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Remedies for Misapplications of Sylvester's Criterion: A Pedagogic Illustration

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Abstract

Sylvester’s criterion, which verifies the positive definiteness of any real symmetric matrix by examining the signs of all leading principal minors, is an excellent analytical tool. However, as noted by several authors in various academic fields, misapplications of the tool include its unjustified use for non-symmetric matrices and its unjustified extension for verifying the positive semidefiniteness of matrices. As remedies are available, this paper provides a pedagogic illustration that connects the corresponding tests for the positive definiteness and the positive semidefiniteness of a matrix to the underlying concepts. Drawing on standard materials in linear algebra, this paper uses self-contained Excel worksheets to illustrate the concepts involved and to generate matrices that are suitable for use in courses covering Sylvester’s criterion. This paper also suggests the use of some Excel-based exercises for students, as an alternative approach to cover the same topic.

Keywords: positive semidefinite matrices, positive definite matrices, Sylvester's criterion

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1 Introduction

In linear algebra, an $n \times n$ real symmetric matrix $\boldsymbol{A}$ is said to be positive definite if, for any arbitrary $n \times 1$ matrix $\boldsymbol{x}$ with only real elements, excluding the case where all elements of $\boldsymbol{x}$ are zeros, the scalar that $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ represents is always strictly positive. Here, $\boldsymbol{x}$ is also called an n-element column vector, and the prime stands for matrix transposition. If $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ is always nonnegative instead, where $\boldsymbol{x}$ can have all zero elements, then $\boldsymbol{A}$ is said to be positive semidefinite. Given the above definitions, a positive definite matrix is also a positive semidefinite matrix; however, the converse of the statement is false.

According to Sylvester's criterion, a real symmetric matrix is positive definite if and only if all of its leading principal minors are positive. This matrix property is named after James Joseph Sylvester (1814-1897). For an $n \times n$ matrix, there are n leading principal minors, each of which is the determinant of the submatrix containing the first k rows and the first k columns of the matrix, for $k = 1, 2, \ldots, n$. Implicitly, the n-th leading principal minor is the determinant of the matrix itself. Various proofs of Sylvester's criterion are available, though not entirely from the mathematics literature (see, for example, Giorgi [2017] for a survey and Kwan [2010, Appendix B] for a proof where the algebraic tools involved are confined to familiar matrix operations).

As a general rule in mathematical proofs, the conditions under which a statement holds are precise. However, some users of Sylvester's criterion who are primarily interested in its applications have misinterpreted the conditions for its applicability, and its misapplications have been reported in various academic fields. A notable example is that, as the definition of positive definiteness need not require the matrix involved to be symmetric (see, for example, Johnson [1970]), Sylvester's criterion has been applied to non-symmetric matrices as well. In an engineering note, Bose [1968] has illustrated with a 2×2 matrix that it is inappropriate to apply Sylvester's criterion directly to non-symmetric matrices.

In view of the relevance of positive semidefinite matrices in various academic fields (see, for example, Hiriart-Urruty and Malick [2012] for a review of its applications in optimization), Sylvester's criterion has been extended without justification by some users for testing the positive semidefiniteness of matrices. Specifically, a real symmetric matrix is deemed positive semidefinite if all leading principal minors are non-negative. Several authors have warned against using Sylvester's criterion in such a manner, by using 2×2 or 3×3 matrices as examples to reveal some undesirable consequences (see, for example, Swamy [1973], Prussing [1986], Kerr [1990], Bhatia [2007, Chapter 1, page 7], and Ghorpade and Limaye [2007]).

As no principal minor of a positive semidefinite matrix can be negative, Prussing [1986] and Ghorpade and Limaye [2007] have correctly called for the examination of the signs of all principal minors of the symmetric matrix considered when testing its positive semidefiniteness. A principal minor of an $n \times n$ matrix $\boldsymbol{A}$ is the determinant of a $p \times p$ matrix, for $p = 1, 2, \ldots, n$, which is generated by deleting $(n - p)$ rows of $\boldsymbol{A}$ and the corresponding $(n - p)$ columns of $\boldsymbol{A}$, where the deleted rows and columns need not be contiguous. To generate a complete set of principal minors of $\boldsymbol{A}$, as each row can be either retained or deleted, there are 2^n such choices before ruling out the case where none of n rows is retained. For an $n \times n$ matrix, while there are only n leading principal minors, there are $2^n - 1$ principal minors. As 2^n increases exponentially with n, to verify the positive semidefiniteness of $\boldsymbol{A}$ by checking the signs of all principal minors is a tedious task if n is large. However, it is still practically feasible to perform such a task for pedagogic purposes by using some small-scale matrices, such as cases where $2 \leq n \leq 5$.

It is worth noting that the relevance of positive definite and positive semidefinite matrices is not confined to those academic fields where misapplications of Sylvester's criterion have been reported. For example, in some finance courses covering portfolio optimization models for assisting investment decisions, students learn about the importance of verifying the positive definiteness and the positive semidefiniteness of the covariance matrix of returns used for the models involved (see, for example, Kwan [2010, 2018] for some relevant analytical issues). Covariance matrices are real symmetric matrices. With $\boldsymbol{A}$ and $\boldsymbol{x}$ representing an $n \times n$ covariance matrix and an n-element column vector of portfolio weights, respectively, the scalar $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ is the variance of portfolio returns, which is a risk measure. As variance can never be negative, the positive semidefiniteness of $\boldsymbol{A}$ is required. A positive definite $\boldsymbol{A}$ implies that there is always some risk that cannot be diversified away in a portfolio setting.

Small-scale covariance matrices are suitable for illustrative purposes. They are often based on artificial data with the implied correlations of returns all confined in the permissible range of -1 to 1. However, as such a requirement alone does not ensure the positive definiteness and the positive semidefiniteness of the covariance matrices involved, proper verification will still be necessary. Further, when a portfolio optimization model is implemented empirically, as the true variances and covariances of asset returns are unknown, they must be estimated. Covariance matrices estimated with past asset return observations under the assumption of stationary joint distribution of returns — commonly known as sample covariance matrices — are always positive semidefinite, and invertible sample covariance matrices are always positive definite. However, as results from portfolio optimization models are highly input sensitive, revisions to sample covariance matrices are often deemed necessary. If some elements of a sample covariance matrix have been revised for potential improvements, the positive definiteness and the positive semidefiniteness of the resulting matrix will have to be examined before it can be used for implementing any portfolio optimization model.

For courses covering Sylvester's criterion, it is beneficial to students if the misapplications as noted above and the corresponding remedies are also part of the coverage. A task for each instructor involved, therefore, is to generate various real matrices for use in illustrative examples, exercises, and examination questions, unless the same matrices generated by others are used repeatedly instead. Matrices where arbitrary values are assigned to the individual elements tend to be neither positive definite nor positive semidefinite. To generate a real symmetric matrix that is known in advance to be positive definite, positive semidefinite, or neither does require the use of an algebraic relationship for the matrix, its eigenvalues, and the corresponding orthonormal eigenvectors. (What eigenvalues, eigenvectors, and orthonormal eigenvectors represent, as well as how a set of linearly independent eigenvectors can be made orthonormal, are provided in Section 3 and Appendices A and B.) Although such an algebraic relationship is usually part of the standard coverage in linear algebra courses, its coverage in this paper will help students recognize its relevance in the context of applications and misapplications of Sylvester's criterion.

The use of Excel's matrix tools — either for a self-contained Excel worksheet or in combination with computational results from some free online resources — can facilitate the attainment of various real matrices. (A description of such online resources will be provided later in this paper, when the context becomes clearer.) Indeed, once the above-mentioned algebraic relationship has been identified, the use of Excel's matrix tools can lead to as many suitable matrices as needed for pedagogic purposes. Illustrations based on the matrices thus generated, with each matrix containing some specific algebraic properties, can also improve the depth of coverage of Sylvester's criterion in the courses involved.

The remainder of this paper is organized as follows: Section 2 illustrates in more detail the misapplications of Sylvester's criterion as noted earlier, with the example from Bose [1968] and a common example from Prussing [1986], Bhatia [2007], Ghorpade and Limaye [2007], and Hiriart-Urruty and Malick [2012]. Section 3 covers, in several subsections, the core analytical materials of this paper. In addition to generating real symmetric matrices known in advance to be positive definite, positive semidefinite, or neither, based on a specific algebraic relationship, this section provides a simple remedy for the type of misapplications noted by Bose [1968], as well as a simple way to recognize some real symmetric matrices that are neither positive definite nor positive semidefinite. This section also illustrates how some non-symmetric matrices suitable for pedagogic illustrations in courses where Sylvester's criterion is covered can be generated. Further, in support of the approach by Prussing [1986] and Ghorpade and Limaye [2007] for verifying the positive semidefiniteness of any given real symmetric matrix, this section provides analytical justification for examining the signs of all principal minors as well.

Section 4 provides an Excel-based illustration, which is intended to achieve two related pedagogic objectives. The first objective is to illustrate a direct connection between whether a given matrix is positive definite, positive semidefinite, or neither and the eigenvalues, which can be deduced from the matrix itself. Such a connection will help students understand more fully the usefulness of Sylvester's criterion, as well as its limitations. The second objective pertains to generating real symmetric matrices and related non-symmetric matrices given the eigenvalues and the corresponding eigenvectors. The use of Excel tools will allow students to focus on the underlying concepts without being distracted from the computational chores involved. Finally, Section 5 concludes this paper.

As the analytical materials in Section 3 and Appendices A and B are essential for the coverage of Sylvester's criterion and remedies for its misapplications, the two pedagogic objectives of Section 4 can also be achieved via some Excel-based exercises for students. The exercises can be set up in such a way that students not only will generate some small-scale matrices based on some eigenvalues and the corresponding eigenvectors, but also will perform positive definiteness and positive semidefiniteness tests for matrices provided by others. Such exercises can help students connect directly the analytical tools involved and the underlying concepts.

2 Simple Illustrations of Two Common Misapplications of Sylvester's Criterion

Real Non-Symmetric Matrix: The matrix in the example provided by Bose [1968] is

$$\boldsymbol{B} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}. \tag{1}$$

As each leading principal minor is 2 and thus is strictly positive, the use of Sylvester's criterion would erroneously indicate the positive definiteness of $\boldsymbol{B}$. Such a test result is erroneous because, for

$$\boldsymbol{x} = \begin{bmatrix} 1 & s \end{bmatrix}', \tag{2}$$

where s is a real number,

$$\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x} = \begin{bmatrix} 1 & s \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} = 2 + 3s + s^2 = (s+1)(s+2) \tag{3}$$

is negative for $-2 < s < -1$, confirming that $\boldsymbol{B}$ cannot be positive definite.

Real Symmetric Matrix: Unlike the correct use of Sylvester's criterion to confirm the positive definiteness of a symmetric matrix, its use to confirm the positive semidefiniteness instead, by requiring that all leading principal minors be non-negative, is a conjecture that can easily be refuted. In the common example from Prussing [1986], Bhatia [2007], Ghorpade and Limaye [2007], and Hiriart-Urruty and Malick [2012], as

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \tag{4}$$

both leading principal minors are zeros and thus non-negative. However, the use of Sylvester's criterion in such a manner would erroneously confirm its positive semidefiniteness.

For

$$\boldsymbol{x} = \begin{bmatrix} r & s \end{bmatrix}', \tag{5}$$

where r and s are any real numbers, as

$$\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} r & s \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = -s^2, \tag{6}$$

which is negative for $s \neq 0$, the above conjecture of $\boldsymbol{A}$ being positive semidefinite must be false. Further, the three principal submatrices of $\boldsymbol{A}$ are $[0]$, $[-1]$, and $\boldsymbol{A}$ itself, for which the corresponding determinants are 0, -1, and 0. Such a result can also be used to reject the positive semidefiniteness of $\boldsymbol{A}$.

3 Some Relevant Matrix Materials

The five subsections below cover those analytical materials that are directly related to the pedagogic objectives of this paper. They include the following: (*i*) a remedy for misapplying Sylvester's criterion to non-symmetric matrices, (*ii*) justification for immediate rejections of the positive definiteness and the positive semidefiniteness of real symmetric matrices with some specific features, without having to apply Sylvester's criterion explicitly or the remedy according to Prussing [1986] and Ghorpade and Limaye [2007], (*iii*) a procedure to orthonormalize linearly independent eigenvectors for generating suitable symmetric matrices for use in applying Sylvester's criterion, (*iv*) a simple approach to generate suitable non-symmetric matrices for further illustrations, by drawing on the analytical materials in (*i*), and (*v*) justification for the examination of the signs of all principal minors for verifying the positive semidefiniteness of a given matrix. To avoid digressions, some analytical materials, though also relevant for the pedagogic objectives of this paper, are provided in Appendices A and B instead.

3.1 Real Non-Symmetric Matrices

For an $n \times n$ real non-symmetric matrix $\boldsymbol{B}$ and any n-element column vector $\boldsymbol{x}$ with all real elements, the matrix product $\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x}$ — which is a 1×1 matrix — is the same as its transpose. As we can write

$$\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x} = (\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x})' = \boldsymbol{x}'\boldsymbol{B}'\boldsymbol{x} \tag{7}$$

and then

$$2\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x} = \boldsymbol{x}'(\boldsymbol{B} + \boldsymbol{B}')\boldsymbol{x} = \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}, \tag{8}$$

where

$$\boldsymbol{A} = \boldsymbol{B} + \boldsymbol{B}', \tag{9}$$

$\boldsymbol{B}$ is positive definite (positive semidefinite) if $\boldsymbol{A}$ is positive definite (positive semidefinite). Further, as $\boldsymbol{A}$ is symmetric, Sylvester's criterion can be used to verify whether $\boldsymbol{A}$ is positive definite, thus allowing us to reach the corresponding conclusion about $\boldsymbol{B}$ itself.

If it turns out that $\boldsymbol{B}$ is positive definite, it must also be positive semidefinite, as the occurrence of $\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x} = 0$ corresponds to the trivial case where $\boldsymbol{x}$ has all zero elements. However, if the application of Sylvester's criterion leads to the rejection of the positive definiteness of

$\boldsymbol{A}$, all we can infer from the result is that $\boldsymbol{B}$ is not positive definite. To verify the positive semidefiniteness of $\boldsymbol{B}$ still requires the subsequent analytical materials in this section, unless we simply treat the remedy provided by Prussing [1986] and Ghorpade and Limaye [2007] as a recipe that works well.

In the example provided by Bose [1968],

$$\boldsymbol{A} = \boldsymbol{B} + \boldsymbol{B}' = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \tag{10}$$

being symmetric, Sylvester's criterion is applicable. As the two leading principal minors of $\boldsymbol{A}$ are 4 and -1, the positive definiteness of $\boldsymbol{A}$ is rejected. Thus, so is $\boldsymbol{B}$. As $\boldsymbol{A}$ has three principal submatrices, consisting of $[4]$, $[2]$, and $\boldsymbol{A}$ itself, for which the corresponding determinants are 4, 2, and -1, its positive semidefiniteness must be rejected according to the recipe. The same conclusion applies to $\boldsymbol{B}$ as well.

3.2 Real Symmetric Matrices with Zero or Negative Diagonal Elements

If a real $n \times n$ symmetric matrix $\boldsymbol{A}$ has at least one zero diagonal element, the matrix cannot be positive definite. Suppose that the (i,i)-element of $\boldsymbol{A}$ is zero. For any n-element column vector $\boldsymbol{x}$ where the only non-zero element is in its i-th entry, we always have

$$\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} = 0, \tag{11}$$

which directly leads to the rejection of the positive definiteness of $\boldsymbol{A}$. If so, an explicit application of Sylvester's criterion to verify this rejection is unnecessary.

If $\boldsymbol{A}$ has at least one negative diagonal elements instead, the matrix can neither be positive definite nor positive semidefinite. If so, neither an explicit application of Sylvester's criterion for verifying its positive definiteness nor the use of the remedy as proposed by Prussing [1986] and Ghorpade and Limaye [2007] for verifying its positive semidefiniteness is necessary. To see this, suppose that the (i,i)-element of $\boldsymbol{A}$, denoted as a_{ii}, is negative. All that we have to do is to attempt an n-element column vector $\boldsymbol{x}$ where all elements are zeros, except for its non-zero i-th entry, denoted as x_i. In such a case, as

$$\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} = a_{ii}x_i^2 < 0, \tag{12}$$

$\boldsymbol{A}$ can be neither positive definite nor positive semidefinite. Notice that the 2×2 real symmetric matrix in equation (4) is a simple example that covers both situations above.

The same conclusion about $\boldsymbol{A}$ can also be reached, if the inverse of $\boldsymbol{A}$ exists and has at least one negative diagonal element. To see this, let us denote the (i, i)-element of $\boldsymbol{A}^{-1}$, which is negative, as c_{ii}. As $\boldsymbol{A}\boldsymbol{A}^{-1}$ is an identity matrix, we can write, for an arbitrary n-element column vector $\boldsymbol{x}$,

$$\boldsymbol{x}'\boldsymbol{A}^{-1}\boldsymbol{x} = \boldsymbol{x}'\boldsymbol{A}^{-1}(\boldsymbol{A}\boldsymbol{A}^{-1})\boldsymbol{x} = \boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}, \tag{13}$$

where

$$\boldsymbol{y} = \boldsymbol{A}^{-1}\boldsymbol{x}. \tag{14}$$

We can choose $\boldsymbol{x}$ in such a way that all of its elements are zeros, except for its non-zero i-th entry, denoted as x_i. As there is a corresponding $\boldsymbol{y}$, for which

$$\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y} = \boldsymbol{x}'\boldsymbol{A}^{-1}\boldsymbol{x} = c_{ii}x_i^2 < 0, \tag{15}$$

$\boldsymbol{A}$ can be neither positive definite nor positive semidefinite.

3.3 Symmetric Matrix Corresponding to Given Eigenvalues and Linearly Independent Eigenvectors

For the purpose of generating a symmetric matrix $\boldsymbol{A}$ by using its n eigenvalues and the corresponding n linearly independent eigenvectors, an orthonormalization procedure for such eigenvectors is required. See Appendix A for a brief introduction to eigenvalues and eigenvectors. See also Appendix B for the detail of the orthonormalization procedure. In essence, we let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the n eigenvalues and $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ be the corresponding linearly independent eigenvectors and write

$$\boldsymbol{A}\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i, \text{ for } i = 1, 2, \ldots, n, \tag{16}$$

succinctly as

$$\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{\Lambda}. \tag{17}$$

Here, $\boldsymbol{\Lambda}$ is an $n \times n$ diagonal matrix with the individual diagonal elements being $\lambda_1, \lambda_2, \ldots, \lambda_n$, and

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_n \end{bmatrix} \tag{18}$$

is an $n \times n$ matrix, where each column i is the corresponding eigenvector $\boldsymbol{x}_i$. If $\boldsymbol{X}\boldsymbol{X}'$ is an identity matrix, $\boldsymbol{X}$ is said to be orthonormal.

The orthonormalization of the original set of linearly independent eigenvectors can be achieved by using the well-known Gram-Schmidt process (see, for example, Strang [2016]), which is named after Jϕrgen Pedersen Gram (1850-1916) and Erhard Schmidt (1876-1959). Upon the completion of the Gram-Schmidt process, $\boldsymbol{X}\boldsymbol{X}'$ will become an identity matrix, which implies that

$$\boldsymbol{X}' = \boldsymbol{X}^{-1}. \tag{19}$$

It follows that

$$\boldsymbol{A} = \boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1} = \boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}'. \tag{20}$$

As

$$\boldsymbol{A}' = (\boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}')' = \boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}' = \boldsymbol{A}, \tag{21}$$

the symmetry of $\boldsymbol{A}$ thus generated is assured. Any changes in $\boldsymbol{\Lambda}$ and the corresponding $\boldsymbol{X}$ resulted from the orthonormalization procedure will lead to a different $\boldsymbol{A}$. Such symmetric matrices thus generated are suitable examples for use in illustrating positive definiteness and positive semidefiniteness tests.

Notice that, for a small-scale case such as $n = 4$, the orthonormal matrix $\boldsymbol{X}$ can also be obtained directly. Specifically, the use of a Hadamard matrix — which is named after Jacques Solomon Hadamard (1865-1963) — will directly lead to an orthonormal matrix. By definition, each element of an $n \times n$ Hadamard matrix, if the matrix exists, must be 1 or -1, and each pair of different rows of the matrix must be orthogonal vectors. For example,

$$\boldsymbol{H} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{22}$$

is a 4×4 Hadamard matrix, as

$$\boldsymbol{H}\boldsymbol{H}' = 4\boldsymbol{I}, \tag{23}$$

where $\boldsymbol{I}$ is a 4×4 identity matrix. Thus, if $\boldsymbol{\Lambda}$ is a 4×4 diagonal matrix, $\boldsymbol{H}/2$ can be used directly as $\boldsymbol{X}$ in equation (21) for the purpose of generating a 4×4 symmetric matrix $\boldsymbol{A}$; that is, there is no need to go through the Gram-Schmidt process for the same purpose. However, for other small-scale cases such as $n = 3$, 5, 6, or 7, the corresponding Hadamard matrices do not exist.

Notice also that, for the purpose of generating an $n \times n$ symmetric matrix $\boldsymbol{A}$, the starting point can still be n eigenvalues and the corresponding linearly independent eigenvectors, but with the technical detail of the Gram-Schmidt process bypassed entirely as well. This is because, for small-scale cases, several online Gram-Schmidt calculators are available; for example, *eMathHelp* and *dCode* offer free orthonormalization for three vectors and for two, three, and four vectors, respectively.[1] If such online resources are utilized, the users will still have to perform subsequent computations to reach the corresponding symmetric matrix $\boldsymbol{A}$ via equation (21).

3.4 Non-Symmetric Matrices Generated from a Given Symmetric Matrix

Once we have generated a real symmetric matrix by following the procedure as described in the preceding subsection and in Appendix B, we can obtain as many real non-symmetric matrices as needed for the purpose of testing their positive definiteness or positive semidefiniteness. To see how, suppose that a real $n \times n$ symmetric matrix $\boldsymbol{A}$ has already been generated. Let $\boldsymbol{S}$ be an arbitrary $n \times n$ real skew-symmetric matrix, which is a matrix where all diagonal elements are zeros and the (i, j)-element, denoted as s_{ij}, is the negative of s_{ji}, for $i, j = 1, 2, \ldots, n$ and $i \neq j$. With $\boldsymbol{S}$ being skew-symmetric, we always have

$$\boldsymbol{S}' = -\boldsymbol{S}. \tag{24}$$

Now, let

$$\boldsymbol{B} = \frac{1}{2}\boldsymbol{A} + \boldsymbol{S}. \tag{25}$$

As

$$\boldsymbol{B}' = \frac{1}{2}\boldsymbol{A}' + \boldsymbol{S}' = \frac{1}{2}\boldsymbol{A} - \boldsymbol{S}, \tag{26}$$

we always have

$$\boldsymbol{B} + \boldsymbol{B}' = \boldsymbol{A}, \tag{27}$$

[1]The corresponding electronic links are https://www.emathhelp.net/calculators/linear-algebra/gram-schmidt-calculator/ and https://www.dcode.fr/gram-schmidt-orthonormalization .

regardless of the choice of $\boldsymbol{S}$ for the purpose of generating the corresponding $\boldsymbol{B}$. We have established in equation (8) from Subsection 3.1 that, for any real n-element column vector $\boldsymbol{x}$,

$$2\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x} = \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}. \tag{28}$$

An implication is that the positive definiteness (positive semidefiniteness) of $\boldsymbol{A}$ indicates the same about $\boldsymbol{B}$ and that, if $\boldsymbol{A}$ is not positive definite (positive semidefinite), neither is $\boldsymbol{B}$. As the choice of $\boldsymbol{S}$ is arbitrary for the same real symmetric matrix $\boldsymbol{A}$, we can generate as many non-symmetric matrices as needed for pedagogic purposes.

3.5 Signs of Individual Principal Minors and the Positive Semidefiniteness of a Real Symmetric Matrix

Suppose that a given $n \times n$ real symmetric matrix $\boldsymbol{A}$ is partitioned as follows:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}, \tag{29}$$

where $\boldsymbol{A}_{11}$, $\boldsymbol{A}_{12}$, $\boldsymbol{A}_{21}$, and $\boldsymbol{A}_{22}$ are $(n-p)\times(n-p)$, $(n-p)\times p$, $p\times(n-p)$, and $p\times p$ matrices, respectively, with p being any integer among $1, 2, \ldots, n-1$. For each p, the n-element column vector $\boldsymbol{x}$ is partitioned conformally as

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix}, \tag{30}$$

where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are $(n-p)$-element and p-element column vectors, respectively. If $\boldsymbol{x}_1$ is set to be a vector of zeros, we can write

$$\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1' & \boldsymbol{x}_2' \end{bmatrix} \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \boldsymbol{x}_2'\boldsymbol{A}_{22}\boldsymbol{x}_2. \tag{31}$$

With $\boldsymbol{A}$ partitioned in the above manner, $\boldsymbol{A}_{22}$ can be viewed as a principal submatrix resulted from the deletion of its first $(n-p)$ rows and its first $(n-p)$ columns. The corresponding principal minor is $|\boldsymbol{A}_{22}|$. If $|\boldsymbol{A}_{22}| < 0$, $\boldsymbol{A}_{22}$ must have at least one negative eigenvalue. If so, $\boldsymbol{A}_{22}$ cannot be positive semidefinite. Further, in view of equation (31), neither can $\boldsymbol{A}$.

The above idea can be extended, by allowing any of the $(n-p)$ deleted rows and the corresponding $(n-p)$ columns not to be contiguous. All that we have to do is to set the elements of $\boldsymbol{x}$ corresponding to the deleted rows (or columns) of $\boldsymbol{A}$ to zeros. Let $\boldsymbol{x}_p$ be the p-element column vector that contains the remaining elements of $\boldsymbol{x}$. Let also $\boldsymbol{A}_{pp}$ be the corresponding

$p \times p$ principal submatrix of $\boldsymbol{A}$ after such row and column deletions, with $|\boldsymbol{A}_{pp}|$ being the corresponding principal minor.

For each p, there are

$$\binom{n}{n-p} = \binom{n}{p} = \frac{n!}{(n-p)!p!} \tag{32}$$

ways to delete $(n-p)$ rows and the corresponding $(n-p)$ columns for obtaining a $p \times p$ principal submatrix $\boldsymbol{A}_{pp}$. In each case, we have

$$\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}_p'\boldsymbol{A}_{pp}\boldsymbol{x}_p. \tag{33}$$

If $|\boldsymbol{A}_{pp}| < 0$, $\boldsymbol{A}_{pp}$ cannot be positive semidefinite, and neither can $\boldsymbol{A}$.

Suppose that the positive definiteness of a given $n \times n$ real symmetric matrix $\boldsymbol{A}$ has already been rejected according to Sylvester's criterion. Suppose also that none of the leading principal minors of $\boldsymbol{A}$ is negative. Then, to verify the positive semidefiniteness of $\boldsymbol{A}$ requires that the $2^n - 1$ principal submatrices be examined. If the determinant of any of these principal submatrices is negative, the positive semidefiniteness of $\boldsymbol{A}$ must be rejected as well.

4 An Excel-Based Illustration

As indicated in the Introduction, there are two related pedagogic objectives of the Excel-based illustration. The Excel worksheets for Figures 1 and 2 are for illustrating these two objectives separately. Specifically, Figure 1 is focused on tests of the positive definiteness and the positive semidefiniteness of a given matrix; Figure 2 is focused on generating a matrix given its eigenvalues and the corresponding eigenvectors instead.

Although both figures are based on 4×4 matrices, the corresponding Excel worksheets can easily be revised to accommodate a 3×3 case or a 5×5 case. In either case, the corresponding changes to the worksheet for Figure 1 will be straightforward. For changes to the worksheet for Figure 2 for the 5×5 case, the cell formulas for orthonormalizing the given eigenvectors via the Gram-Schmidt process will inevitably be more tedious.

4.1 Figure 1

The 4×4 real symmetric matrix for the illustration in Figure 1 is

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0.8 & 0.4 \\ 1 & 1 & 0.4 & 0.5 \end{bmatrix}, \tag{34}$$

which is a variant of the 3×3 real symmetric matrix

$$\boldsymbol{Z} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & a \end{bmatrix}, \tag{35}$$

where $a = 0$ or $a < 1$, as considered in different notations by Swamy [1973], Prussing [1986], and Kerr [1990]. As none of the diagonal elements of $\boldsymbol{A}$ is zero or negative, we must rely on the usual procedures to verify whether the matrix is positive definite, positive semidefinite, or neither. With the second leading principal minor being zero, it is obvious that $\boldsymbol{A}$ cannot be positive definite according to Sylvester's criterion. The next step is to verify the positive semidefiniteness of $\boldsymbol{A}$; this task requires that the signs of the 15 ($= 2^4 - 1$) principal minors be examined.

In Figure 1, the 15 principal submatrices are displayed in B8:E53. The computations of the corresponding principal minors in G8:G53 are straightforward. The principal minors corresponding to the four single-element submatrices require no computations. The computations for the remaining principal minors are by using the Excel function MDETERM for determinants. The four leading principal minors, as displayed in B8, B16:C17, B34:D36, and B50:E53, are shaded. They are 1, 0, 0, and 0, indicating that $\boldsymbol{A}$ cannot be positive definite, as expected. As the lowest value of the 15 principal minors is -0.5, as displayed in H6, the positive semidefiniteness of $\boldsymbol{A}$ must also be rejected. Such results imply that at least one of the four eigenvalues of $\boldsymbol{A}$ is negative; however, an examination of the signs of the four leading principal minors alone has been unable to reveal such a feature. (See the fourth property in Appendix A.)

To connect directly the rejection of the positive semidefiniteness of $\boldsymbol{A}$ to the signs of its four eigenvalues requires that they be determined first. For the 4×4 real symmetric matrix $\boldsymbol{A}$ in equation (34), combining equations (A5), (A6), and (A8) in Appendix A. leads to

$$(-\lambda)^4 + 3.3(-\lambda)^3 - 1.16(-\lambda)^2 - 0.52(-\lambda) + 0 = 0 \tag{36}$$

	A	B	C	D	E	F	G	H
1	Symmetric Matrix A	1	1	1	1			
2		1	1	1	1			
3		1	1	0.8	0.4			
4		1	1	0.4	0.5			
5								
6							Principal Minor, Lowest	-0.5
7								
8	Principal Submatrices	1					Principal Minors	1
9								
10		1						1
11								
12		0.8						0.8
13								
14		0.5						0.5
15								
16		1	1					
17		1	1					0
18								
19		1	1					
20		1	0.8					-0.2
21								
22		1	1					
23		1	0.5					-0.5
24								
25		1	1					
26		1	0.8					-0.2
27								
28		1	1					
29		1	0.5					-0.5
30								
31		0.8	0.4					
32		0.4	0.5					0.24
33								
34		1	1	1				
35		1	1	1				
36		1	1	0.8				0
37								
38		1	1	1				
39		1	1	1				
40		1	1	0.5				0
41								
42		1	1	1				
43		1	0.8	0.4				
44		1	0.4	0.5				-0.26
45								
46		1	1	1				
47		1	0.8	0.4				
48		1	0.4	0.5				-0.26
49								
50		1	1	1	1			
51		1	1	1	1			
52		1	1	0.8	0.4			
53		1	1	0.4	0.5			0

Figure 1: An Excel-Based Illustration of Positive Definiteness and Positive Semidefiniteness Tests for a Given Real Symmetric Matrix.

or, equivalently,

$$\lambda\left(\lambda^3 - 3.3\lambda^2 - 1.16\lambda + 0.52\right) = 0. \tag{37}$$

As expected, the coefficient of the $(-\lambda)^3$ term in equation (36), which is 3.3, is the trace of $\boldsymbol{A}$. Also as expected is that the constant term on the left hand side of the same equation, which is zero, is the determinant of $\boldsymbol{A}$. In equation (37), one of the four eigenvalues is zero, and the remaining three eigenvalues are −0.54803, 0.26480, and 3.58323, which have been solved and verified by using the free online cubic equation calculators offered by *CalculatorSoup* or *keisan*.[2] (See the first three properties in Appendix A.)

The validity of equation (37) has also been confirmed, as the four eigenvalues based on it all satisfy the characteristic equation, which is equation (A5) in Appendix A. The computations of

$$|\boldsymbol{A} - \lambda_i \boldsymbol{I}| = \begin{vmatrix} (1-\lambda_i) & 1 & 1 & 1 \\ 1 & (1-\lambda_i) & 1 & 1 \\ 1 & 1 & (0.8-\lambda_i) & 0.4 \\ 1 & 1 & 0.4 & (0.5-\lambda_i) \end{vmatrix}, \tag{38}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the four solved eigenvalues, are by using directly the Excel function MDETERM. As one of the four eigenvalues based on equation (37) is zero, $\boldsymbol{A}^{-1}$ does not exist, thus confirming that $\boldsymbol{A}$ cannot be positive definite. Further, as one of the remaining eigenvalues is negative, the positive semidefiniteness of $\boldsymbol{A}$ must also be rejected, thus confirming the same conclusion as reached earlier. (See the fourth property in Appendix A.)

The same worksheet for Figure 1 can also be used for any 4×4 real symmetric matrices. Further, for the same matrix $\boldsymbol{A}$ in equation (34), by letting the skew-symmetric matrix be, for example,

$$\boldsymbol{S} = \begin{bmatrix} 0 & 0.2 & -0.1 & 0.1 \\ -0.2 & 0 & 0.1 & -0.2 \\ 0.1 & -0.1 & 0 & 0.3 \\ -0.1 & 0.2 & -0.3 & 0 \end{bmatrix}, \tag{39}$$

we can deduce — via equation (25) in Subsection 3.4 of the preceding section — a non-symmetric matrix

$$\boldsymbol{B} = \begin{bmatrix} 0.5 & 0.7 & 0.4 & 0.6 \\ 0.3 & 0.5 & 0.6 & 0.3 \\ 0.6 & 0.4 & 0.4 & 0.5 \\ 0.4 & 0.7 & -0.1 & 0.25 \end{bmatrix}, \tag{40}$$

[2] The corresponding electronic links are https://www.calculatorsoup.com/calculators/algebra/cubicequation.php and https://keisan.casio.com/exec/system/1181809414 .

for which equation (27) holds. Notice that a different choice of $\boldsymbol{S}$ will result in a different $\boldsymbol{B}$. As explained earlier, to address the concern raised by Bose [1968], the use of same worksheet for Figure 1 to check the positive definiteness or the positive semidefiniteness of $\boldsymbol{B}$ ought to be via $\boldsymbol{B} + \boldsymbol{B}'$ instead. Notice also that the end results pertaining to the positive definiteness and the positive semidefiniteness of $\boldsymbol{B} + \boldsymbol{B}'$ will be unaffected by the different choices of $\boldsymbol{S}$.

4.2 Figure 2

We now turn our attention to generating a matrix given its eigenvalues and the corresponding eigenvectors. Of special interest, therefore, is how the same 4×4 real symmetric matrix $\boldsymbol{A}$ in equation (34) can be generated from knowing its eigenvalues and the corresponding eigenvectors, instead of modifying an available 3×3 real symmetric matrix from the literature. In the illustration in Figure 1, as no eigenvectors have been deduced, they must be determined first. A set of four eigenvectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$, and $\boldsymbol{x}_4$ corresponding to λ_1, λ_2, λ_3, and λ_4, respectively, can be determined from equation (16), where $n = 4$.

As it turns out that the three eigenvectors corresponding to the three non-zero eigenvalues of the same matrix $\boldsymbol{A}$ in equation (34) are already orthogonal, they are unsuitable for illustrating the Gram-Schmidt process. Thus, it is better that the illustration in Figure 2 be based on a different set of input data. However, the worksheet showing how the same matrix $\boldsymbol{A}$ in equation (34) has been generated, from knowing its eigenvalues and the corresponding eigenvectors, is still in the Excel file accompanying this paper.

The Excel worksheet for Figure 2 is self-contained; that is, no online Gram-Schmidt calculator has been used. An obvious advantage of using a self-contained Excel worksheet is that there is no need to copy the online results manually to the worksheet for any subsequent computations. Figure 2 shows how both a 4×4 real symmetric matrix $\boldsymbol{A}$ and a related real non-symmetric matrix $\boldsymbol{B}$, as mentioned at the end of the preceding subsection, can be generated. The given data, including the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, the corresponding linearly independent eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$, and the skew-symmetric matrix $\boldsymbol{S}$, are displayed in B25:E28 as its four diagonal elements, in B1:E4 as its four columns, and in H30:K33, respectively. The cells containing such data are shaded. Notice that, on purpose, the four eigenvectors in B1:E4 are not based on a Hadamard matrix, although the Excel worksheet can accommodate such a matrix as given data for the same cells.

	A	B	C	D	E	F	G	H	I	J	K
1	Original Vectors v	1	0	0	2						
2		1	1	0	0						
3		0	1	1	0						
4		0	0	1	0						
5											
6	Lengths	1.414	1.414	1.414	2						
7											
8	Orthogonal Vec. W	1	-0.5	0.333	0.5						
9		1	0.5	-0.33	-0.5						
10		0	1	0.333	0.5						
11		0	0	1	-0.5						
12											
13	Lengths	1.414	1.225	1.155	1						
14											
15	Orthonormal Vec. X	0.707	-0.41	0.289	0.5		Transpose, X'	0.707	0.707	0	0
16		0.707	0.408	-0.29	-0.5			-0.41	0.408	0.816	0
17		0	0.816	0.289	0.5			0.289	-0.29	0.289	0.866
18		0	0	0.866	-0.5			0.5	-0.5	0.5	-0.5
19											
20	Product XX'	1	0	0	0						
21		0	1	6E-17	-0						
22		0	6E-17	1	2E-16						
23		0	-0	2E-16	1						
24											
25	Diag. Mat. Lambda	1	0	0	0						
26		0	3	0	0						
27		0	0	0	0						
28		0	0	0	2						
29											
30	Symmetric Mat. A	1.5	-0.5	-0.5	-0.5		Skew-Sym Mat. S	0	0.3	-0.1	0.1
31		-0.5	1.5	0.5	0.5			-0.3	0	0.15	-0.2
32		-0.5	0.5	2.5	-0.5			0.1	-0.15	0	0.2
33		-0.5	0.5	-0.5	0.5			-0.1	0.2	-0.2	0
34											
35	Lead. Prin. Min., A	1.5					Non-Sym. Mat. B	0.75	0.05	-0.35	-0.15
36			2					-0.55	0.75	0.4	0.05
37				4.5				-0.15	0.1	1.25	-0.05
38					4E-16			-0.35	0.45	-0.45	0.25

Figure 2: An Excel-Based Illustration of the Attainment of a Real Symmetric Matrix and a Related Non-Symmetric Matrix Given the Eigenvalues and the Corresponding Linearly Independent Eigenvectors of the Former Matrix.

The length of each eigenvector in B6:E6 is the square root of the sum of squares of the four individual elements of the eigenvector involved. The orthogonal vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3, \boldsymbol{w}_4$ and the corresponding orthonormal vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4$, as displayed in B8:E11 and B15:E18, respectively, have been obtained via the Gram-Schmidt process as described in Appendix B. The difference between these two sets of vectors is that the latter set has been normalized so that each vector is of unit length. This is indicated by $\boldsymbol{X}\boldsymbol{X}'$ being an identity matrix, as displayed in B20:E23, where

$$\boldsymbol{X} = \left[\begin{array}{cccc} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 & \boldsymbol{x}_n \end{array} \right]. \tag{41}$$

To avoid digressions, all cell formulas for the Excel worksheet in Figure 2 are listed in Appendix C instead.

As shown in B35:E38, the fourth leading principal minor of $\boldsymbol{A}$, which is also its determinant, is zero for practical purposes, thus ruling out its positive definiteness. Such an outcome is as expected, because the four eigenvalues have been set at 0, 1, 2, and 3, for which $\boldsymbol{A}^{-1}$ does not exist. The same set of eigenvalues also ensures that $\boldsymbol{A}$ be positive semidefinite. However, without knowing these eigenvalues in advance, one cannot make such a conclusion before first checking the signs of all principal minors. The task can easily be performed by copying the values of the cells in B30:E33 of the worksheet for Figure 2 to B1:E4 of the worksheet for Figure 1. As expected, the lowest value of the 15 principal minors in H6 of Figure 1 will become 0, thus confirming the positive semidefiniteness of this symmetric matrix $\boldsymbol{A}$.

5 Conclusion

Sylvester's criterion is an excellent analytical tool for verifying the positive definiteness of real symmetric matrices. Its misapplications have also been noted across various academic fields; so have its remedies. However, the corresponding illustrative examples from various published works have been confined to only a few 2×2 or 3×3 matrices. Indeed, there are not many available examples that instructors of courses covering Sylvester's criterion can use as additional examples, exercises, and examination questions for students. This paper has used self-contained Excel worksheets for the core computations to generate suitable 4×4 matrices for pedagogic purposes, intending to improve the depth of coverage of Sylvester's criterion in the courses involved.

This paper has replicated those materials in linear algebra that are essential for properly understanding Sylvester's criterion and remedies for its misapplications. Thus, this paper is also intended to help students connect directly the analytical tools involved and the underlying concepts. An effective approach to help students make the connection is via some Excel-based exercises. Such exercises will involve generating small-scale matrices of various dimensions, as well as performing positive definiteness and positive semidefiniteness tests for matrices provided by others, by using those analytical materials as replicated in this paper. The experience that students will gain from such exercises does go beyond learning the technical detail of the tasks involved; it also helps students appreciate more fully the usefulness and the limitations of Sylvester's criterion as an analytical tool.

Appendix A: Determinants, Eigenvalues, Eigenvectors, and Four Matrix Properties

The materials in this appendix is part of the standard coverage of introductory linear algebra courses. After briefly introducing determinants, eigenvalues, and eigenvectors, we identify four matrix properties pertaining to Sylvester's criterion and remedies for its misapplications. For each property, if the detail of a proof is directly related to the materials in the main text, it is included; otherwise, references are provided instead.

A Brief Introduction to Determinants, Eigenvalues, and Eigenvectors: Formally, the determinant of an $n \times n$ matrix $\boldsymbol{A}$, denoted as $|\boldsymbol{A}|$, where a_{ij} is its (i,j)-element, for $i, j = 1, 2, \ldots, n$, is

$$|\boldsymbol{A}| = \sum (\pm 1) a_{1i_1} a_{2i_2} \cdots a_{ni_n}, \tag{A1}$$

where the summation is over all $n!$ permutations of $i_1, i_2, \ldots, i_n$. The n different integers that $i_1, i_2, \ldots, i_n$ represent can be any permutation of $1, 2, \ldots, n$. If it takes an even number of interchanges, involving an adjacent pair of integers for each interchange, to rearrange these integers as $1, 2, \ldots, n$, then we use the multiplicative factor $(+1)$ for $a_{1i_1} a_{2i_2} \cdots a_{ni_n}$. If it takes an odd number of interchanges, then we use the multiplicative factor (-1) instead.

The determinant of $\boldsymbol{A}$ and the inverse of $\boldsymbol{A}$, denoted as $\boldsymbol{A}^{-1}$, are related by

$$\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|} \; adj \; \boldsymbol{A}, \tag{A2}$$

where *adj* $\boldsymbol{A}$, known as the adjoint of $\boldsymbol{A}$, is the transpose of an $n \times n$ matrix of cofactors of $\boldsymbol{A}$. Here, the (i,j)-element of the matrix of cofactors of $\boldsymbol{A}$ is $(-1)^{i+j}$ times the determinant of the $(n-1) \times (n-1)$ matrix resulted from the deletion of row i and row j of $\boldsymbol{A}$. The latter determinant is also known as the (i,j) minor, and principal minors pertain to cases where $i = j$. The existence of $\boldsymbol{A}^{-1}$, therefore, requires that $|\boldsymbol{A}|$ be non-zero. A non-zero $|\boldsymbol{A}|$ ensures that

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}, \tag{A3}$$

where $\boldsymbol{I}$ is an $n \times n$ identity matrix.

There is a non-zero n-element column vector $\boldsymbol{x}$, which satisfies the equation

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x} \tag{A4}$$

for some scalar λ. To ensure that $\boldsymbol{x}$ be a vector where not all n elements are zeros, the following equation, known as the characteristic equation, must hold:

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0. \tag{A5}$$

For the n-th order polynomial function

$$P(\lambda) = |\boldsymbol{A} - \lambda \boldsymbol{I}|, \tag{A6}$$

which is also known as the characteristic polynomial, the n roots of λ, denoted as $\lambda_1, \lambda_2, \ldots, \lambda_n$, are the n eigenvalues of $\boldsymbol{A}$.

In

$$\boldsymbol{A}\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i, \text{ for } i = 1, 2, \ldots, n, \tag{A7}$$

the n eigenvectors corresponding to the n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$. Eigenvectors are not unique. Of special interest are those that are linearly independent; that is, none of $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ can be replicated by a linear combination of the remaining eigenvectors.

First Property: The n eigenvalues of an $n \times n$ real symmetric matrix $\boldsymbol{A}$ are all real. This property can be proven by first assuming that they are complex instead. A proof by contradiction in the context here is to rule out such an assumption at the end of the proof. The corresponding detail of the proof, which is omitted here, is available in Kwan [2010, Appendix B]. This property has provided an important first step to connect Sylvester's criterion and symmetric matrices. It allows us to work with real eigenvalues and real eigenvectors for the tasks involved.

Second Property: The determinant of an $n \times n$ real matrix $\boldsymbol{A}$, which need not be symmetric, is the product of its n eigenvalues. To prove this property, we start with expressing the polynomial function $P(\lambda)$ in equation (A6) as

$$\begin{aligned} P(\lambda) &= (-\lambda)^n + b_{n-1}(-\lambda)^{n-1} + b_{n-2}(-\lambda)^{n-2} + \cdots + b_1(-\lambda) + b_0 \\ &= (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n), \end{aligned} \tag{A8}$$

where $b_0, b_1, \ldots, b_{n-2}, b_{n-1}$ are coefficients. It follows directly that

$$|\boldsymbol{A}| = P(0) = b_0 = \lambda_1\lambda_2\cdots\lambda_n, \tag{A9}$$

which is the product of the n eigenvalues.

Third Property: The trace of an $n \times n$ real matrix $\boldsymbol{A}$, which need not be symmetric, is the sum of its n eigenvalues. Here, the trace of $\boldsymbol{A}$, usually denoted as $tr(\boldsymbol{A})$, is the sum of the n diagonal elements of $\boldsymbol{A}$. It is also the coefficient b_{n-1} in equation (A8). That is,

$$tr(\boldsymbol{A}) = b_{n-1} = \lambda_1 + \lambda_2 + \cdots + \lambda_n. \tag{A10}$$

To prove this property requires an explicit expansion of $|\boldsymbol{A} - \lambda\boldsymbol{I}|$, with special attention to the $b_{n-1}(-\lambda)^{n-1}$ term in equation (A8). The detail of a proof, which draws on properties of the determinant of the sum of two matrices, is available in Marcus [1990].

Fourth Property: If the n eigenvalues of an $n \times n$ real symmetric matrix $\boldsymbol{A}$ are all positive (non-negative), it must be positive definite (positive semidefinite). To prove this property, we start with equation (A4), which leads to

$$\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}'\boldsymbol{x}. \tag{A11}$$

As the n elements of $\boldsymbol{x}$ in equation (A4) are not all zeros, the scalar that $\boldsymbol{x}'\boldsymbol{x}$ represents is always positive. Thus, $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ and λ must be of the same sign. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all positive (non-negative), $\boldsymbol{A}$ must be positive definite (positive semidefinite). Notice that, if any of $\lambda_1, \lambda_2, \ldots, \lambda_n$ is zero, as $|\boldsymbol{A}| = 0$, $\boldsymbol{A}^{-1}$ does not exist. The positive definiteness of $\boldsymbol{A}$ requires the existence of $\boldsymbol{A}^{-1}$; however, the existence of $\boldsymbol{A}^{-1}$ does not imply that $\boldsymbol{A}$ is positive definite. If any of $\lambda_1, \lambda_2, \ldots, \lambda_n$ is negative, $\boldsymbol{A}$ must be neither positive definite nor positive semidefinite.

Appendix B: Orthonormalization of Linearly Independent Eigenvectors

Given equation (17) in the main text, we can write

$$\boldsymbol{A} = \boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1}. \tag{B1}$$

However, the symmetry of the $n \times n$ matrix $\boldsymbol{A}$ thus generated is not guaranteed. To ensure that $\boldsymbol{A}$ be symmetric and that the original set of eigenvalues be retained, the original set of linearly independent eigenvectors must first be transformed into a set of orthonormal vectors.

The Gram-Schmidt process for such a task starts with the eigenvector that corresponds to λ_1. We go through the process iteratively, by removing the part of each eigenvector that is not orthogonal to the previous ones, until all eigenvectors become orthogonal to each other. The final step of the process is to normalize the resulting eigenvectors, so that each one is of a unit length.

For the original set of n linearly independent eigenvectors, denoted as $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$, we first obtain the orthogonal set $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$ iteratively, by letting

$$\begin{aligned}
\boldsymbol{w}_1 &= \boldsymbol{v}_1 \\
\boldsymbol{w}_2 &= \boldsymbol{v}_2 - \frac{\langle \boldsymbol{v}_2, \boldsymbol{w}_1 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle}\boldsymbol{w}_1, \\
\boldsymbol{w}_3 &= \boldsymbol{v}_3 - \frac{\langle \boldsymbol{v}_3, \boldsymbol{w}_1 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle}\boldsymbol{w}_1 - \frac{\langle \boldsymbol{v}_3, \boldsymbol{w}_2 \rangle}{\langle \boldsymbol{w}_2, \boldsymbol{w}_2 \rangle}\boldsymbol{w}_2, \\
&\vdots \\
\boldsymbol{w}_n &= \boldsymbol{v}_n - \frac{\langle \boldsymbol{v}_n, \boldsymbol{w}_1 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle}\boldsymbol{w}_1 - \frac{\langle \boldsymbol{v}_n, \boldsymbol{w}_2 \rangle}{\langle \boldsymbol{w}_2, \boldsymbol{w}_2 \rangle}\boldsymbol{w}_2 - \cdots - \frac{\langle \boldsymbol{v}_n, \boldsymbol{w}_{n-1} \rangle}{\langle \boldsymbol{w}_{n-1}, \boldsymbol{w}_{n-1} \rangle}\boldsymbol{w}_{n-1}.
\end{aligned} \tag{B2}$$

Here, $\langle \boldsymbol{v}_j, \boldsymbol{w}_k \rangle$ and $\langle \boldsymbol{w}_k, \boldsymbol{w}_k \rangle$ are scalars captured by the 1×1 matrices $\boldsymbol{v}_j' \boldsymbol{w}_k$ and $\boldsymbol{w}_k' \boldsymbol{w}_k$, respectively, for $j = 2, 3, \ldots, n$ and $k = 1, 2, \ldots, n-1$. The set of n orthonormal vectors, denoted as $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$, is obtained by letting

$$\boldsymbol{x}_i = \frac{\boldsymbol{w}_i}{\sqrt{\langle \boldsymbol{w}_i, \boldsymbol{w}_i \rangle}}, \text{ for } i = 1, 2, \ldots, n. \tag{B3}$$

Appendix C: Cell Formulas in the Worksheet for Figure 2

The cell formulas in Figure 2 are as follows:

- B6, =SQRT(MMULT(TRANSPOSE(B1:B4),B1:B4)), copied to B6:E6 and B13:E13.
- C8:C11, =C1:C4-(MMULT(TRANSPOSE(C1:C4),B8:B11))/(MMULT(TRANSPOSE (B8:B11),B8:B11))*B8:B11.
- D8:D11, =D1:D4-(MMULT(TRANSPOSE(D1:D4),B8:B11))/(MMULT(TRANSPOSE (B8:B11),B8:B11))*B8:B11-(MMULT(TRANSPOSE(D1:D4),C8:C11))/(MMULT(TRANSPOSE(C8:C11),C8:C11))*C8:C11.
- F8:F11, =E1:E4-(MMULT(TRANSPOSE(E1:E4),B8:B11))/(MMULT(TRANSPOSE (B8:B11),B8:B11))*B8:B11-(MMULT(TRANSPOSE(E1:E4),C8:C11))/(MMULT(TRANSPOSE(C8:C11),C8:C11))*C8:C11-(MMULT(TRANSPOSE(E1:E4),D8: D11))/(MMULT(TRANSPOSE(D8:D11),D8:D11))*D8:D11.
- H15:K18, =TRANSPOSE(B15:E18).
- B20:E23, =MMULT(H15:K18,B15:E18).
- B30:E33, =MMULT(B15:E18,MMULT(B25:E28,H15:K18)).
- B35, =MDETERM(B30:B30) copied to C36, D37, and E38.
- H35, =B30/2+H30 copied to H35:K38.

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