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A Richardson's Extrapolation Spreadsheet Calculator for Numerical Differentiation

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A Richardson's Extrapolation Spreadsheet Calculator for Numerical Differentiation

Abstract

Even though approximating the definite differentiation by Richardson's extrapolation method is straight forward, but its repetitive calculations are quite boring. Hence, there is a need to develop a suitable tool in teaching and learning for this method. Here, we developed a Richardson's extrapolation Excel spreadsheet calculator to approximate the numerical differentiation that can be used by students and educators who need its full solution. With an appropriate function entered by users using 3-point central formula and a starting step size h , the full solution of Richardson's extrapolation table up to level 4 can be obtained quickly and easily. In this spreadsheet calculator, users can extend the level to any desired level by themselves.

Keywords

Excel spreadsheet, Richardson's extrapolation

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Cover Page Footnote

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A Richardson's Extrapolation Spreadsheet Calculator for Definite Numerical Differentiation

Abstract

Even though approximating the definite differentiation by Richardson's extrapolation method is straight forward, but its repetitive calculations are quite boring. Hence, there is a need to develop a suitable tool in teaching and learning for this method. Here, we developed a Richardson's extrapolation Excel spreadsheet calculator to approximate the numerical differentiation that can be used by students and educators who need its full solution. With an appropriate function entered by users using 3-point central formula and a starting step size h , the full solution of Richardson's extrapolation table up to level 4 can be obtained quickly and easily. In this spreadsheet calculator, users can extend the level to any desired level by themselves.

Keywords: Excel spreadsheet, Richardson's extrapolation

1 Introduction

Many physical problems involving the rate of change need differentiation. For example, velocity is the rate of change of distance with respect to time, while acceleration is the rate of change of velocity with respect to time. The rate of change of y with respect to x can be calculated from the gradient of the curve in a graph of y against x .

For a function that is not easily differentiated analytically or only when a set of data is given, numerical differentiation can be used. Normally, numerical differentiation can be calculated using the difference formulae based on Taylor's series expansions. To obtain the desired accuracy of the numerical differentiation, we may use higher order difference formula, a smaller step size h or Richardson's extrapolation method. Approximating the differentiation numerically by using the Richardson's extrapolation method is straight forward, but its repetitive calculations are quite tedious. Hence, there is a need to design a calculator to help students and educators who need its full solutions.

A series of papers working on solving numerical methods in classroom and examination situations using spreadsheet which focus on systems of nonlinear and linear equations, approximation of interpolation, computing of eigenvalues, ordinary differential equations (ODEs) by the Fourth-order Runge-Kutta (RK4) and the Laplace equation can be seen in [1-9].

In the text book [10], some numerical algorithm implementations with Excel are included, but not the Richardson's extrapolation method in Excel. In [11], even though numerical algorithms are implemented using Excel, VBA programming is used and Richardson's extrapolation implementation method in Excel is not covered. Similarly, topic of numerical differentiations implementation in Excel is covered in [12] but Richardson's extrapolation is not included.

As such, literature works dealing with spreadsheet calculator for approximating the definite differentiation by using the Richardson's extrapolation method in Excel have not been explored as yet. Thus, in this paper, we developed a Richardson's extrapolation spreadsheet calculator to approximate the numerical differentiation.

This paper is organized as follows. Section 2 provides the background of Richardson's extrapolation method for numerical differentiation. While a numerical example is given in

Section 3, the spreadsheet calculator, its numerical solution and Excel commands are discussed in Section 4. Finally, a conclusion is given in the final part of this paper, Section 5.

2 Richardson’s Extrapolation

There are two ways to improve derivative estimates when employing finite divided differences; decreasing the step size or using a higher-order formula that employs more points. A third approach, based on Richardson’s extrapolation which uses two derivatives, estimates a more accurate approximation. [10]

For a given function of f , approximations D , to the derivative of f at a specified value of x , can be computed for a chosen value of h , using Richardson’s extrapolation method as follows: [13]

$$D(J, K) = \frac{4^K D(J, K-1) - D(J-1, K-1)}{4^K - 1} + O(h^{2K+2}), \quad J = 1, 2, \dots, \quad K = 1, 2, \dots \tag{1}$$

$$D(J, 0) = \frac{f(x+h_j) - f(x-h_j)}{2h_j}, \quad J = 0, 1, 2, \dots \tag{2}$$

$$h_0 = h, h_1 = \frac{h}{2}, h_2 = \frac{h}{4}, \dots, h_j = \frac{h}{2^j} \tag{3}$$

The iteration process (1) is repeated until $|D(J, J) - D(J, J-1)| < \epsilon$ or $|D(J, J) - D(J-1, J-1)| < \epsilon$ for a specified value of ϵ . The value of $D(J, J)$ then approximates $f'(x)$ at this level.

Richardson’s extrapolation table for K from 1 to 4 is shown in Table 1.

Table 1: Richardson’s Extrapolation table for level K values from 1 to 4

J	h_j	K				
		0	1	2	3	4
		$D(J, 0)$	$D(J, 1)$	$D(J, 2)$	$D(J, 3)$	$D(J, 4)$
0	h_0	$D(0, 0)$				
1	h_1	$D(1, 0)$	$D(1, 1)$			
2	h_2	$D(2, 0)$	$D(2, 1)$	$D(2, 2)$		
3	h_3	$D(3, 0)$	$D(3, 1)$	$D(3, 2)$	$D(3, 3)$	
4	h_4	$D(4, 0)$	$D(4, 1)$	$D(4, 2)$	$D(4, 3)$	$D(4, 4)$

3 Numerical Example

In this section, a numerical example to be solved by Richardson's extrapolation is provided.

Question

The velocity (meter per second, m/s) of an upward rocket at time t seconds is given by

$$v(t) = u \ln \left(\frac{m_0}{m_0 - qt} \right) - gt,$$

where m_0 is the initial mass (kilogram, kg) of a rocket at time $t = 0$ second, q is the rate at which fuel is expelled (kilogram per second, kg/s), and u is the velocity at which the fuel is being expelled (meter per second, m/s). The initial mass of the rocket is 100,000 kg and the rocket expels fuel at a velocity of 1400 m/s at a consumption rate of 1500 kg/s. Assume that g is the gravity force with $g = 9.81$ m/s².

By using Richardson's extrapolation,

- find the acceleration of the rocket at time $t = 8$ seconds with a step size of $h = 1$.
- find the absolute error if the exact acceleration is

$$a(t) = \frac{dv(t)}{dt} = \frac{uq}{m_0 - qt} - g.$$

Solution

- Given $m_0 = 100000$, $u = 1400$, $q = 1500$ and let $f(t)$ be the velocity function, $v(t)$, then

$$f(t) = v(t) = u \ln \left(\frac{m_0}{m_0 - qt} \right) - gt = 1400 \ln \left(\frac{100000}{100000 - 1500t} \right) - 9.81t.$$

Thus acceleration function is given by $a(t) = f'(t) = v'(t)$ can be approximated by Richardson's extrapolation. By using the 3-point central formula in (2), we get

$$D(J,0) = \frac{\left[1400 \ln \left(\frac{1000}{1000 - 15(t+h_1)} \right) - 9.81(t+h_1) \right] - \left[1400 \ln \left(\frac{1000}{1000 - 15(t-h_1)} \right) - 9.81(t-h_1) \right]}{2h_1}$$

with $t = 8$.

The numerical solution of this example is shown in Figure 1 in Section 4, where $a(8)$ is approximately 14.0536

- The absolute error is shown in cell F13 in Figure 1

4 The Excel Commands for Richardson’s Extrapolation

Figure 1 illustrates Richardson’s extrapolation spreadsheet calculator. The spreadsheet calculator is able to compute the Richardson’s extrapolation values up to level 4. The users need to enter the given function by using 3-point central formula into the cell C7, h into the cell B2 and a chosen point of x into cell D2. To calculate the absolute error, users need to enter the exact differentiated function into cell C13. Then, the absolute error will be calculated automatically in cell F13. To get the changes on the Richardson’s extrapolation table, users need to click on the **APPLY** button and the values in the Richardson’s extrapolation table will be calculated automatically. The final answer is shown in cell G11. Therefore, the spreadsheet calculator is very easy to be used by any users. If the desired solution is not yet achieved at level 4, users can extend the level up to the accuracy they need. The **APPLY** button is associated with the **MACRO** option in Excel spreadsheet recording the action of copy cell C7 to cells C8:C11.

	A	B	C	D	E	F	G	H	I	
1	Richardson's Extrapolation Spreadsheet Calculator									
2	h	1	x	8						
3										
4			K							
5			0	1	2	3	4			
6	J	h_J	D(J,0)	D(J,1)	D(J,2)	D(J,3)	D(J,4)	APPLY		
7	0	1	14.0559							
8	1	0.5	14.0542	14.0536						
9	2	0.25	14.0538	14.0536	14.0536					
10	3	0.125	14.0537	14.0536	14.0536	14.0536	Answer			
11	4	0.0625	14.0536	14.0536	14.0536	14.0536	14.0536			
12										
13	Exact		14.0536		 Error 	0.0000				

Figure 1: Solution of Richardson’s extrapolation spreadsheet calculator

Table 2 shows the Excel commands in cells B7, B8, C7 and D8 used in the spreadsheet calculator. The corresponding formulae within the same column can then be copied by dragging the formulae into the respective target cells. The rest of formulae in cell D9:G11 can be copied from cell D8.

Table 2: Excel Commands in the spreadsheet calculator

Cells	Excel Commands
B7	=B2
B8	=B7/2
C7	=(1400*LN(1000/(1000-15*(\$D\$2+B7)))-9.81*(\$D\$2+B7)-(1400*LN(1000/(1000-15*(\$D\$2-B7)))-9.81*(\$D\$2-B7)))/(2*B7)
D8	=(4^D\$5*C8-C7)/(4^D\$5-1)

5 Conclusion

A Richardson's extrapolation spreadsheet calculator has been developed to approximate the derivative of the given function at a chosen point with a step size h . The Excel commands that calculate Richardson's extrapolation values are given in Table 1 and the layout of the spreadsheet calculator is shown in Figure 1. The spreadsheet calculator is very user-friendly and it provides an alternative tool for approximating the numerical differentiation by Richardson's extrapolation. It can be used as a marking scheme for educators and students who need its full solutions. Moreover, it reduces the calculation time and hopefully can increase the students' learning ability.

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