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# Using an Unsolved Problem to Motivate Student Interest in Mathematics

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## Abstract

This "in the classroom" note stems from using a spreadsheet to explore an open problem in mathematics by pre-college students of different grade levels. While being accessible to the upper elementary and middle school students, the activities suggested reveal the tool's potential for some deeper investigations towards fostering student interest in more advanced mathematics. Due to a simple access to the problem combined with its rich spreadsheet-enhanced milieu for questioning and conjecturing, many collateral problems can be posed by teachers and their students alike. A possibility of extending and deepening one's experience in mathematical observation and computational verification which goes beyond an average mathematics classroom and aimed at "special needs" of mathematically advanced students is discussed.



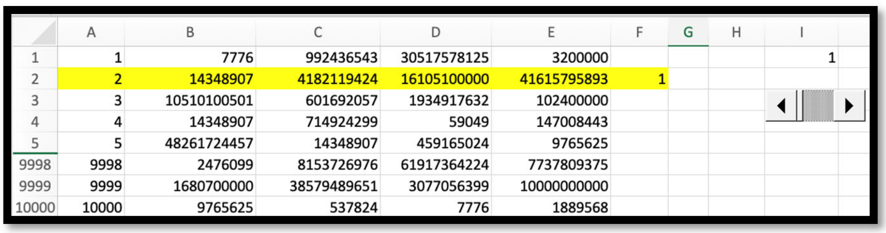
**Key words**: spreadsheets, number theory, verification, observation, Collatz conjecture, Fibonacci numbers


## 1. Introduction

In mathematics, a problem-solver often deals with a statement which needs to be proved. When a proof is found, the statement is typically called a theorem; otherwise, the statement is called a conjecture. In mathematics education, we often ask students to find some evidence of the truth of a statement and eventually validate that a statement is true with a proof. In doing so, students use both informal and formal reasoning. For example, an elementary school student can be asked to prove that increasing an odd/even number by one yields an even/odd number. An informal reasoning may deal with representing an odd (or even) number through a collection of counters among which only one counter cannot find a pair (or all counters are paired). Therefore, adding just one counter to the former collection makes all counters paired; in the case of the latter collection, the added counter does not find a pair. The student sees that their action alters the binary label, paired – nonpaired, describing each collection and uses this observation as evidence that the statement to be proved is true. A more difficult task is to prove that the product of any two or more odd numbers yields an odd number. Because the problem involves multiplication, some (modern-day) students might try to use a calculator as a way of "proving" the last statement. They can multiply two odd numbers, say, 83 and 13 to get 1079 and check that the last number is not divisible by two. Trying other similar examples, they might become convinced through empirical evidence that the statement is true. Their use of a calculator does not provide a proof, though. Rather, it is a proof-related activity associated with verification [5]. In high school geometry, especially with the advent of dynamic geometry software, students' justification activities have gravitated away from formal proof towards visual justification [12]. Likewise, outside the geometry curriculum, the activity of computational verification has become one of the functions of proof that replaced deductive reasoning with empirically based justification of mathematical statements [14]. Therefore, an important role of a mathematics teacher using technology in the classroom is to make sure students understand the difference between computational verification and mathematical proof.

A conceptually different task is to find out whether the statement that any natural number greater than two can be represented as a sum of consecutive natural numbers is true. The use of the word *any* is critical in the formulation of the task as already the number four provides a counterexample to this statement. Working on an informal, hands-on demonstration of the counterexample, a student may find it not possible to put four counters in two or more groups the cardinalities of which differ by one counter. Unlike the first two tasks, a single counterexample is enough to refute the statement through verification.

As the last example illustrated, the activity of verification of a mathematical statement can bear fruit by providing a counterexample to the statement. Yet, a counterexample is not always easy to find. For example, the equality $144^5 = 27^5 + 84^5 + 110^5 + 133^5$ [20, p. 46], found through a computer search, serves as a counterexample to the statement (known as Euler's conjecture which, if true, would have proved Fermat's Last Theorem) that it is not possible to represent a perfect $n$-th power through the sum of fewer than $n$ like powers, so that already the equation $a^3 = b^3 + c^3$ does not possess integer solutions because the third power of an integer, due to Euler's conjecture, cannot be represented by fewer than three third powers of integers. At the same time, one can see that the fifth power of 144 is represented through the sum of *four* fifth powers of other integers. The computational complexity of this counterexample can be demonstrated by trying to use a spreadsheet (Figure 1), albeit, most likely, with no success, to find the quadruple (27, 84, 110, 133) through randomly generating, say, $10^4$ quadruples of integers using the random number generator RANDBETWEEN(1, 150) to see whether the sum of the fifth powers of those integers is the fifth power of a larger integer and repeating such random trial and error approach another $10^4$ times. The programming of the spreadsheet of Figure 1 is included in the Appendix.



| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 7776 | 992436543 | 30517578125 | 3200000 | | | | 1 |
| 2 | 2 | 14348907 | 4182119424 | 16105100000 | 41615795893 | 1 | | | |
| 3 | 3 | 10510100501 | 601692057 | 1934917632 | 102400000 | | | | |
| 4 | 4 | 14348907 | 714924299 | 59049 | 147008443 | | | | |
| 5 | 5 | 48261724457 | 14348907 | 459165024 | 9765625 | | | | |
| 9998 | 9998 | 2476099 | 8153726976 | 61917364224 | 7737809375 | | | | |
| 9999 | 9999 | 1680700000 | 38579489651 | 3077056399 | 10000000000 | | | | |
| 10000 | 10000 | 9765625 | 537824 | 7776 | 1889568 | | | | |

Figure 1. Random search for a counterexample to Euler's conjecture.

Both computational and conceptual complexity of this counterexample, being indicative of the complexity of Fermat's Last Theorem, should not be taken to mean that the activity of verification of mathematical statements, the proof of which is either unknown or too complex to be offered as a classroom activity, has to be avoided. On the contrary, there are notable cases when such an activity has to be recommended as a motivation for students to enjoy mathematics and ultimately to develop "the habits of mind of a mathematical thinker" [7, p. 19]. As was noted in [21, p. 187], "the fact that proof is important for the professional mathematician does not imply that the teaching of mathematics to a given audience must be limited to ideas whose proofs are accessible to that audience". Because there are quite a few unsolved problems in contemporary mathematics [24], we can add to the last quote that teaching of mathematics may even be associated with ideas whose proofs are not known.

In the age of technology, the activity of verification in a mathematics classroom can not only be enhanced by digital tools, but those tools can turn such activity into a deep conceptual exploration of complex yet grade-appropriate for understanding mathematical statements. Regardless, whether those statements have been proved or remain unproved, the use of technology enables many previously not thought of questions to be either answered or finding an answer to be considered as a genuine and even fun activity. For one, Fermat's Last Theorem, the proof of which was found after some 350 years of efforts (including those of Euler), can be introduced by asking a question whether it is possible to extend the idea of representing a square as a sum of two squares to higher powers? One such technological tool the use of which allows for a kind of visualization of Fermat's Last Theorem is a spreadsheet.

This paper offers a possible use of a spreadsheet by schoolchildren in exploring one open problem in mathematics due to the ease of its formulation and relatively unsophisticated spreadsheet programming needed for its treatment.

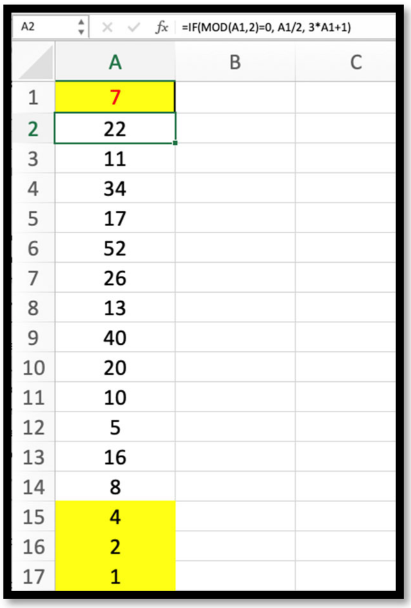
Although at the tertiary level the use of famous unsolved or solved problems as a motivation for the learning of mathematics is a well-known pedagogical approach [2], it is rare that schoolchildren, especially at the elementary level, get a taste of real mathematics dealing with problems that attracted mathematicians from over the world in search for proofs. Also rare are opportunities to explore problems which remain unsolved. Nonetheless, with the advent of digital technology, such opportunities have become available at the primary and secondary levels. For example, in the context of spreadsheets, Baker [4] discussed the Goldbach conjecture [22] and the Palindrome conjecture (Weisstein, 1999b) was discussed in [1].

This classroom note deals with the so-called *Collatz conjecture* named after Lothar Collatz, a German mathematician (1910–1990) who is believed to be the first to come up with it in 1937, although, according to [16], no evidence could be found in Collatz's publications. The introduction of this conjecture in terms of rules which allow one to build interesting number sequences, is accessible to even young learners of mathematics who can be encouraged to try to guess, after initial explorations, what those sequences have in common. Yet, the conjecture turned out to be very difficult to prove and it has the status of an open problem although many renowned mathematicians have left their traces on its solution.

The main argument of this paper is that students' exploration of the Collatz conjecture using technology is pedagogically worthwhile. In particular, a potential of spreadsheets to enhance and deepen students' mathematical investigations will be demonstrated. Before turning to the use of spreadsheets with schoolchildren, some historical roots of mathematicians' work on the Collatz conjecture will be discussed.

## 2. Formulation of the Collatz conjecture and its history

Another name of the Collatz conjecture used in the literature is the *3n + 1 problem*. It deals with a sequence defined as follows. Start with any positive integer $n$ and if $n$ is an even number divide it by two to get $n/2$; otherwise, multiply $n$ by three and add one to get $3n + 1$. Whatever the outcome, apply the same rule to either $n/2$ or $3n + 1$. The conjecture states that regardless of the starting number $n$, the sequence always converges to a three-cycle 4, 2, 1. Put another way, the sequence always reaches the number 1 from which it goes to 4 followed by 2 and 1, thus repeating the cycle (4, 2, 1). For example, as shown in the spreadsheet of Figure 2, starting from the number 7, the cycle (4, 2, 1) is reached on the 14$^{th}$ iteration. A simple formula that does computations is shown in the formula bar of the spreadsheet.


A2 | fx =IF(MOD(A1,2)=0, A1/2, 3*A1+1)

| | A | B | C |
|---|---|---|---|
| 1 | 7 | | |
| 2 | 22 | | |
| 3 | 11 | | |
| 4 | 34 | | |
| 5 | 17 | | |
| 6 | 52 | | |
| 7 | 26 | | |
| 8 | 13 | | |
| 9 | 40 | | |
| 10 | 20 | | |
| 11 | 10 | | |
| 12 | 5 | | |
| 13 | 16 | | |
| 14 | 8 | | |
| 15 | 4 | | |
| 16 | 2 | | |
| 17 | 1 | | |

Figure 2. The number 7 is attracted by the cycle (4, 2, 1).

The problem is said to be circulated in the 1950s by “word of mouth”, among others, during the 1950 International Congress of Mathematicians [16]. Yet, it appeared to be somewhat disconnected from the mainstream of mathematical theories and until 1970s no printed publication dealt with it. It was first published by Coxeter [8] based on his 1970 lecture, yet as a “piece of mathematical gossip” [16, p. 6]. It was further publicized by Gardner [11] in his *Scientific American* column. The interest to the problem has grown connecting it to another class of problems “studying sets of integers closed under iteration of affine maps” [16, p. 6]. This is how the problem initially believed to be a pure curiosity gradually became a part of investigation of mathematicians linking it to number theory, dynamic systems and theory of computation [16].

Labelle [15] asks why the problem is so interesting citing its unpredictable behavior in terms of the sequence generated by different numbers which looks very random. For example, the number 27 which is relatively small requires 112 steps before it gets into the loop 4, 2, 1 whereas the number 84 requires only 10 steps to reach the number 1 counting 84 as the first step (Figure 3). Labelle [15] also mentions the number 15,733,919 which was tried by a computer program resulting into a sequence of 705 iterations before it reaches 1. This randomness in the behavior, according to [15] makes the conjecture very difficult to prove. In addition, a call for computing technology to help with investigations brings about a lot of difficulties related to decidability of associated algorithms [15].

While the investigation of mathematics involved in the (still ongoing; it attracts such renowned mathematicians as Terence Tao, see, for example, https://www.irishtimes.com/news/science/maths-prodigy-terence-tao-does-battle-with-conjecture-1.4373396) search for a proof (or impossibility to produce one) is too complicated for schoolchildren, they can still ask some deeper questions and make investigations, especially when being supported by such a commonly available tool as a spreadsheet. They can also make some small discoveries which could nurture their interest in studying more advanced mathematics.

For instance, the length of the sequence is of course a very interesting aspect to investigate. While it is quite easy to determine the shortest length (yet still interesting to do), the search for the longest paths would be an open-ended problem that also can be investigated (for example, the number 27 produced a very long sequence, can this one be increased?). Is there something to do with parity (odd vs even number), divisibility, etc.? What is the biggest number one can get in the sequence? The biggest amplitude (increase in value)? In the next section, we will show some possibilities of investigation of these and eventually other questions using a spreadsheet.

## 3. Introducing the rule: initial steps in conjecturing

Using a ‘low entry – high ceiling’ strategy for mathematical enrichment activities accessible for all students, Collatz conjecture was explored by one of the authors with schoolchildren of ages 9 – 13 in a number of Canadian schools as part of enriching and challenging activities [10], [17]. Prior to the use of a spreadsheet, the students were asked to explore all numbers in the range 1 through 10 according to the following rule: choose a number and if it is even, divide it by 2; if it is odd, multiply it by 3 and add 1; step by step apply the same rule to a so developed number and continue until they recognize something special. Very soon, all students reported repetition of the (4, 2, 1) triple. Then they were asked to describe how the process of arriving to this triple was different for each starting number. Already, some observations were made, comparing the number 8 (directly going down, 4, 2, 1, to stop at 1) with the number 9 which produces quite a long sequence (28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 10, 5, 16, 8, 4, 2, 1). Collectively, by checking all numbers from 1 to 10, students see that the rule always produces the triple (4, 2, 1) at the end of the sequence. They become curious to know if this is true for larger numbers and can be motivated to continue investigating for integers greater than 10.

After exploring integers greater than 10 (something that is still possible to do with paper-and-pencil) and coming to the same conclusion that the sequence seems to always end with the triple (4, 2, 1), students were making observations based on this initial investigation. For instance, some of them shared that an odd number in the sequence is always followed by an even number (e.g., the number 7 produces $3 \times 7 + 1 = 22$) and once a power of two is reached, it descends along the smaller powers of two to stop at 1 (e.g., the number 16 ($= 2^4$) goes down through 8, 4, 2, 1).

Other questions were discussed prompting further investigation. For example: Does it (descending to 4, 2, 1) always work? Here, trying to find one number to disprove conjecture leads to a need to use technology; but some observations could (and did) eventually narrow the search. For example, if the number is already in the sequence generated by a smaller number, there is no need to check it: based on the sequence for the number 9, the students recognized that 52 does not need to be checked. These small yet surprising discoveries seemed to trigger student interest to go further. At that point, a spreadsheet shown in Figure 2 was useful to model the behavior of the sequences for different starting numbers, thus increasing opportunities for new questions and discoveries. Following are examples of such open-ended questions that could be investigated by students.

1. Is an even number a good candidate for a longer sequence? An odd number? A number divisible by 3? What would be other ‘good’ candidates?
2. How ‘high’ can the sequence go? (This means the largest number in a sequence).
3. What would be the biggest increase (drop) from one number to the next one?
4. What are consecutive numbers that produce a ‘summit’ (big increase – big drop – big increase)?

Then, the students already familiar with a spreadsheet (Figure 2), were introduced to the more complex spreadsheet pictured in Figure 3 to refine and deepen their investigations.

## 4. Deepening investigation: some possible paths using a spreadsheet

Using a spreadsheet, one can do much more than just generate the sequences converging to the three-cycle 4, 2, 1. It can interactively record and, most importantly, save another sequence – the number of steps, $S(n)$, it takes the process to reach the cycle starting from number $n$. For example, as shown in the spreadsheet of Figure 3, the number 10 (cell B2) is attracted by the three-cycle after four steps (cell E2).

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | 1000 nums | the tested number | | | The step 4 is reached | | Steps to reach 4 |
| 2 | 1 | 10 | | | 4 | | 1 |
| 3 | 2 | 5 | | | | | 2 |
| 4 | 3 | 16 | | | | | 5 |
| 5 | 4 | 8 | | | | | 0 |
| 6 | 5 | 4 | | | | | 3 |
| 7 | 6 | 2 | | | | | 6 |
| 8 | 7 | 1 | | | | | 14 |
| 9 | 8 | 4 | | | | | 1 |
| 10 | 9 | 2 | | | | | 17 |
| 11 | 10 | 1 | | | | | 4 |
| 12 | 11 | 4 | | | | | |

Figure 3. The number 10 reaches the number 4 in four steps.

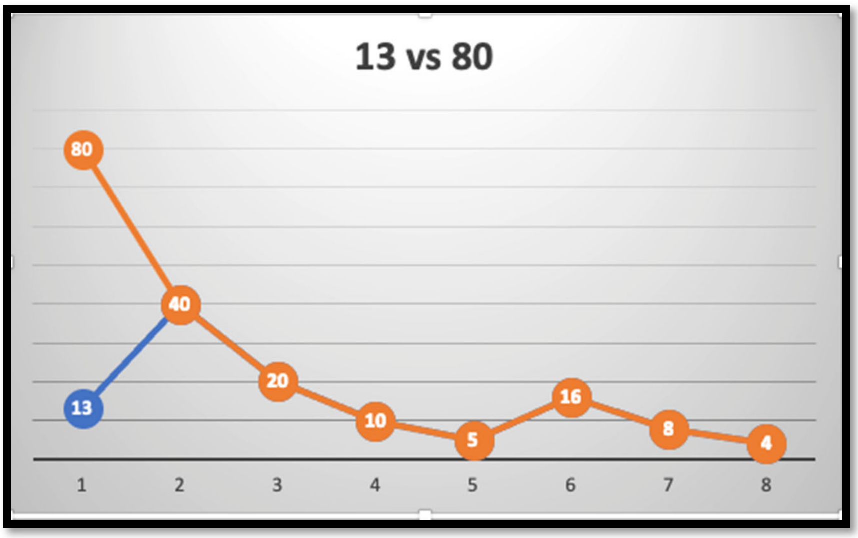



Figure 4. Coming to 40 from different sides.

Below is a list of challenging questions that can be explored by students under the guidance of their instructor using the spreadsheet of Figure 3. Although challenging questions mostly come from teachers as 'more knowledgeable others', in the digital era, such questions also can be asked by capable and aspiring students, whose "special need" should not be neglected. Starting from seemingly low-level questions requiring, nonetheless, high level thinking, one should not consider such questions as coming from nowhere as they are naturally afforded by the students' appropriate use of technology. Students' ability to control a digital tool by modifying numeric data involved enables them to reconstruct the entire thinking about a problem and, through this process of reconstruction, to begin asking questions. In particular, classroom teaching, enhanced by the appropriate use of spreadsheets, should be organized along the lines of pedagogy that motivates students to ask questions and expects teachers, by providing prompts, to assist them in finding answers.

1. Does zero appear only once in this sequence?
2. Is it possible to have two (three, four, five, and so on) equal numbers as consecutive terms of this sequence? What are these numbers?
3. How many steps does the $n$-th power of 2, $n > 2$, require reaching the cycle (4, 2, 1)?
4. What is the difference between the number of steps found for the numbers 341 and 85?
5. What is the smallest five-term sequence of consecutive numbers each of which has the same number of steps to reach the number 4? What is the first number they all meet in this process?
6. How to explain the equality $S(13) = S(80)$? (Figure 4).
7. It is known that $S(11) = 12$. Find the smallest $n > 11$ such that $S(n) = 12$.
8. Without developing the entire path starting from the number 85, determine the number of steps required reaching the cycle (4, 2, 1).
9. Without developing the entire path starting from the number 341, determine the number of steps required reaching the cycle (4, 2, 1).
10. Which monotonically increasing sequence $a_n$ have the number of steps $S(a_n)$ forming the sequence of consecutive natural numbers starting from the number 1? How can such sequence be described by a formula?
11. Find a formula for a sequence $a_n$ so that $S(a_{n+1})=S(a_n)+2$.

The last two questions appear being the most challenging ones on the above list and, following advice of one of the reviewers, the authors provide answers to them. To answer question 10, one can use the spreadsheet of Figure 3 (changing the value of the slider-controlled cell B2 and analyzing entries of cell E2 recorded in column G) from where the following equalities can be written down: $S(1) = 1$, $S(2) = 2$, $S(5) = 3$, $S(10) = 4$, $S(20) = 5$, $S(40) = 6$, $S(80) = 7$, $S(160) = 8$. Thus, the first eight terms of the sequence $a_n$

sought are: 1, 2, 5, 10, 20, 40, 80, 160. One can recognize a pattern in the development of this sequence: every term beginning from $a_4$ is twice the previous term. Thus, the sequence $a_n$ have can defined as follows:

$$a_1 = 1, a_2 = 2, a_3 = 5, a_n = 2a_{n-1}, n \geq 4. \qquad (1)$$

A more sophisticated way to describe sequence (1) is to use a closed formula involving the greatest integer function (available in the tool kit of spreadsheet formulas, so it can be verified through spreadsheet modeling)

$$a_n = INT(5 \cdot 2^{n-3}), n = 1, 2, 3, \dots . \qquad (2)$$

To answer question 11, one can use the result of question 10 and develop the following equalities: $S(1) = 1$, $S(5) = 3$, $S(20) = 5$, $S(80) = 7$, $S(320) = 9$. From here, the sequence 1, 5, 20, 80, 320, ... sought in question 11 can be written down and then generalized to the form

$$a_1 = 1, a_2 = 5, a_n = 4 \cdot a_{n-1}, n \geq 3. \qquad (3)$$

A more sophisticated way to describe sequence (3) is to use the formula

$$a_n = INT(5 \cdot 4^{n-2}), n = 1, 2, 3, \dots , \qquad (4)$$

which, once again, can be easily modeled within a spreadsheet. As an aside, note that finding formulas (2) and (4) can be due to the use of the On-line Encyclopedia of Integer Sequences (OEIS®). This, however, is beyond the scope of this classroom note.

## 5. Fibonacci numbers emerge

It is interesting to note that through exploring the $3n + 1$ problem, students can see another context where Fibonacci numbers emerge. As one practicing middle school teacher once inquired, "*The wonder of the Fibonacci numbers ... they pop up everywhere ... and what does it mean?*" This wonder of mathematical concepts, whether mathematical or not, appearing in seemingly unrelated contexts was noted already by Dewey [9, p. 86] who emphasized the educational importance of an "empirical situation in which [familiar] objects are differently related to one another". In the specific context of mathematics, Pólya [19, p. 15] argued that "One of the first and foremost duties of the teacher is not to give his students the impression that mathematical problems have little connection with each other, and no connection at all with anything else". This unexpected appearance of Fibonacci numbers means that the notion of mathematical connections is truly the most common thread permeating the entire school mathematics curriculum. Indeed, the starting odd number always generates an even number (see the first two tasks mentioned in the introduction); the starting even number generates either an even number or an odd number. Consider Figure 5 in which the process is presented in the form of a tree diagram: An even number when divided by two yields either even or odd number; an odd number when multiplied by three and increased by one yields an even number.

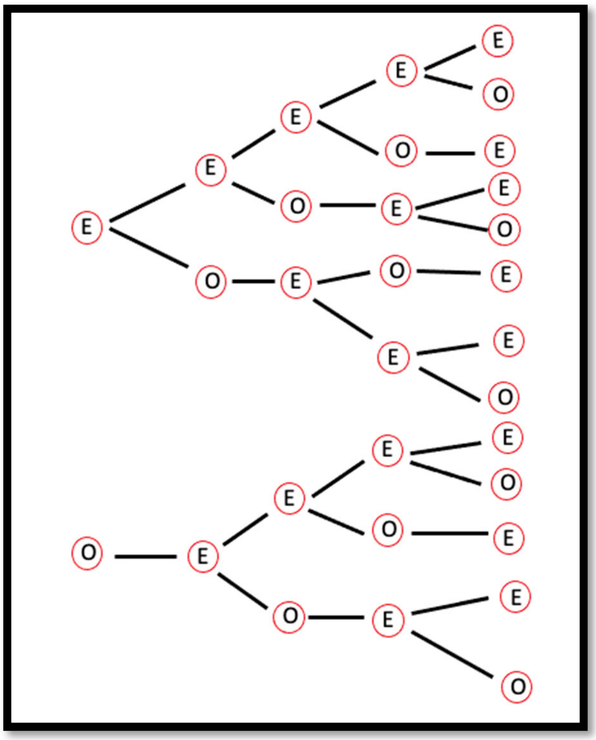


Figure 5. The tree diagram develops like Fibonacci numbers.

This observation can be expressed through the diagram:

$$E + O \rightarrow 2E + O \rightarrow 3E + 2O \rightarrow 5E + 3O \rightarrow 8E + 5O \rightarrow \cdots$$

Here, the elements develop as Fibonacci-like numbers starting from E and O. By assigning to the letters E and O the numeric value 1, we have the sequence 2, 3, 5, 8, 13, ... . That is, the first step in the tree diagram is described by the third Fibonacci number $F_3 = 2$, the second step by the fourth Fibonacci number $F_4 = 3$ , and so on. Let us assume that step $n$ in the tree diagram is described by the Fibonacci number $F_{n+2} = F_{n+1} + F_n$ , where $F_{n+1}$ and $F_n$ describe the number of possibilities for even and odd numbers, respectively. Then, on step $n+1$ there are $F_{n+1}$ possibilities for even numbers from evens on step $n$ and $F_n$ possibilities for even numbers from odds on step $n$. Therefore, there are $F_{n+2} = F_{n+1} + F_n$ possibilities for even numbers and $F_{n+1}$ possibilities for odd numbers on step $n + 1$ in the tree diagram. Therefore, the sum $F_{n+2} + F_{n+1} = F_{n+3}$ represents the total number of possibilities on step $n+1$. This testing of "transition from $n$ to $n + 1$" [19, p. 111] represents the inductive step in a mathematical induction proof that on step $n$ in the tree diagram of Figure 5 there are $F_{n+1}$ possibilities for even numbers out of $F_{n+2}$ total possibilities.

Recognizing a connection that exists between Fibonacci numbers and the $3n + 1$ problem, students (at the middle and secondary school levels) may become interested to use the spreadsheet of Figure 3 in finding the number of steps it takes a Fibonacci number to reach the (4, 2, 1) cycle. For example, the number 3 requires 5 steps, the number 5 requires 3 steps, the number 8 requires 1 step, the number 13 requires 7 steps and the number 21 requires 5 steps. A simple question is: How can one describe these findings mathematically? Is there something special about the number of steps? Furthermore, it can be seen that the number 55 (the 10th Fibonacci number) requires 110 steps. Is there something interesting about the latter finding? Thinking about such simple questions/problems, as was already mentioned in the introduction, develops "the habits of mind of a mathematical thinker" [7, p. 19]. In this regard, almost a century ago, in the general context of education, Dewey [9] delineated one's love to think as an interest in solving a problem. Mathematics provides ample opportunities for fostering one's natural curiosity.

## 6. Conclusion

This classroom note stemmed from using a spreadsheet with students of different ages in exploring the $3n+1$ problem (known as the Collatz conjecture). Being posed in the first part of the 20th century, the problem introduces a simple rule for forming number sequences that seem to be always terminating with a loop (4, 2, 1). Despite the simplicity of the rules, this conjecture still remains unproved despite significant efforts of professional mathematicians. It presents an example of an open-ended task attractive for students even at the elementary level, especially with the use of technology (e.g., a computer spreadsheet) that enhances investigations. Using a spreadsheet, one creates modeling data enabling many questions to be raised that would not be feasible otherwise [6]. A classroom activity associated with exploration of the behavior of sequences generated by different starting numbers prompts questions leading to a variety of conjectures. In turn, the verification of conjectures can develop students' interest in mathematics. While triggering a positive relationship to mathematics in all students, especially at a younger age, these experiences can lead some of them to more advanced mathematical studies [3], [13].

The role of a teacher in such a classroom is critical for motivating students' questioning and prompting further investigations. But with each new question asked and tried by students (with the help of a teacher when needed), students learn the art of asking mathematical questions and looking for the answers by experimenting and modeling (a process where technology can be particularly useful). Therefore, some unexpected (by both students and teachers) observations and questions can arise. In that way, mathematics learning enhanced by the use of a spreadsheet becomes a reciprocal process when students learn from teachers and teachers learn from students. The more a teacher learns today from students, the more tomorrow's students would learn from the teacher and through such learning reciprocity both parties epistemically develop, hopefully bringing mathematics closer to the solution of unsolved problems.

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**Appendix**

The notation (A1): is used to present a formula defined in cell A1.

### Programming of the spreadsheet of Figure 1.

Column A includes the first $10^4$ natural numbers. Cell I1 is controlled by a slider allowing one to generate consecutive natural numbers in the range [1, $10^4$]. Each click of the slider generates a new combination of random numbers in columns B, C, D, and E.

(B2), (C2), (D2), (E2): =(RANDBETWEEN(1,150))^5 – replicated down to row $10^4$.

(F1): =IF(INT((SUM(B1:E1))^0.2)=(SUM(B1:E1))^0.2, 1, " ") – replicated down to row $10^4$. The formula generates the number 1 (as a marker) in the case when the sum of four fifth powers of randomly generated integers is the fifth power itself. Note that the range [B1:E1] is entered with the fifth powers of the quadruple (27, 84, 110, 133) the sum of which is equal to $144^5$ (see the counterexample to Euler's conjecture mentioned in the Introduction); thus, cell F1 generates the number 1 as a confirmation of the existence of the counterexample.

(G1): =IF(I$1=A1,F1,G1) – replicated down to row $10^4$. The formula includes circular reference, that is, a reference to the cell in which it is defined; this technique allows for the preservation of the results displayed in column G as the value of the slider-controlled cell I1 changes.

(H1): =COUNTIF(G1:G10000,1). The formula counts the number of ones (markers) appearing in column G.

### Programming of the spreadsheet of Figure 3.

(A1): =1, (A2): = A1+1 – replicated down to cell A1001.

(B2) is slider-controlled cell, the range is from 0 to 1000.

(B3): =IF(MOD(B2, 2)=0, B2/2, 3*B2+1) – replicated down to cell B1001.

(E2): =IF(B2=0," ", IF(B2=4, 0, XLOOKUP(4, B$2:B$1001,A$2:A$1001)-1)) – this formula, by using the XLOOKUP function instructs the spreadsheet to look for the number 4 in column B and display the corresponding number in column A diminished by one thereby calculating the number of steps needed to reach the number 4 by starting with a number displaying in cell B2.

(G2): =IF(B$2=0, " ", IF(B$2=A2, E$2, G2)) – replicated down to cell G1001; this formula, by using a circular reference, that is, referencing a cell in which the formula is defined, makes it possible to preserve results in column G obtained at each step of changing the content of the slider-controlled cell B2.