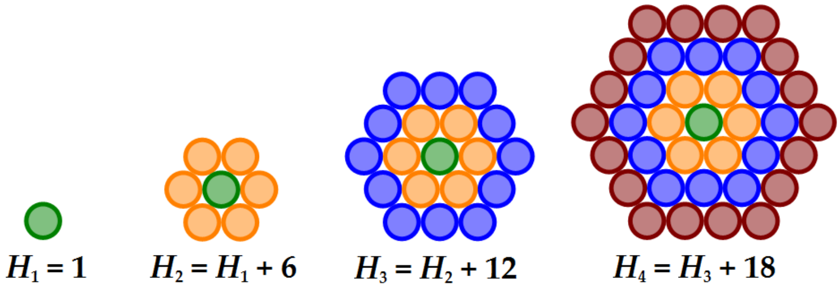


Figure 11: Modeling hex numbers

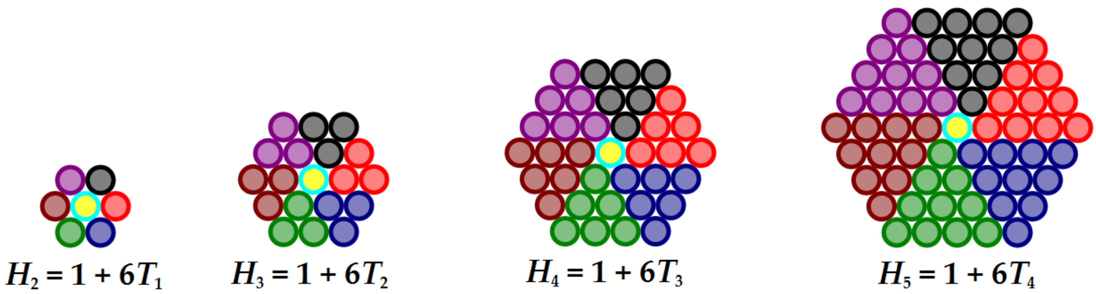


Figure 12: Modeling hex numbers (cont.)

The sequence of hex numbers along with some of their interesting properties in spreadsheets can be obtained recursively as follows. Natural numbers in Column A as usual; the syntax B1=1 followed by B2: =B1+6*A1 are entered in Column B. This is the first way of obtaining hex numbers introduced above (Fig.13a). The second way could be achieved in a similar manner by first entering trianular numbers in Column C either recursively or explicitly as demonstrated before (Fig.13b). The hex numbers are then entered in Column D via the syntax D1=1 followed by D2: =1+6*C1 (Fig.13c). Finally, the syntax E1=1 followed by E2: =E1+D2 demonstrates the interesting property that the sum of the first n hex numbers is a perfect cube (Fig.13d).

a (Number **B2**: B1 + 6A1)

A	B
1	1
2	7
3	19
4	37
5	61
6	91
7	127
8	169
9	217
10	271
11	331
12	397

b

A	B	C
1	1	=A1*(A1+1)/2
2	7	3
3	19	6
4	37	10
5	61	15
6	91	21
7	127	28
8	169	36
9	217	45
10	271	55
11	331	66
12	397	78

c

A	B	C	D
1	1	1	1
2	7	3	=1+6*C1
3	19	6	19
4	37	10	37
5	61	15	61
6	91	21	91
7	127	28	127
8	169	36	169
9	217	45	217
10	271	55	271
11	331	66	331
12	397	78	397

d

A	B	C	D	E
1	1	1	1	1
2	7	3	7	=E1+D2
3	19	6	19	27
4	37	10	37	64
5	61	15	61	125
6	91	21	91	216
7	127	28	127	343
8	169	36	169	512
9	217	45	217	729
10	271	55	271	1000
11	331	66	331	1331
12	397	78	397	1728

Figure 13: Hex numbers in two different ways

4.7. Fibonacci numbers

For each natural number n, the n^{th} Fibonacci number f_n can be defined recursively by $f_1 = 1, f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for $n \geq 1$. Fibonacci numbers can also be defined explicitly via the equation $f_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}$ for $n \geq 1$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the well-known golden ratio. Via either approach, it can be shown that $f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8$ and so on. To obtain the Fibonacci sequence recursively, set B1=1, B2=1 and B3: =B1+B2 (Fig.14a). To obtain the Fibonacci sequence explicitly, enter the syntax C1: =(((1 + sqrt(5))/2)^A1 – ((1 – sqrt(5))/2)^A1) /sqrt(5) and drag down to the bottommost cell (Fig.14b).

a (Number **B3**: B1 + B2)

A	B
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
9	34
10	55
11	89
12	144

b (Number **C1**: (((1 + sqrt(5)) / 2)^A1 - ((1 - sqrt(5)) / 2)^A1) / sqrt(5))

A	B	C
1	1	1
2	1	1
3	2	2
4	3	3
5	5	5
6	8	8
7	13	13
8	21	21
9	34	34
10	55	55
11	89	89
12	144	144

Figure 14: Fibonacci numbers in two different ways

Spreadsheets could be used to demonstrate interesting properties of Fibonacci sequence. After defining the terms of the sequence in Column A, for instance, to demonstrate the famous sum of squares identity $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$, Column B and Column C are used to visualize the LHS and the RHS, respectively. In Column B, we set B1: =A1^2 and B2: =B1+A2^2 (Fig.15a). In Column C, we set C1: =A1*A2 and drag it down (Fig.15b). The famous Simson's identity $f_{n-1}f_{n+1} - f_n = (-1)^n$ can be demonstrated in a similar manner: Set D2: =A1*A3-A2^2 and drag down (Fig.15c).

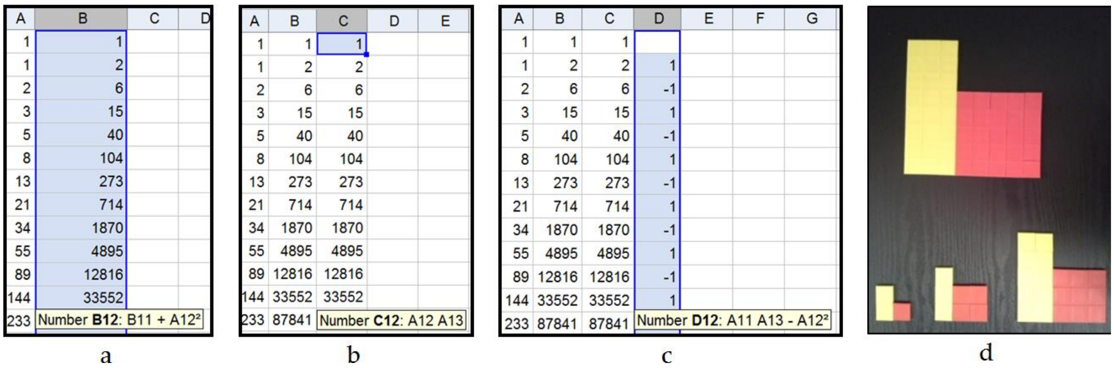


Figure 15: Fibonacci sequence identities

Simson's identity $f_{n-1}f_{n+1} - f_n = (-1)^n$ can also be demonstrated physically using color tiles. Because a subtraction is involved, only two colors are suggested in such a way that one color represents positives and the other color represents negatives. Fig.15d demonstrates Simson's identity for $n = 2,3,4,5$. Specifically we have the physical representation of four Simson identites, which are in agreement with the rectangular arrangments demonstrated in Fig.15d:

- $n = 2$: $f_1 f_3 = f_2^2 + 1$ (that is, $1{\times}2 = 1^2 + 1$);
- $n = 3$: $f_2 f_4 = f_3^2 - 1$ (that is, $1{\times}3 = 2^2 - 1$);
- $n = 4$: $f_3 f_5 = f_4^2 + 1$ (that is, $2{\times}5 = 3^2 + 1$);
- $n = 5$: $f_4 f_6 = f_5^2 - 1$ (that is, $3{\times}8 = 5^2 - 1$);
- For even $n \geq 2$: $f_{n-1}f_{n+1} = f_n^2 + 1$ (i.e., rectangle exceeds square by 1 unit);
- For odd $n \geq 3$: $f_{n-1}f_{n+1} = f_n^2 - 1$ (i.e., square exceeds rectangle by 1 unit).

4.8. A Recurrence relation of factorials

The next example is the famous recurrence relation of factorials $\sum_{k=1}^{N} k \cdot k! = (N+1)N! - 1$, which can also be demonstrated physically with Cuisenaire rods or color tiles for the first couple of terms as follows (Fig.16-17):

- $N = 1: 1 \cdot 1! = 2 \cdot 1! - 1$;
- $N = 2: 1 \cdot 1! + 2 \cdot 2! = 3 \cdot 2! - 1$;
- $N = 3: 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! = 4 \cdot 3! - 1$;
- $N = 4: 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! = 5 \cdot 4! - 1$;
- Thus $1 \cdot 1! + 2 \cdot 2! + \cdots + (N-1) \cdot (N-1)! + N \cdot N! = (N+1) \cdot N! - 1$.

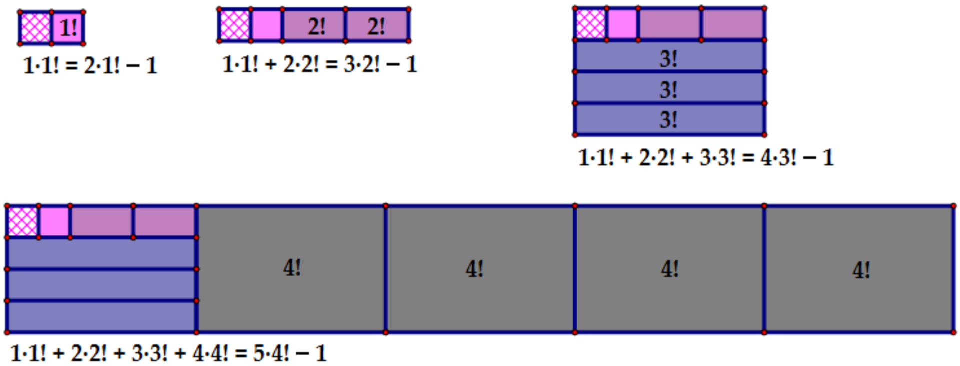


Figure 16: A recurrence relation of factorials

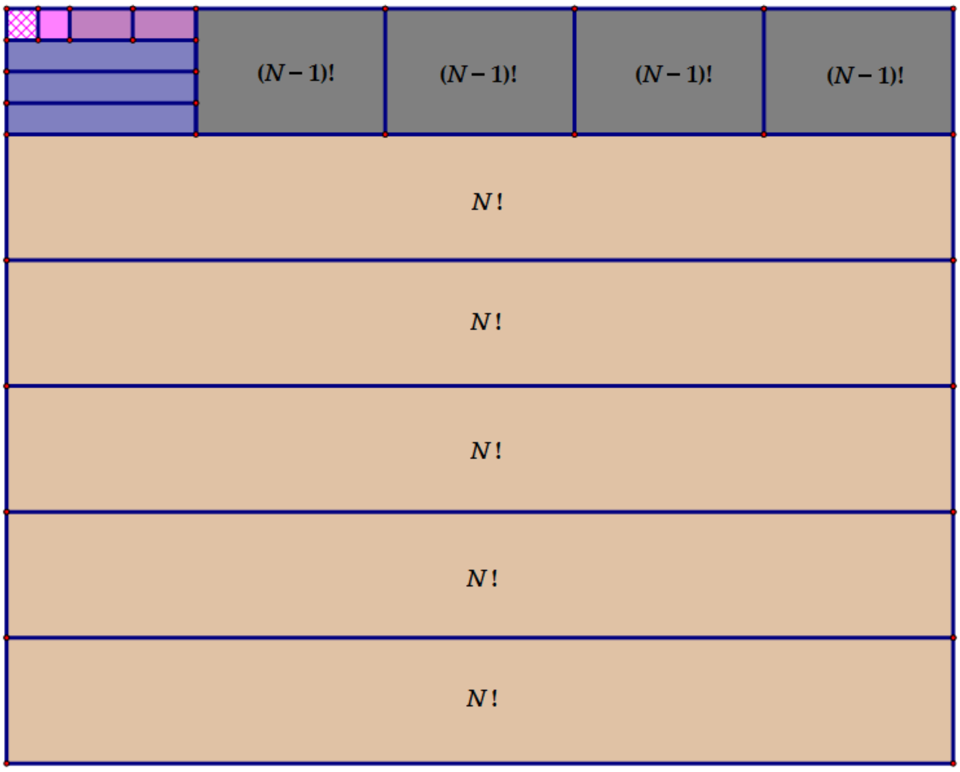


Figure 17: A recurrence relation of factorials (cont.)

The spreadsheets demonstration of this relation is based on the factorial notation that is applied as follows: Upon entering the natural numbers in Column A as before, in Column B, type B1: =A1*A1! and drag it down to bottom cell (Fig.18a). Column C is then used to demonstrate the summation on the LHS of the identity using the previously described "add them diagonally" technique: C1=B1 and C2: =C1+B2, which is then dragged down to the bottom cell (Fig.18b). Finally, Column D is used to demonstrate the RHS of the identity: D1: =(A1+1)*A1!–1 which is then applied to all Column D (Fig.18c). The equivalence of Column C and Column D concludes the demonstration of the LHS=RHS identity.

Number **B1**: A1 A1!

A	B
1	1
2	4
3	18
4	96
5	600
6	4320
7	35280
8	322560
9	3265920
10	36288000
11	439084800
12	5748019200
13	80951270400

a

Number **C2**: C1 + B2

A	B	C
1		1
2	4	5
3	18	23
4	96	119
5	600	719
6	4320	5039
7	35280	40319
8	322560	362879
9	3265920	3628799
10	36288000	39916799
11	439084800	479001599
12	5748019200	6227020799
13	80951270400	87178291199

b

Number **D1**: (A1 + 1) A1! - 1

C	D
1	1
5	5
23	23
119	119
719	719
5039	5039
40319	40319
362879	362879
3628799	3628799
39916799	39916799
479001599	479001599
6227020799	6227020799
87178291199	87178291199

c

Figure 18: A recurrence relation of factorials on spreadsheets

4.9. A Recurrence relation of consecutive powers of 2

Another interesting formula that can be both recursively and explicitly modeled is the recurrence relation of consecutive powers of 2, which is given by $\sum_{k=1}^{N}(k+1)\cdot 2^k = N\cdot 2^{N+1}$. This formula can also be visualized physically with square color tiles or Cuisenaire rods, as depicted in Fig.19:

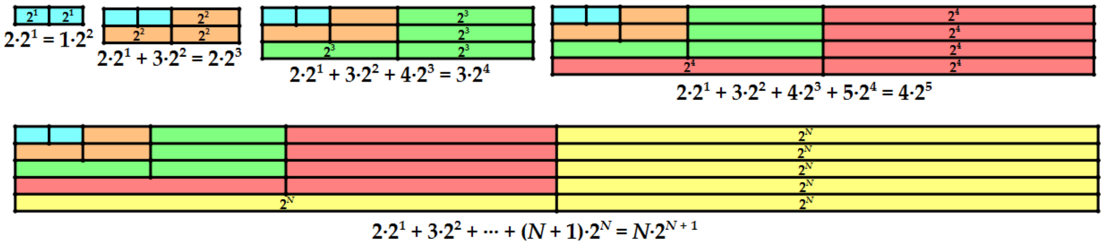


Figure 19: A recurrence relation of consecutive powers of 2

The spreadsheets demonstration starts with the introduction of the natural numbers in Column A, as usual. The next step is to enter the LHS of the identity, that is, the recursive form: Set B1: =(A1+1)*2^A1; C1=B1; C2: =B2+C1 (diagonal addition technique) and drag them all down to the bottom cell (Fig.20a). To demonstrate the RHS of the identity: Set D1: =A1*2^(A1+1) and drag it down to the bottom cell (the explicit form), which is depicted in Fig.20b.

Number **C2**: B2 + C1

A	B	C
1	4	4
2	12	16
3	32	48
4	80	128
5	192	320
6	448	768
7	1024	1792
8	2304	4096
9	5120	9216
10	11264	20480
11	24576	45056
12	53248	98304
13	114688	212992

a

Number **D1**: A1 (2^(A1 + 1))

A	B	C	D
1	4	4	4
2	12	16	16
3	32	48	48
4	80	128	128
5	192	320	320
6	448	768	768
7	1024	1792	1792
8	2304	4096	4096
9	5120	9216	9216
10	11264	20480	20480
11	24576	45056	45056
12	53248	98304	98304
13	114688	212992	212992

b

Figure 20: A recurrence relation of consecutive powers of 2 on spreadsheets

4.10. Sums of consecutive powers of 3

The sum of consecutive powers of 3 formula $\sum_{k=0}^{N} 3^k = \frac{3^{N+1}-1}{2}$ can be modeled both recursively and explicitly as demonstrated below. I start with the color tiles representation demonstrated in Fig.21.

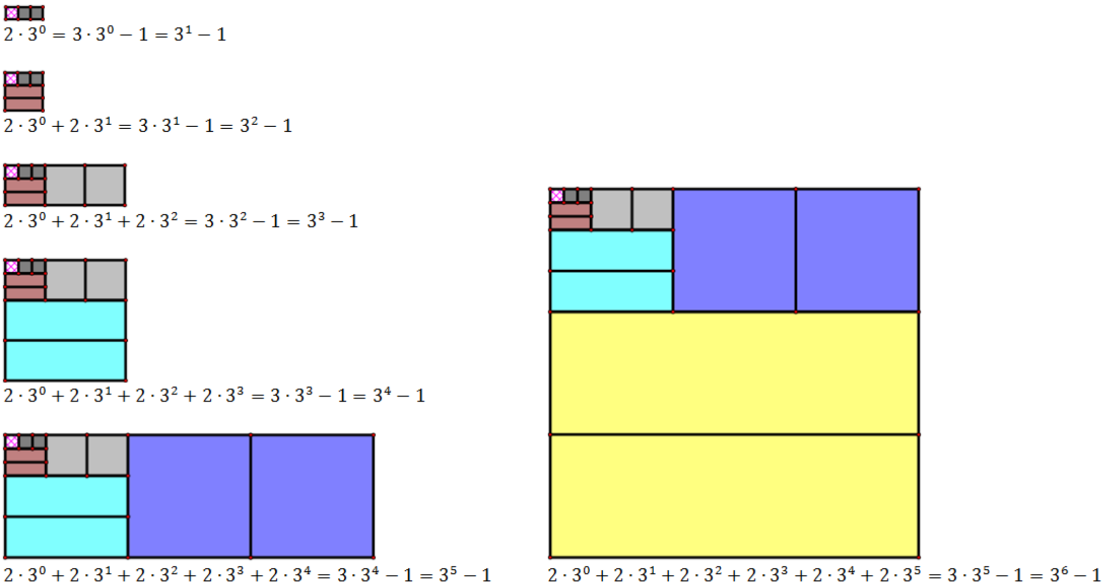


Figure 21: A recurrence relation of consecutive powers of 3

In spreadsheets, the LHS of the identity (the recursive form) is visualized as follows: Set B1: =3^A1; C1=B1; C2: =B2+C1 (diagonal addition technique) and drag them all down to the bottom cell (Fig.22a). To demonstrate the RHS of the identity, that is, the explicit form: Set D1: =(3^(A1+1)–1)/2 and drag it down to the bottom cell (Fig.22b).

Number C2: B2 + C1

A	B	C
0	1	1
1	3	4
2	9	13
3	27	40
4	81	121
5	243	364
6	729	1093
7	2187	3280
8	6561	9841
9	19683	29524
10	59049	88573
11	177147	265720
12	531441	797161

a

Number D1: (3^(A1 + 1) - 1) / 2

A	B	C	D
0	1	1	1
1	3	4	4
2	9	13	13
3	27	40	40
4	81	121	121
5	243	364	364
6	729	1093	1093
7	2187	3280	3280
8	6561	9841	9841
9	19683	29524	29524
10	59049	88573	88573
11	177147	265720	265720
12	531441	797161	797161

b

Figure 22: A recurrence relation of consecutive powers of 3 on spreadsheets

5. Transition from Numerical Evidence to Formal Mathematical Proof

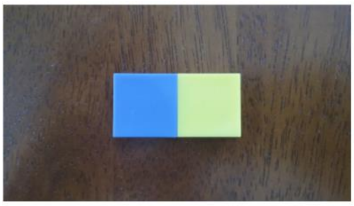
Although terminologically classified as recursive and explicit, there are those "sum=product" identities that are recursive in their nature and those explicit in their nature, yet some of these formulas still cannot be strictly classified as either implicit or recursive in the mathematical sense. Spreadsheets proved necessary in not only visualizing the LHS and RHS of "sum=product" identities in a coordinative manner, but in understanding and distinguishing between the recursive forms and explicit forms of such LHS=RHS identities as well. In coordination with physical manipulatives, spreadsheets could also prove very useful in helping students transition from numerical evidence to formal mathematical proof in many ways. This concluding section aims to highlight these many ways via which students of elementary number theory can be helped to achieve that transition.

5.1. Visualizing the Steps of Mathematical Induction

Most of the examples used in this article can be proved via either direct proof or mathematical induction. En route to a thorough understanding of mathematical induction proof, in particular, the primary role of spreadsheets and physical manipulatives in coordination could facilitate the steps of mathematical induction, including the base case and the inductive step. Though trivial, it is fundamental to start with the demonstration of the base case in all three representations (physical, spreadsheets, and algebraic). To explain this, I choose one of the identities introduced above: the recurrence relation of factorials $1+\sum_{k=1}^{N} k\cdot k! = (N+1)N!$.

5.1.1. Step 1: $\boldsymbol{N=1 \Rightarrow 1+1\cdot 1!=2\cdot 1!}$

In this step, the LHS of the identity consists of the constant 1 plus one 1!, that is, one piece of a blue square tile (or a Cuisenaire rod of value 1) modeling the constant 1 plus one more yellow piece modeling the addend one 1!. The LHS of the identity, in a sense, is viewed as the "area as a sum" of a growing rectangle. The RHS of the same identity at this elementary stage can be viewed as "area as a repeated addition" as two 1! (Fig.23a). It is important, at this elementary stage, to note the slight difference between the two interpretations of the area of the growing rectangle. At the same time, on spreadsheets, we first enter the natural numbers in Column A as usual. We then type B1: =A1*A1! which models the number of factorials to be added at each step followed by C1: =1+B1 which models the "area as a sum" of the growing rectangle (that is, the LHS of the identity). Finally the syntax D1: =(A1+1)*A1! is entered, which models the "area as a repeated addition;" that is, the RHS of the first step of the identity $1+1\cdot 1! = 2\cdot 1!$ (Fig.23b).



a

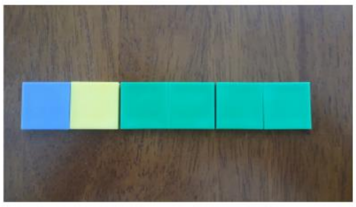
A	B	C	D
1	1	2	2
2	Number **B1**: A1 A1!	Number **C1**: 1 + B1	Number **D1**: (A1 + 1) A1!
3			
4			
5			

b

Figure 23: Identity $1+\sum_{k=1}^{N} k\cdot k! = (N+1)N!$ Step 1: $\boldsymbol{N=1} \Rightarrow 1+\boldsymbol{1}\cdot\boldsymbol{1}! = \boldsymbol{2}\cdot\boldsymbol{1}!$

5.1.2. Step 2: $\boldsymbol{N=2 \Rightarrow 1+1\cdot 1!+2\cdot 2!=3\cdot 2!}$

In the second step, we keep what is already there from the previous step and just add the new addend, that is, two sets of two green square tiles modeling the two 2!. The LHS of the identity is once again checked and viewed as the "area as a sum" of a growing rectangle sequence. The RHS of the identity at this second step is interpreted as "area as a repeated addition" as three sets of 2! (Fig.24a). As before, B2 can be obtained via spreadsheet functionality by dragging cell B1 down. The "diagonal adding" approach is to be applied to Column C as this is the column in which we add all terms in agreement with the LHS of the identity. For that purpose, we type C2: =C1+B2, which models the "area as a sum" of the growing rectangle at the second step. D2 too can be obtained by dragging cell D1 down as Column D can be thought of as the explicit form the identity which models the RHS of the identity (Fig.24b). Columns C and D could be thought of as reflecting the implicit (recursive) and the explicit nature of the identity, respectively.



a

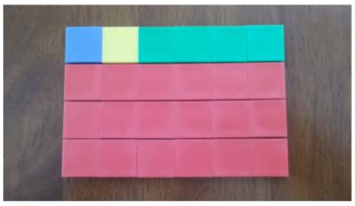
A	B	C	D
1	1	2	2
2	4	6	6
3	Number **B2**: A2 A2!	Number **C2**: C1 + B2	Number **D2**: (A2 + 1) A2!
4			
5			

b

Figure 24: Identity $1+\sum_{k=1}^{N} k\cdot k! = (N+1)N!$ Step 2: $\boldsymbol{N=2} \Longrightarrow 1+1\cdot 1!+2\cdot 2! = 3\cdot 2!$

5.1.3. Step 3: $N = 3 \Longrightarrow 1 + 1\cdot 1! + 2\cdot 2! + 3\cdot 3! = 4\cdot 3!$

In the third step, the new addend will be three sets of what is already present, namely three sets of 6! in red color (Fig.25a). The LHS is then recursively interpreted as an area as a sum of $1+1\cdot 1!+2\cdot 2!+3\cdot 3!$ whereas the RHS is explicitly interpreted as an area as a repeated addition as $4\cdot 3!$ because there are four sets of 3! altogether (Fig.25a). Spreadsheet approach in the third step is easier than before because at this stage all three cells B2, C2, and D2 are draggable (Fig.25b). Once again the term being added in B3 is color-coded to emphasize the connection between the two representations.



a

A	B	C	D
1	1	2	2
2	4	6	6
3	18	24	24
4	Number **B3**: A3 A3!	Number **C3**: C2 + B3	Number **D3**: (A3 + 1) A3!
5			

b

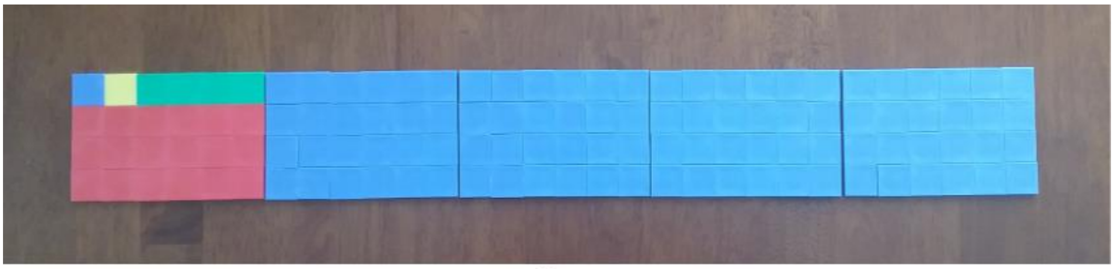
Figure 25: Identity $1+\sum_{k=1}^{N} k\cdot k! = (N+1)N!$ Step 3: $N=3 \Longrightarrow 1+1\cdot 1!+2\cdot 2!+3\cdot 3! = 4\cdot 3!$

5.1.4. Step 4: $N = 4 \Longrightarrow 1 + 1\cdot 1! + 2\cdot 2! + 3\cdot 3! + 4\cdot 4! = 5\cdot 4!$

The fourth stage is demonstrated in a similar manner, keeping in mind the setting of the stage for transition from numerical evidence to formal mathematical proof (Fig.26a-b). It is important to emphasize the respective roles of the LHS and the RHS of the identity as the area of the growing rectangle sequence (Fig.26a) and how these roles manifest themselves in spreadsheets in coordination (Fig.26b).

5.2. Formal Mathematical Proof

Now that the two representations (physical manipulatives and spreadsheets) are explored in a reconciliatory manner in coordination at each step, it remains to show how these two representations can be used to help students transition to the actual formal mathematical proof (algebraic representation) via mathematical induction. Given that the base case was already verified, I move to the inductive step, that is, to show that the implication $P(N = n) \Longrightarrow P(N = n + 1)$ is true.



a

A	B	C	D
1	1	2	2
2	4	6	6
3	18	24	24
4	96	120	120
5	Number **B4**: A4 A4!	Number **C4**: C3 + B4	Number **D4**: (A4 + 1) A4!

b

Figure 26: Identity $1 + \sum_{k=1}^{N} k \cdot k! = (N + 1)N!$ Step 4: $N = 4 \Longrightarrow 1 + 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! = 5 \cdot 4!$

$P(N = n)$ refers to the identity $1 + \sum_{k=1}^{n} k \cdot k! = (n + 1)n!$. To move to the next stage, that is, the $N = n + 1$ stage, by experience with the discussion in Section 5.1 along with Figures 24-27, we add the "new" addend $(n + 1)(n + 1)!$ to both sides of the identity $1 + \sum_{k=1}^{n} k \cdot k! = (n + 1)n!$. The new LHS becomes $1 + \sum_{k=1}^{n} k \cdot k! + (n + 1)(n + 1)!$ whereas the new RHS becomes $(n + 1)n! + (n + 1)(n + 1)!$. By experience once again, the LHS is viewed as a certain number of addends which corresponds to the "area as a sum" of the growing rectangle sequence. The new LHS, on the other hand, can be rewritten by using properties of factorials as: $(n + 1)n! + (n + 1)(n + 1)! = (n + 1)n! + (n + 1)(n + 1)n!$. Finally, factorization, simplifying, and the commutative property of multiplication yield the LHS in the desired format: $(n + 1)n! + (n + 1)(n + 1)n! = (n + 1)n!\,[1 + (n + 1)] = (n + 1)n!\,[(n + 1) + 1] = (n + 1)!\,[(n + 1) + 1] = [(n + 1) + 1](n + 1)!$.

6. Conclusion

The activities presented in this article are a compilation of lecture notes that I used in the past years in the teaching of sequences and series along with recursive and explicit formulas in various classes such as Sequences and Series, Mathematics for Elementary School Teachers, Elementary Number Theory, etc. A multi-representational pedagogy is considered to explain (i) how to help students understand summation identities of the form LHS = RHS; and (ii) how to use spreadsheet activities to lead to a better understanding of how to develop proofs for these formulas. The multi-representational pedagogy proposed in the article comprised physical manipulatives, diagrams or drawn representations, visual proofs, algebraic representations, formal proofs, and spreadsheets. The spreadsheets component stood as a valuable component in (i) visualizing the LHS and RHS of "sum=product" identities in a coordinative manner; (ii) understanding and distinguishing between the recursive forms and explicit forms of such LHS = RHS identities; and (iii) connecting the visualizations to the understanding of mathematical induction and the formal proof.

7. Acknowledgements

I would like to thank the *Spreadsheets in Education* editors and anonymous reviewers for their helpful comments on earlier drafts.

8. References

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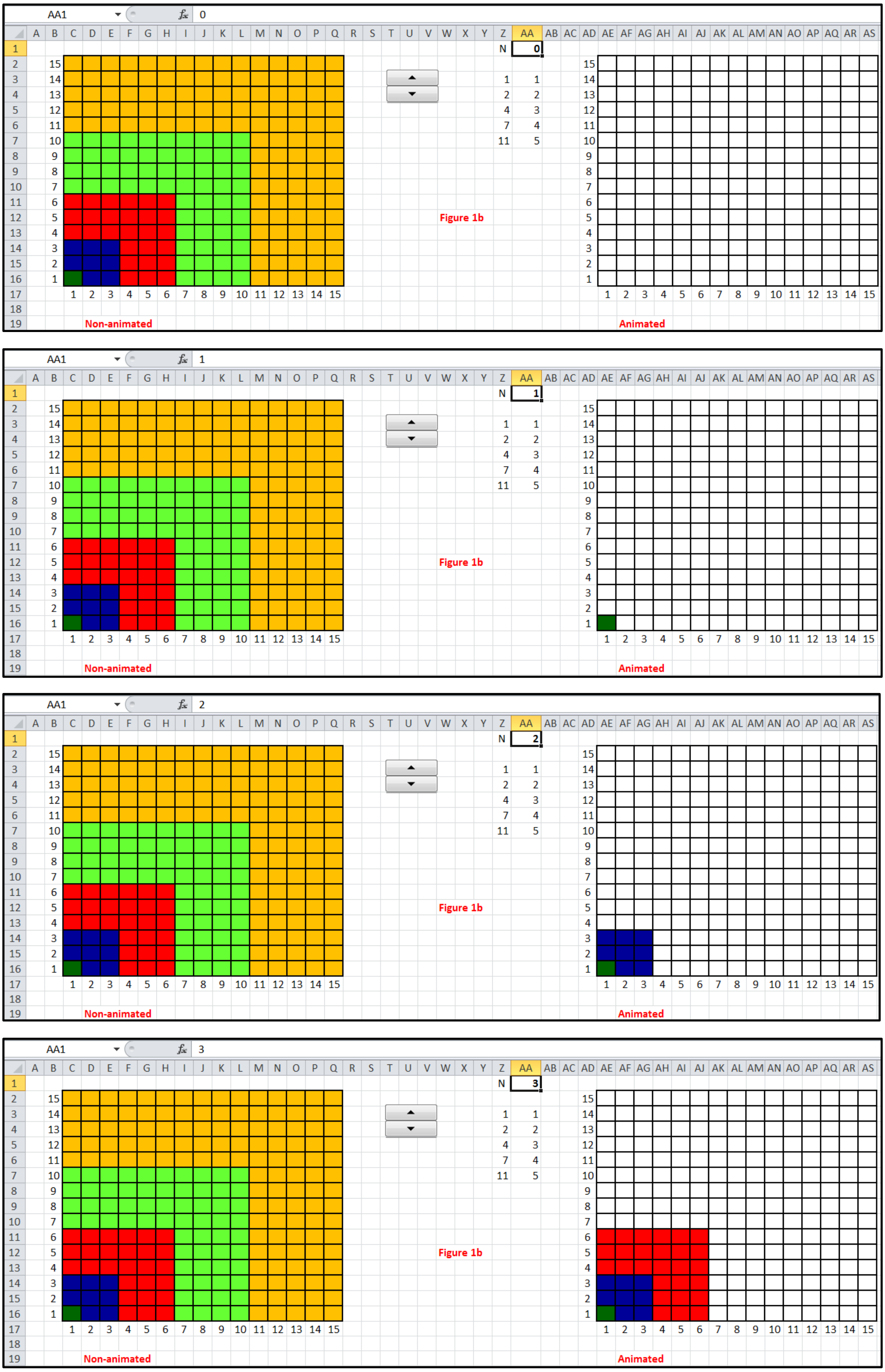
APPENDIX A



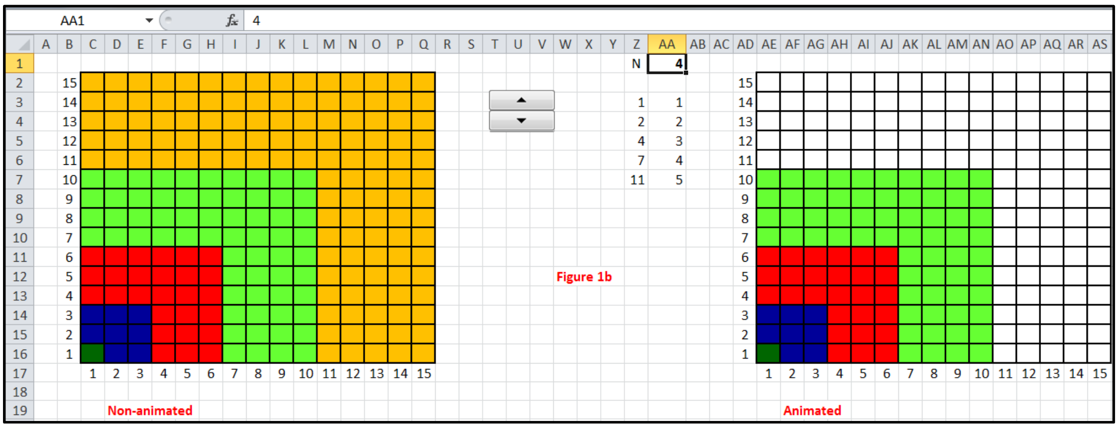


Figure 1b
Non-animated
Animated

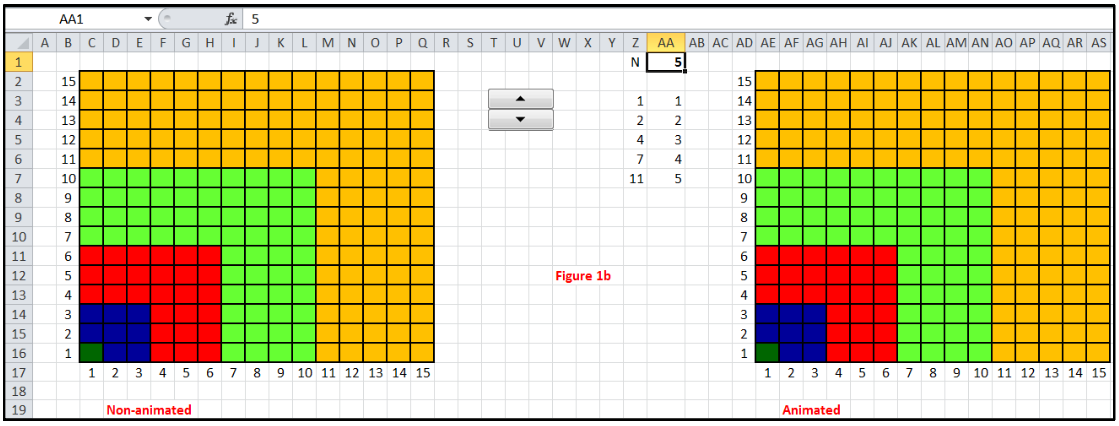


Figure 1b
Non-animated
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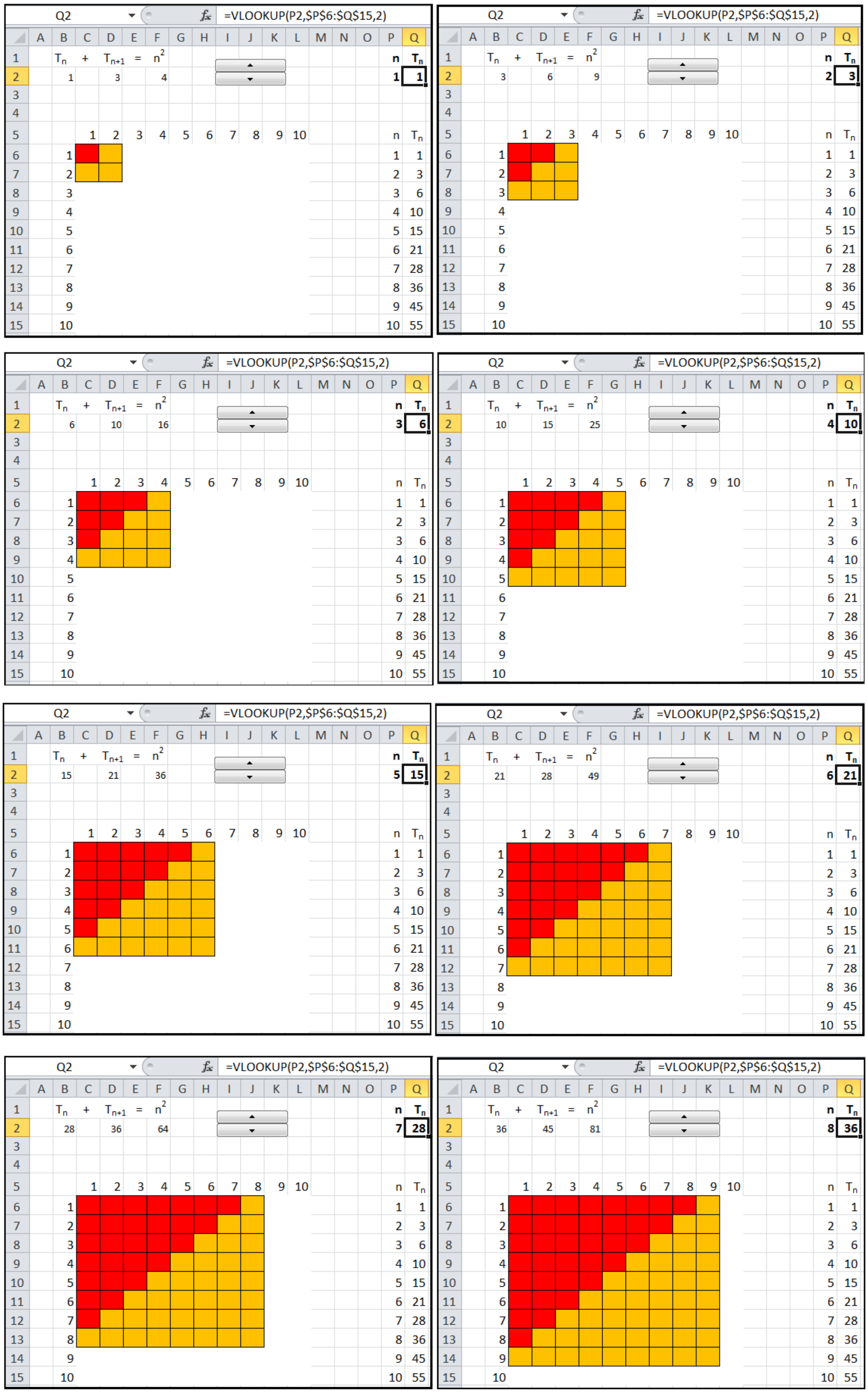
APPENDIX B


Q2 =VLOOKUP(P2,$P$6:$Q$15,2)
Tn + Tn+1 = n2
1 3 4
n Tn
1 1
Q2 =VLOOKUP(P2,$P$6:$Q$15,2)
Tn + Tn+1 = n2
3 6 9
n Tn
2 3
Q2 =VLOOKUP(P2,$P$6:$Q$15,2)
Tn + Tn+1 = n2
6 10 16
n Tn
3 6
Q2 =VLOOKUP(P2,$P$6:$Q$15,2)
Tn + Tn+1 = n2
10 15 25
n Tn
4 10
Q2 =VLOOKUP(P2,$P$6:$Q$15,2)
Tn + Tn+1 = n2
15 21 36
n Tn
5 15
Q2 =VLOOKUP(P2,$P$6:$Q$15,2)
Tn + Tn+1 = n2
21 28 49
n Tn
6 21
Q2 =VLOOKUP(P2,$P$6:$Q$15,2)
Tn + Tn+1 = n2
28 36 64
n Tn
7 28
Q2 =VLOOKUP(P2,$P$6:$Q$15,2)
Tn + Tn+1 = n2
36 45 81
n Tn
8 36